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Certain theorems involving differential superordination and sandwich-type results

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Abstract. To obtain the main result of the present paper, we use the technique of differential superordination. As special cases of our main result, we obtain sufficient conditions for $f \in \mathcal{A}$ to be $\phi-$like, parabolic $\phi-$like, starlike, parabolic starlike, close-to-convex and uniform close-to-convex. We also obtain sandwich-type results regarding these functions. For demonstration of the results, we have plotted the images of open unit disk under certain functions using Mathematica 7.0.

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1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of the functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...$$

Let $\mathcal{A}$ be the class of functions f, analytic in the unit disk $\mathbb{E}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$.

Let $\mathcal{S}$ denote the class of all analytic univalent functions f defined in the open unit disk $\mathbb{E}$ which are normalized by the conditions $f(0) = f'(0) - 1 = 0$. The Taylor series expansion of any function $f \in \mathcal{S}$ is

$$f(z) = z + a_2 z^2 + a_3 z^3 + ...$$

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Let the functions f and g be analytic in $\mathbb{E}$. We say that f is subordinate to g written as $f \prec g$ in $\mathbb{E}$, if there exists a Schwarz function ϕ in $\mathbb{E}$ (i.e. ϕ is regular in $|z| < 1$, $\phi(0) = 0$ and $|\phi(z)| \leq |z| < 1$) such that

$$f(z) = g(\phi(z)),\ |z| < 1.$$

Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic function, p an analytic function in $\mathbb{E}$ with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and h be univalent in $\mathbb{E}$. Then the function p is said to satisfy first order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z),\ \Phi(p(0), 0; 0) = h(0). \tag{1.1}$$

A univalent function q is called dominant of the differential subordination (1.1) if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.1), is said to be the best dominant of (1.1). The best dominant is unique up to the rotation of $\mathbb{E}$.

Let $\Psi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic and univalent function in domain $\mathbb{C}^2 \times \mathbb{E}$, h be analytic function in $\mathbb{E}$, p be analytic and univalent in $\mathbb{E}$ with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$. Then p is called the solution of the first order differential superordination if

$$h(z) \prec \Psi(p(z), zp'(z); z), h(0) = \Psi(p(0), 0; 0). \tag{1.2}$$

An analytic function q is called a subordinant of the differential superordination (1.2) if $q \prec p$ for all p satisfying (1.2). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (1.2), is said to be the best subordinant of (1.2). The best subordinant is unique up to the rotation of $\mathbb{E}$.

A function $f \in \mathcal{A}$ is said to be starlike in the open unit disk $\mathbb{E}$, if it is univalent in $\mathbb{E}$ and $f(\mathbb{E})$ is a starlike domain. The well known condition for the members of class $\mathcal{A}$ to be starlike is that

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in \mathbb{E}.$$

Let $\mathcal{S}^*$ denote the subclass of $\mathcal{S}$ consisting of all univalent starlike functions with respect to the origin.
A function $f \in \mathcal{A}$ is said to be close-to-convex in $\mathbb{E}$, if there exists a starlike function g (not necessarily normalized) such that

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > 0,\ z \in \mathbb{E}.$$

In addition, if g is normalized by the conditions $g(0) = 0 = g'(0) - 1$, then the class of close-to-convex functions is denoted by $\mathcal{C}$.
A function $f \in \mathcal{A}$ is called parabolic starlike in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right|,\ z \in \mathbb{E}, \tag{1.3}$$

and the class of such functions is denoted by S_P.
A function $f \in \mathcal{A}$ is said to be uniformly close-to-convex in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > \left|\frac{zf'(z)}{g(z)} - 1\right|, \ z \in \mathbb{E}, \tag{1.4}$$

for some $g \in \mathcal{S}_P$. Let UCC denote the class of all such functions. Note that the function $g(z) \equiv z \in \mathcal{S}_P$. Therefore, for $g(z) \equiv z$, condition (1.4) becomes:

$$\Re\left(f'(z)\right) > \left|f'(z) - 1\right|, \ z \in \mathbb{E}. \tag{1.5}$$

Ronning [11] and Ma and Minda [6] studied the domain Ω and the function $q(z)$ defined below:

$$\Omega = \left\{u + iv : u > \sqrt{(u-1)^2 + v^2}\right\}.$$

Clearly the function

$$q(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$$

maps the unit disk $\mathbb{E}$ onto the domain Ω. Hence the conditions (1.3) and (1.5) are, respectively, equivalent to

$$\frac{zf'(z)}{f(z)} \prec q(z), \ z \in \mathbb{E},$$

and

$$f'(z) \prec q(z).$$

Let ϕ be analytic in a domain containing $f(\mathbb{E})$, $\phi(0) = 0$ and $\Re(\phi'(0)) > 0$. Then, the function $f \in \mathcal{A}$ is said to be $\phi-$ like in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{\phi(f(z))}\right) > 0, \ z \in \mathbb{E}.$$

This concept was introduced by Brickman [2]. He proved that an analytic function $f \in \mathcal{A}$ is univalent if and only if f is $\phi-$ like for some analytic function ϕ. Later, Ruscheweyh [12] investigated the following general class of $\phi-$like functions:
Let ϕ be analytic in a domain containing $f(\mathbb{E})$, where $\phi(0) = 0$, $\phi'(0) = 1$ and $\phi(w) \neq 0$ for some $w \in f(\mathbb{E})\backslash\{0\}$, then the function $f \in \mathcal{A}$ is called $\phi-$like with respect to a univalent function q, $q(0) = 1$, if

