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Asymptotic behavior of generalized $CR-$iteration algorithm and application to common zeros of accretive operators

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Abstract. The purpose of this study is to provide a generalized $CR-$iteration algorithm for finding common fixed points (CFP_s) for nonself quasi-nonexpansive mappings ($QNEMs$) in a uniformly convex Banach space. The suggested algorithm's convergence analysis is analyzed in uniformly convex Banach spaces.

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1. Introduction

Let $\mathfrak{B}$ be a Banach space, $\emptyset \neq \mathfrak{B}_s \subseteq \mathfrak{B}$ be closed and convex, and $\Upsilon : \mathfrak{B}_s \to \mathfrak{B}_s$ be an operator which has at least one fixed point. Then, for the initial value $\mathfrak{a}_0 \in \mathfrak{B}_s$:

(i) Picard's iteration algorithm [16] is defined as :
$\mathfrak{a}_{\eta+1} = \Upsilon \mathfrak{a}_\eta, \ \forall \ \eta \in \mathbb{N}_0.$

(ii) Mann's iteration algorithm [13] is defined as:
$\mathfrak{a}_{\eta+1} = (1-\kappa_\eta)\mathfrak{a}_\eta + \kappa_\eta \Upsilon \mathfrak{a}_\eta, \ \forall \ \eta \in \mathbb{N}_0,$
where $\{\kappa_\eta\} \in (0,1)$.

(iii) Ishikawa's iteration algorithm [8] is defined as:
$\mathfrak{a}_{\eta+1} = (1-\kappa^1_\eta)\mathfrak{a}_\eta + \kappa^1_\eta \Upsilon[(1-\kappa^2_\eta)\mathfrak{a}_\eta + \kappa^2_\eta \Upsilon \mathfrak{a}_\eta], \ \forall \ \eta \in \mathbb{N}_0,$
where $\{\kappa^1_\eta\}$ and $\{\kappa^2_\eta\} \in (0,1)$.

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For nonexpansive operators, it is very well established that the Picard iteration algorithm often does not work effectively. As a result, for the estimation of FPs for nonexpansive type mappings in ambient spaces, the Mann and Ishikawa iterative algorithms have been extensively studied (see [1, 3, 6]).
On the other side, Chug et al. [5] introduced the $CR-$iteration algorithm in a Banach space in 2012. The structure of the $CR-$iterative algorithm differs significantly from that of the Mann and Ishikawa iterative algorithms, making it absolutely independent of both. Several mathematicians have been intrigued by the $CR-$iterative algorithm as an alternative iterative algorithm for fixed point analysis in recent years (see [9, 2]), and it has opened up a substantial field of research in various aspects (see [11, 12]).

Let Υ be a self map on $\mathfrak{B}$. Then the sequence $\{\mathfrak{a}_\eta\}_{n=0}^{\infty}$ defined as follows:

$$\begin{cases} \mathfrak{a}_0 \in \mathfrak{B} \\ \mathfrak{a}_{\eta+1} = (1-\kappa_\eta^1)\mathfrak{b}_\eta + \kappa_\eta^1 \Upsilon \mathfrak{b}_\eta, \\ \mathfrak{b}_\eta = (1-\kappa_\eta^2)\Upsilon \mathfrak{a}_\eta + \kappa_\eta^2 \Upsilon \mathfrak{c}_\eta, \\ \mathfrak{c}_\eta = (1-\kappa_\eta^3)\mathfrak{a}_\eta + \kappa_\eta^3 \Upsilon \mathfrak{a}_\eta, \end{cases} \tag{CR}$$

where $\{\kappa_\eta^1\}$, $\{\kappa_\eta^2\}$ and $\{\kappa_\eta^3\} \in (0,1)$ is called $CR-$iteration. The $CR-$iteration method is a three-step iteration method. For contraction mappings, $CR-$iterative algorithms perform better than Picard and Ishikawa iterative algorithms, and behave well for nonexpansive mappings.
We are concerned with two quasi-nonexpansive nonself mappings $\mathcal{M}_1,\ \mathcal{M}_2 : \mathfrak{B}_s \to \mathfrak{B}$, where $\mathfrak{B}_s$ is a nonempty subset of the Banach space $\mathfrak{B}$, the iterative location and weak limits of the proposed iterative algorithm for these types of functions in the context of current research [19]. Our findings are applied to the zeros of accretive operators in some different ways.

2. Tools and notations

In this section, we discuss the notations which we are going to use in the entire manuscript. The framework in which we shall prove our results from now on is a Banach space $\mathfrak{B}$. Υ is a mapping. $\mathbb{N}_0$ represents the set of natural numbers including 0, whereas the terminology $\mathbb{R}$ is used to represent the set of real numbers. The notation 'for all'is represented by '$\forall$'and 'such that'is represented by '$\ni$'. The symbol $\in$ represents 'belongs to'. The terminology $\mathcal{H}_s$ is used to represent the 'Hilbert space'with the inner product $\langle \cdot, \cdot \rangle$ and whereas $\mathcal{Q}_{\mathfrak{B}_s}$ is a retraction of $\mathfrak{B}$ onto $\mathfrak{B}_s$. $\mathcal{P}_{\mathfrak{B}_s}$ is used to represent the projection from $\mathfrak{B}$ to $\mathfrak{B}_s$. $\mathcal{H}_s^{'} \subseteq \mathcal{H}_s$. $Dom(\mathcal{A})$ represents the domain