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z), \ z \in \mathbb{E}.$$

A function $f \in \mathcal{A}$ is said to be parabolic $\phi-$ like in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{\phi(f(z))}\right) > \left|\frac{zf'(z)}{\phi(f(z))} - 1\right|, \ z \in \mathbb{E}. \tag{1.6}$$

Equivalently, condition (1.6) can be written as:

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2.$$

In 2005, Ravichandran et al. [10] proved the following result for ϕ-like functions: Let $\alpha \neq 0$ be a complex number and $q(z)$ be a convex univalent function in $\mathbb{E}$. Suppose $h(z) = \alpha q^2(z) + (1-\alpha)q(z) + \alpha z q'(z)$ and

$$\Re\left\{\frac{1-\alpha}{\alpha} + 2q(z) + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0, \ z \in \mathbb{E}.$$

If $f \in \mathcal{A}$ satisfies

$$\frac{zf'(z)}{\phi(f(z))}\left(1 + \frac{\alpha z f''(z)}{f'(z)} + \frac{\alpha(f'(z) - (\phi(f(z)))'}{\phi(f(z))}\right) \prec h(z),$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z), \ z \in \mathbb{E},$$

and $q(z)$ is best dominant. Later on, Shanmugam et al. [13] and Ibrahim [9] also obtained the results for ϕ-like functions similar to the above mentioned results of Ravichandran [10].

In 2017, Kaur and Billing [4] investigated the following operator

$$a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)$$

to obtain $\phi-$likeness, starlikeness and close-to-convexity of normalized analytic functions.

Later, in 2019, Adegani et al. [1] studied the operator

$$\frac{\lambda z f'(z)}{g(z)}\left(1 + \frac{1}{\lambda} + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)}\right)$$

and derived criteria for close-to-convexity of normalized analytic functions.

Recently, Mohammed et al. [8] studied the geometric properties of some subfamilies of holomorphic functions in this direction.

In this paper, we obtain the superordination theorem for the differential operator

$$\left(\frac{zf'(z)}{\phi(g(z))}\right)^{\gamma}\left[a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)\right]^{\beta}$$

where $f, \ g \in \mathcal{A}$ and $\beta, \ \gamma$ be complex numbers such that $\beta \neq 0$. Also ϕ is an analytic function in a domain containing $g(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E})\backslash\{0\}$, for real numbers $a, \ b(\neq 0)$. Further, we derive sandwich-type theorem. As consequences of our main results, we obtain sufficient conditions for ϕ-like, parabolic $\phi-$like, starlike, parabolic starlike, close-to-convex, and uniform close-to-convex functions.

2. Preliminaries

We shall need the following definition and lemma to prove our main result.

Definition 2.1. ([7], Definition 2, p.817) Denote by $\mathbb{Q}$, the set of all functions $f(z)$ that are analytic and injective on $\bar{\mathbb{E}} \setminus \mathbb{E}(f)$, where

$$\mathbb{E}(f) = \left\{ \zeta \in \partial\mathbb{E} \ : \ \lim_{z \to \zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{E} \setminus \mathbb{E}(f)$.

Lemma 2.2. *([3]). Let q be univalent in $\mathbb{E}$ and let θ and φ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$. Set $Q_1(z) = zq'(z)\varphi[q(z)],\ h(z) = \theta[q(z)] + Q_1(z)$ and suppose that either*
(i) Q_1 is starlike and
(ii) $\Re\left(\frac{\theta' q(z)}{\varphi(q(z)})\right) > 0$ for all $z \in \mathbb{E}$.
If $p \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ with $p(\mathbb{E}) \subset \mathbb{D}$ and $\theta[p(z)] + zp'(z)\varphi[p(z)]$ is univalent in $\mathbb{E}$ and

$$\theta[q(z)] + zq'(z)\varphi[q(z)] \prec \theta[p(z)] + zp'(z)\varphi[p(z)],\ z \in \mathbb{E},$$

then $q(z) \prec p(z)$ and q is the best subordinant.

3. A superordination theorem

Theorem 3.1. *Let β and γ be complex numbers such that $\beta \neq 0$ and $a,\ b(\neq 0)$ are real numbers. Let $q(z) \neq 0$ with $q(0) = 1$ be a univalent function in $\mathbb{E}$, such that*
(i) $\Re\left[1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta} - 1\right)\frac{zq'(z)}{q(z)}\right] > 0$ and
(ii) $\Re\left[\frac{a}{b}\left(1 + \frac{\gamma}{\beta}\right)q(z)\right] > 0.$
Let ϕ be analytic function in the domain containing $g(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E}) \backslash \{0\}$. If $f,\ g \in \mathcal{A}$, $\frac{zf'(z)}{\phi(g(z))} \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ and

$$\left(\frac{zf'(z)}{\phi(g(z))}\right)^{\gamma}\left[a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)\right]^{\beta}$$

is univalent in $\mathbb{E}$, satisfy

$$(q(z))^{\gamma}\left[aq(z) + b\frac{zq'(z)}{q(z)}\right]^{\beta} \prec \left(\frac{zf'(z)}{\phi(g(z))}\right)^{\gamma}$$
$$\left[a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)\right]^{\beta} \tag{3.1}$$

then

$$q(z) \prec \frac{zf'(z)}{\phi(g(z))},\ z \in \mathbb{E},$$

and $q(z)$ is the best subordinant.

Proof. On writing $p(z) = \dfrac{zf'(z)}{\phi(g(z))}$, the superordination (3.1) can be rewritten as:

$$(q(z))^{\gamma}\left(aq(z)+b\frac{zq'(z)}{q(z)}\right)^{\beta} \prec (p(z))^{\gamma}\left(ap(z)+b\frac{zp'(z)}{p(z)}\right)^{\beta}$$

or

$$a(q(z))^{\frac{\gamma}{\beta}+1}+b(q(z))^{\frac{\gamma}{\beta}-1}zq'(z) \prec a(p(z))^{\frac{\gamma}{\beta}+1}+b(p(z))^{\frac{\gamma}{\beta}-1}zp'(z)$$

Let us define the functions θ and ϕ as follows:

$$\theta(w) = aw^{\frac{\gamma}{\beta}+1} \text{ and } \phi(w) = bw^{\frac{\gamma}{\beta}-1}$$

Obviously, the functions θ and ϕ are analytic in domain $\mathbb{D} = \mathbb{C}\backslash\{0\}$ and $\phi(w) \neq 0$ in $\mathbb{D}$.
Therefore,

$$Q(z) = \phi(q(z))zq'(z) = b(q(z))^{\frac{\gamma}{\beta}-1}zq'(z)$$

and

$$h(z) = \theta(q(z)) + Q(z) = a(q(z))^{\frac{\gamma}{\beta}+1}+b(q(z))^{\frac{\gamma}{\beta}-1}zq'(z)$$

On differentiating, we obtain

$$\frac{zQ'(z)}{Q(z)} = 1+\frac{zq''(z)}{q'(z)}+\left(\frac{\gamma}{\beta}-1\right)\frac{zq'(z)}{q(z)}$$

and

$$\frac{\theta'(q(z))}{\phi(q(z))} = \frac{zh'(z)}{Q(z)} - \frac{zQ'(z)}{Q(z)} = \frac{a}{b}\left(1+\frac{\gamma}{\beta}\right)q(z).$$

In view of the given condition (i) and (ii), we see that Q is starlike and

$$\Re\left(\frac{\theta'(q(z))}{\phi(q(z))}\right) > 0.$$

Therefore, the proof, now follows from the Lemma [2.2]. □

Remark 3.2. Together with the corresponding result for differential subordination (see Kaur et al. [5]), we get the following "sandwich result".

4. Sandwich-type result and its applications

Theorem 4.1. *Let β and γ be complex numbers such that $\beta \neq 0$ and $a,\ b(\neq 0)$ are real numbers. Let $q_1,\ q_2$ $(q_1(z) \neq 0,\ q_2(z) \neq 0,\ z \in \mathbb{E})$, be univalent functions in $\mathbb{E}$, such that*

(i) $\Re\left[1+\dfrac{zq_i''(z)}{q_i'(z)}+\left(\dfrac{\gamma}{\beta}-1\right)\dfrac{zq_i'(z)}{q_i(z)}\right] > 0$ *and*

(ii) $\Re\left[\dfrac{a}{b}\left(1+\dfrac{\gamma}{\beta}\right)q_i(z)\right] > 0;\ i = 1,\ 2.$

Let ϕ be analytic function in the domain containing $g(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0)-1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E})\backslash\{0\}$. If $f,\ g \in \mathcal{A},\ \dfrac{zf'(z)}{\phi(g(z))} \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ and

$$\left(\frac{zf'(z)}{\phi(g(z))}\right)^{\gamma}\left[a\frac{zf'(z)}{\phi(g(z))}+b\left(1+\frac{zf''(z)}{f'(z)}-\frac{z(\phi(g(z)))'}{\phi(g(z))}\right)\right]^{\beta}$$

is univalent in $\mathbb{E}$, satisfy

$$(q_1(z))^{\gamma}\left[aq_1(z)+b\frac{zq_1'(z)}{q_1(z)}\right]^{\beta}$$

$$\prec \left(\frac{zf'(z)}{\phi(g(z))}\right)^{\gamma}\left[a\frac{zf'(z)}{\phi(g(z))}+b\left(1+\frac{zf''(z)}{f'(z)}-\frac{z(\phi(g(z)))'}{\phi(g(z))}\right)\right]^{\beta}$$

$$\prec (q_2(z))^{\gamma}\left[aq_2(z)+b\frac{zq_2'(z)}{q_2(z)}\right]^{\beta} \tag{4.1}$$

then

$$q_1(z) \prec \frac{zf'(z)}{\phi(g(z))} \prec q_2(z),\ z \in \mathbb{E},$$

where $q_1(z)$ and $q_2(z)$ are the best subordinant and the best dominant respectively.

Remark 4.2. When we select $q_1(z) = 1 + m_1 z$, $q_2(z) = 1 + m_2 z$; $0 < m_1 < m_2 \leq 1$, $\beta = 1$, $\gamma = 0$ in Theorem 4.1, we obtain:

Corollary 4.3. *Let $a,\ b(\neq 0)$ are real numbers such that $\dfrac{a}{b} > 0$. Let ϕ be analytic function in the domain containing $g(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E})\backslash\{0\}$. If $f,\ g \in \mathcal{A},\ \dfrac{zf'(z)}{\phi(g(z))} \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ with*

$$a\frac{zf'(z)}{\phi(g(z))}+b\left(1+\frac{zf''(z)}{f'(z)}-\frac{z(\phi(g(z)))'}{\phi(g(z))}\right)$$

is univalent in $\mathbb{E}$ and satisfy

$$a(1+m_1z)+\frac{bm_1z}{1+m_1z} \prec \left[a\frac{zf'(z)}{\phi(g(z))}+b\left(1+\frac{zf''(z)}{f'(z)}-\frac{z(\phi(g(z)))'}{\phi(g(z))}\right)\right]$$

$$\prec a(1+m_2z)+\frac{bm_2z}{1+m_2z}$$

then

$$1+m_1z \prec \frac{zf'(z)}{\phi(g(z))} \prec 1+m_2z,\ \textit{where } 0<m_1<m_2\leq 1,\ z\in\mathbb{E}.$$

By selecting $a = 1,\ b = 1,\ m_1 = \frac{1}{3},\ m_2 = 1$ in Corollary 4.3, we get

Example 4.4. Let ϕ be analytic function in the domain containing $g(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E})\backslash\{0\}$. If $f,\ g \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(g(z))} \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ with

$$1 + \frac{zf''(z)}{f'(z)} + \frac{zf'(z)}{\phi(g(z))} - \frac{z(\phi(g(z)))'}{\phi(g(z))}$$

is univalent in $\mathbb{E}$ and satisfy

$$\frac{z^2+9z+9}{3z+9} \prec 1 + \frac{zf''(z)}{f'(z)} + \frac{zf'(z)}{\phi(g(z))} - \frac{z(\phi(g(z)))'}{\phi(g(z))} \prec \frac{z^2+3z+1}{z+1}$$

then

$$1 + \frac{z}{3} \prec \frac{zf'(z)}{\phi(g(z))} \prec 1 + z,\ z \in \mathbb{E}.$$

By selecting $g(z) = f(z)$ in Example 4.4, we have

Example 4.5. Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E})\backslash\{0\}$. If $f,\ g \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(f(z))} \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ with

$$1 + \frac{zf''(z)}{f'(z)} + \frac{zf'(z)}{\phi(f(z))} - \frac{z(\phi(f(z)))'}{\phi(f(z))}$$

is univalent in $\mathbb{E}$ and satisfy

$$\frac{z^2+9z+9}{3z+9} \prec 1 + \frac{zf''(z)}{f'(z)} + \frac{zf'(z)}{\phi(f(z))} - \frac{z(\phi(f(z)))'}{\phi(f(z))} \prec \frac{z^2+3z+1}{z+1}$$

then

$$1 + \frac{z}{3} \prec \frac{zf'(z)}{\phi(f(z))} \prec 1 + z,\ z \in \mathbb{E}.$$

i.e. f is $\phi-$like.

By selecting $\phi(z) = z$ and $g(z) = f(z)$ in Example 4.4, we get

Example 4.6. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ with $1 + \dfrac{zf''(z)}{f'(z)}$ is univalent in $\mathbb{E}$ and satisfies

$$\frac{z^2+9z+9}{3z+9} \prec 1 + \frac{zf''(z)}{f'(z)} \prec \frac{z^2+3z+1}{z+1}$$

then

$$1 + \frac{z}{3} \prec \frac{zf'(z)}{f(z)} \prec 1 + z,\ z \in \mathbb{E},$$

and hence $f(z)$ is starlike.

By selecting $\phi(z) = g(z) = z$ in Example 4.4, we have

Example 4.7. If $f \in \mathcal{A}$, $f'(z) \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$, with $f'(z) + \dfrac{zf''(z)}{f'(z)}$ is univalent in $\mathbb{E}$ and satisfy

$$\frac{z^2+9z+9}{3z+9} \prec f'(z) + \frac{zf''(z)}{f'(z)} \prec \frac{z^2+3z+1}{z+1}$$

then

$$1 + \frac{z}{3} \prec f'(z) \prec 1 + z,\ z \in \mathbb{E},$$

and hence $f(z)$ is close-to-convex.

For illustration, in Figure 4.1, we plot the images of unit disk $\mathbb{E}$ under the functions

$$w_1(z) = \frac{z^2+9z+9}{3z+9} \text{ and } w_2(z) = \frac{z^2+3z+1}{z+1}.$$

In Figure 4.2, the images of unit disk $\mathbb{E}$ under the functions

$$q_1(z) = 1 + \frac{z}{3} \text{ and } q_2(z) = 1 + z$$

are given. In the light of Example 4.4, when the differential operator

$$1 + \frac{zf''(z)}{f'(z)} + \frac{zf'(z)}{\phi(g(z))} - \frac{z(\phi(g(z)))'}{\phi(g(z))}$$

takes values in the light shaded portion as shown in Figure 4.1, then $\dfrac{zf'(z)}{\phi(g(z))}$ takes values in the light shaded region as given in Figure 4.2. Consequently, in view of Example 4.5, Example 4.6, Example 4.7, $f(z)$ is $\phi - like$, starlike and close-to-convex respectively.

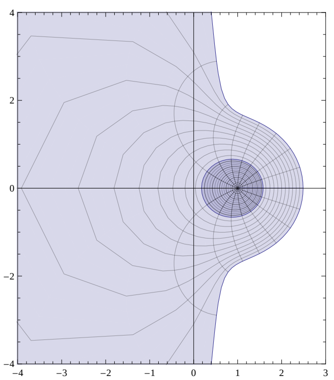

Figure 4.1

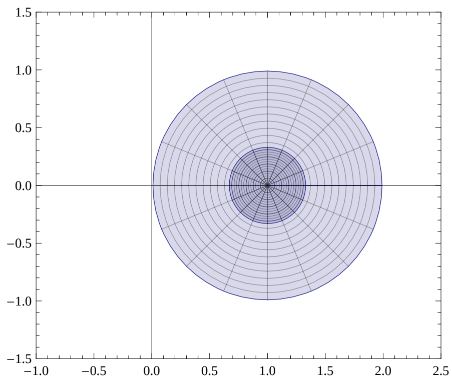

Figure 4.2

Remark 4.8. When we select

$$q_1(z) = \left(\frac{1+z}{1-z}\right)^{\delta_1},\ q_2(z) = \left(\frac{1+z}{1-z}\right)^{\delta_2},\ 0 < \delta_1 < \delta_2 \leq 1,\ \beta = 1,\ \gamma = 0$$

in Theorem 4.1, we obtain the following result:

Corollary 4.9. *For real numbers $a,\ b(\neq 0)$ with same sign. Let ϕ be analytic function in the domain containing $g(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E})\backslash\{0\}$. If $f,\ g \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(g(z))} \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ with*

$$a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)$$

is univalent in $\mathbb{E}$ and satisfy

$$a\left(\frac{1+z}{1-z}\right)^{\delta_1} + \left(\frac{2b\delta_1 z}{1-z^2}\right) \prec a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)$$
$$\prec a\left(\frac{1+z}{1-z}\right)^{\delta_2} + \left(\frac{2b\delta_2 z}{1-z^2}\right),$$

then

$$\left(\frac{1+z}{1-z}\right)^{\delta_1} \prec \frac{zf'(z)}{\phi(g(z))} \prec \left(\frac{1+z}{1-z}\right)^{\delta_2};0 < \delta_1 < \delta_2 \leq 1,\ z \in \mathbb{E}.$$

Selecting $\delta_1 = 0.3,\ \delta_2 = 1$ and $a = 1,\ b = 1$ in Corollary 4.9, we have:

Example 4.10. Let ϕ be analytic function in the domain containing $g(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E})\backslash\{0\}$. If $f,\ g \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(g(z))} \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ with

$$1 + \frac{zf''(z)}{f'(z)} + \frac{zf'(z)}{\phi(g(z))} - \frac{z(\phi(g(z)))'}{\phi(g(z))}$$

is univalent in $\mathbb{E}$ and satisfy

$$\left(\frac{1+z}{1-z}\right)^{0.3} + \left(\frac{0.6z}{1-z^2}\right) \prec \frac{zf'(z)}{\phi(g(z))} + \left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)$$
$$\prec \left(\frac{1+z}{1-z}\right) + \left(\frac{2z}{1-z^2}\right),$$

then

$$\left(\frac{1+z}{1-z}\right)^{0.3} \prec \frac{zf'(z)}{\phi(g(z))} \prec \left(\frac{1+z}{1-z}\right);\ z \in \mathbb{E}.$$

By selecting $g(z) = f(z)$ in Example 4.10, we get

Example 4.11. Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E})\backslash\{0\}$. If $f,\ g \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(f(z))} \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ with

$$1 + \frac{zf''(z)}{f'(z)} + \frac{zf'(z)}{\phi(f(z))} - \frac{z(\phi(f(z)))'}{\phi(f(z))}$$

is univalent in $\mathbb{E}$ and satisfy

$$\left(\frac{1+z}{1-z}\right)^{0.3}+\left(\frac{0.6z}{1-z^2}\right)\prec\frac{zf'(z)}{\phi(f(z))}+\left(1+\frac{zf''(z)}{f'(z)}-\frac{z(\phi(f(z)))'}{\phi(f(z))}\right)$$
$$\prec\left(\frac{1+z}{1-z}\right)+\left(\frac{2z}{1-z^2}\right),$$

then

$$\left(\frac{1+z}{1-z}\right)^{0.3}\prec\frac{zf'(z)}{\phi(f(z))}\prec\left(\frac{1+z}{1-z}\right);\ z\in\mathbb{E}.$$

i.e. f is $\phi-$like.

By selecting $\phi(z)=z$ and $g(z)=f(z)$ in Example 4.10, we obtain

Example 4.12. If $f\in\mathcal{A}$, $\dfrac{zf'(z)}{f(z)}\in\mathcal{H}[1,\ 1]\cap\mathbb{Q}$ with $1+\dfrac{zf''(z)}{f'(z)}$ is univalent in $\mathbb{E}$ and satisfies

$$\left(\frac{1+z}{1-z}\right)^{0.3}+\left(\frac{0.6z}{1-z^2}\right)\prec\left(1+\frac{zf''(z)}{f'(z)}\right)\prec\left(\frac{1+z}{1-z}\right)+\left(\frac{2z}{1-z^2}\right),$$

then

$$\left(\frac{1+z}{1-z}\right)^{0.3}\prec\frac{zf'(z)}{f(z)}\prec\left(\frac{1+z}{1-z}\right);\ z\in\mathbb{E}.$$

i.e. f is starlike.

By selecting $\phi(z)=g(z)=z$ in Example 4.10, we have

Example 4.13. If $f\ \in\mathcal{A}$, $f'(z)\in\mathcal{H}[1,\ 1]\cap\mathbb{Q}$, with $f'(z)+\dfrac{zf''(z)}{f'(z)}$ is univalent in $\mathbb{E}$ and satisfy

$$\left(\frac{1+z}{1-z}\right)^{0.3}+\left(\frac{0.6z}{1-z^2}\right)\prec f'(z)+\frac{zf''(z)}{f'(z)}\prec\left(\frac{1+z}{1-z}\right)+\left(\frac{2z}{1-z^2}\right),$$

then

$$\left(\frac{1+z}{1-z}\right)^{0.3}\prec f'(z)\prec\left(\frac{1+z}{1-z}\right);\ z\in\mathbb{E}.$$

i.e. f is close-to-convex.

Using Mathematica 7.0, we plot the images of unit disk $\mathbb{E}$ under the functions

$$w_3(z)=\left(\frac{1+z}{1-z}\right)^{0.3}+\frac{0.6z}{1-z^2}\ \text{and}\ w_4(z)=\frac{1+z}{1-z}+\frac{2z}{1-z^2},$$

which are given by Figure 4.3 and the images of unit disk $\mathbb{E}$ under the functions

$$q_1(z)=\left(\frac{1+z}{1-z}\right)^{0.3}\ \text{and}\ q_2(z)=\frac{1+z}{1-z},$$

which are shown in Figure 4.4. It follows from Example 4.10 that the differential operator $\dfrac{zf'(z)}{\phi(g(z))}$ takes values in the light shaded region of Figure 4.4 when the differential operator

$$\frac{zf'(z)}{\phi(g(z))} + \left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)$$

takes values in the light shaded region of Figure 4.3. Therefore, from Example 4.11, Example 4.12, Example 4.13, we can say that $f(z)$ is $\phi - like$, starlike and close-to-convex respectively.

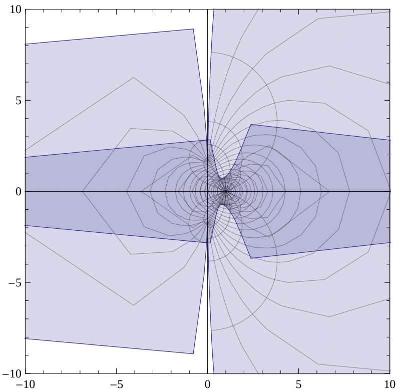

Figure 4.3

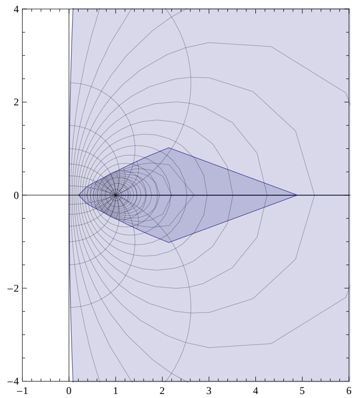

Figure 4.4

Remark 4.14. When we select $q_1(z) = e^{z/2},\ q_2(z) = \dfrac{1+z}{1-z},\ \beta = 1,\ \gamma = 0$ in Theorem 4.1, we get the following result:

Corollary 4.15. *For real numbers* $a,\ b(\neq 0)$ *of same sign. Let* ϕ *be analytic function in the domain containing* $g(\mathbb{E})$ *such that* $\phi(0) = 0 = \phi'(0) - 1$ *and* $\phi(w) \neq 0$ *for* $w \in g(\mathbb{E})\setminus\{0\}$. *If* $f,\ g \in \mathcal{A},\ \dfrac{zf'(z)}{\phi(g(z))} \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ *with*

$$a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)$$

is univalent in $\mathbb{E}$ *and satisfy*

$$ae^{z/2} + \frac{bz}{2} \prec a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)$$
$$\prec a\left(\frac{1+z}{1-z}\right) + \left(\frac{2bz}{1-z^2}\right),$$

then

$$e^{z/2} \prec \frac{zf'(z)}{\phi(g(z))} \prec \frac{1+z}{1-z},\ 0 \leq \delta < 1,\ z \in \mathbb{E}.$$

Selecting $a = 1$ and $b = 1$ in Corollary 4.15, we obtain:

Example 4.16. Let ϕ be analytic function in the domain containing $g(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E})\backslash\{0\}$. If $f,\ g \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(g(z))} \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ with $1 + \dfrac{zf''(z)}{f'(z)} + \dfrac{zf'(z)}{\phi(g(z))} - \dfrac{z(\phi(g(z)))'}{\phi(g(z))}$ is univalent in $\mathbb{E}$ and satisfies

$$e^{z/2} + \frac{z}{2} \prec \frac{zf'(z)}{\phi(g(z))} + \left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right) \prec \frac{z^2+4z+1}{1-z^2},$$

then

$$e^{z/2} \prec \frac{zf'(z)}{\phi(g(z))} \prec \frac{1+z}{1-z},\ 0 \leq \delta < 1,\ z \in \mathbb{E}.$$

By selecting $g(z) = f(z)$ in Example 4.16, we get

Example 4.17. Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E})\backslash\{0\}$. If $f,\ g \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(f(z))} \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ with $1 + \dfrac{zf''(z)}{f'(z)} + \dfrac{zf'(z)}{\phi(f(z))} - \dfrac{z(\phi(f(z)))'}{\phi(f(z))}$ is univalent in $\mathbb{E}$ and satisfy

$$e^{z/2} + \frac{z}{2} \prec \frac{zf'(z)}{\phi(f(z))} + \left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))}\right) \prec \frac{z^2+4z+1}{1-z^2},$$

then

$$e^{z/2} \prec \frac{zf'(z)}{\phi(f(z))} \prec \frac{1+z}{1-z},\ 0 \leq \delta < 1,\ z \in \mathbb{E}.$$

i.e. f is $\phi-$like.

By selecting $\phi(z) = z$ and $g(z) = f(z)$ in Example 4.16, we have

Example 4.18. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ with $1 + \dfrac{zf''(z)}{f'(z)}$ is univalent in $\mathbb{E}$ and satisfies

$$e^{z/2} + \frac{z}{2} \prec \left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \frac{z^2+4z+1}{1-z^2},$$

then

$$e^{z/2} \prec \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z},\ z \in \mathbb{E}.$$

i.e. f is starlike.

By selecting $\phi(z) = g(z) = z$ in Example 4.10, we obtain

Example 4.19. If $f\ \in \mathcal{A}$, $f'(z) \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$, with $f'(z) + \dfrac{zf''(z)}{f'(z)}$ is univalent in $\mathbb{E}$ and satisfy

$$e^{z/2} + \frac{z}{2} \prec f'(z) + \frac{zf''(z)}{f'(z)} \prec \frac{z^2+4z+1}{1-z^2},$$

then

$$e^{z/2} \prec f'(z) \prec \frac{1+z}{1-z}, \ z \in \mathbb{E}.$$

i.e. f is close-to-convex.

For demonstration, we plot the images of unit disk $\mathbb{E}$ under the functions

$$w_5(z) = e^{z/2} + \frac{z}{2} \text{ and } w_6(z) = \frac{z^2+4z+1}{1-z^2},$$

which are shown by Figure 4.5. In Figure 4.6, the images of unit disk $\mathbb{E}$ under the functions

$$q_1(z) = e^{z/2} \text{ and } q_2(z) = \frac{1+z}{1-z}$$

are given. It follows from Example 4.16 that the differential operator $\dfrac{zf'(z)}{\phi(g(z))}$ takes values in the light shaded region of Figure 4.6 when the differential operator

$$\frac{zf'(z)}{\phi(g(z))} + \left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)$$

takes values in the light shaded portion of Figure 4.5. Thus in view of Example 4.17, Example 4.18, Example 4.19, $f(z)$ is $\phi-like$, starlike and close-to-convex respectively.

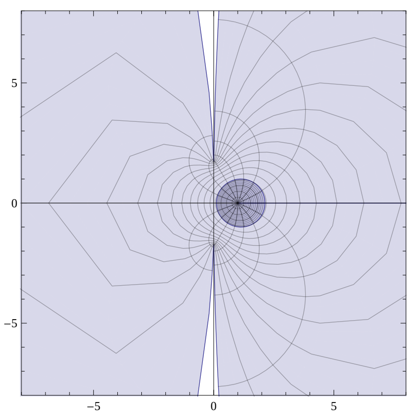

Figure 4.5

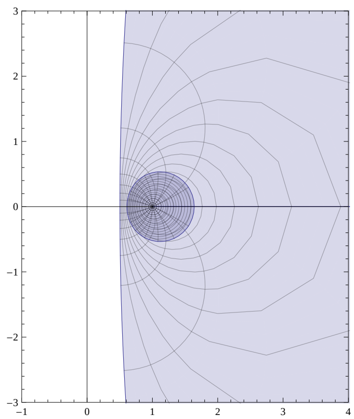

Figure 4.6

Remark 4.20. When we select

$$q_1(z) = e^{z/2}, \ q_2(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2, \ \beta = 1, \ \gamma = 0$$

in Theorem 4.1, we derive the following result:

Corollary 4.21. *For real numbers a, $b(\neq 0)$ of same sign. Let ϕ be analytic function in the domain containing $g(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for*

$w \in g(\mathbb{E})\backslash\{0\}$. If $f,\ g \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(g(z))} \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ with

$$a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)$$

is univalent in $\mathbb{E}$ and satisfy

$$ae^{z/2} + \frac{bz}{2} \prec a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)$$

$$\prec \left\{a + \frac{2a}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 + \frac{\frac{4b\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}$$

then

$$e^{z/2} \prec \frac{zf'(z)}{\phi(g(z))} \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,\ z \in \mathbb{E}.$$

Selecting $a = 1$ and $b = 1$ in Corollary 4.21, we obtain:

Example 4.22. Let ϕ be analytic function in the domain containing $g(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E})\backslash\{0\}$. If $f,\ g \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(g(z))} \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ with

$$1 + \frac{zf''(z)}{f'(z)} + \frac{zf'(z)}{\phi(g(z))} - \frac{z(\phi(g(z)))'}{\phi(g(z))}$$

is univalent in $\mathbb{E}$ and satisfies

$$e^{z/2} + \frac{z}{2} \prec \frac{zf'(z)}{\phi(g(z))} + \left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)$$

$$\prec \left\{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 + \frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}$$

then

$$e^{z/2} \prec \frac{zf'(z)}{\phi(g(z))} \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,\ z \in \mathbb{E}.$$

By selecting $g(z) = f(z)$ in Example 4.22, we get

Example 4.23. Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that

$$\phi(0) = 0 = \phi'(0) - 1 \text{ and } \phi(w) \neq 0 \text{ for } w \in f(\mathbb{E})\backslash\{0\}.$$

If $f,\ g \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(f(z))} \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ with $1 + \dfrac{zf''(z)}{f'(z)} + \dfrac{zf'(z)}{\phi(f(z))} - \dfrac{z(\phi(f(z)))'}{\phi(f(z))}$ is univalent in $\mathbb{E}$ and satisfies

$$e^{z/2} + \frac{z}{2} \prec \frac{zf'(z)}{\phi(f(z))} + \left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))}\right)$$

$$\prec \left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2+\frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}$$

then

$$e^{z/2}\prec\frac{zf'(z)}{\phi(f(z))}\prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,\ z\in\mathbb{E}.$$

i.e. f is parabolic $\phi-$like.

By selecting $\phi(z)=z$ and $g(z)=f(z)$ in Example 4.22, we have

Example 4.24. If $f\in\mathcal{A}$, $\dfrac{zf'(z)}{f(z)}\in\mathcal{H}[1,\ 1]\cap\mathbb{Q}$ with $1+\dfrac{zf''(z)}{f'(z)}$ is univalent in $\mathbb{E}$ and satisfy

$$e^{z/2}+\frac{z}{2}\prec\left(1+\frac{zf''(z)}{f'(z)}\right)\prec\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2+\frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}$$

then

$$e^{z/2}\prec\frac{zf'(z)}{f(z)}\prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,\ z\in\mathbb{E}.$$

i.e. f is parabolic starlike.

By selecting $\phi(z)=g(z)=z$ in Example 4.22, we obtain

Example 4.25. If $f\ \in\mathcal{A}$, $f'(z)\in\mathcal{H}[1,\ 1]\cap\mathbb{Q}$, with $f'(z)+\dfrac{zf''(z)}{f'(z)}$ is univalent in $\mathbb{E}$ and satisfies

$$e^{z/2}+\frac{z}{2}\prec f'(z)+\frac{zf''(z)}{f'(z)}\prec\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2+\frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}$$

then

$$e^{z/2}\prec f'(z)\prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,\ z\in\mathbb{E}.$$

i.e. f is uniform close-to-convex.

Using Mathematica 7.0, we draw the images of unit disk $\mathbb{E}$ under the functions

$$w_7(z)=e^{z/2}+\frac{z}{2}\text{ and }w_8(z)=\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2+\frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\},$$

which are shown by Figure 4.7 and the images of unit disk $\mathbb{E}$ under the functions

$$q_1(z)=e^{z/2}\text{ and }q_2(z)=1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$$

are given by Figure 4.8. Hence from Example 4.22, we can say that the differential operator $\dfrac{zf'(z)}{\phi(g(z))}$ takes values in the light shaded portion of Figure 4.8 when the differential operator $\dfrac{zf'(z)}{\phi(g(z))} + \left(1 + \dfrac{zf''(z)}{f'(z)} - \dfrac{z(\phi(g(z)))'}{\phi(g(z))}\right)$ takes values in the light shaded region of Figure 4.7. Therefore, in light of Example 4.23, Example 4.24, Example 4.25, $f(z)$ is parabolic $\phi-$like, parabolic starlike and uniform close-to-convex respectively.

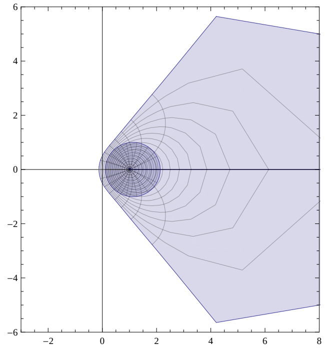

Figure 4.7

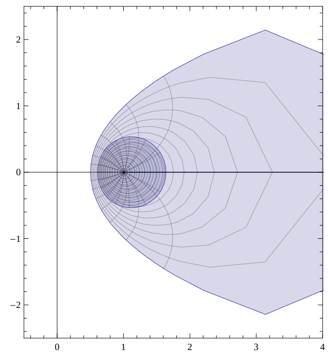

Figure 4.8

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References

[1] Adegani, E.A., Bulboacă T., Motamednezhad, A., *Simple sufficient subordination conditions for close-to-convexity*, Mathematics, **7**(2019), no. 3.

[2] Brickman, L., *$\phi-$like analytic functions*, I, Bull. Amer. Math. Soc., **79**(1973), 555-558.

[3] Bulboacă T., *Differential Subordinations and Superordinations: Recent Results*, House of science Book Publ., Cluj-Napoca 2005.

[4] Kaur, P., Billing, S.S., *Some sandwich type results for $\phi-$ like functions*, Acta Univer. Apul., **51**(2017), 115-134.

[5] Kaur, H., Brar, R., Billing, S.S., *Certain sufficient conditions for $\phi-$ like functions in a parabolic region*, Stud. Univ. Babeş-Bolyai Math., (Accepted).

[6] Ma, W.C., Minda, D., *Uniformly convex functions*, Ann. Polon. Math., **57**(1992), no. 2, 165-175.

[7] Miller, S.S., Mocanu, P.T., *Differential Subordinations: Theory and Applications*, Marcel Dekker, New York and Basel, 2000.

[8] Mohammed, N.H., Adegani, E.A., Bulboaca, T., Cho, N.E., *A family of holomorphic functions defined by differential inequality*, Math. Inequal. Appl., **25**(2022), no. 1, 27-39.

[9] Rabha, W.I., *On certain univalent class associated with first order differential subordinations*, Tamkang J. Math., **42**(2011), no. 4, 445-451.

[10] Ravichandran, V., Mahesh, N., Rajalakshmi, R., *On certain applications of differential subordinations for ϕ-like functions*, Tamkang J. Math., **36**(2005), no. 2, 137-142.

[11] Ronning, F., *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc., **118**(1993), no. 1, 189-196.

[12] Ruscheweyh, St., *A subordination theorem for ϕ-like functions*, J. London Math. Soc., **2**(1976), no. 13, 275-280.

[13] Shanmugam, T.N., Sivassubramanian, S., Darus, M., *Subordination and superordination results for ϕ-like functions*, Journal of Ineq. in Pure and Applied Mathematics, **8**(2007), no. 1, Art. 20, 1-6.

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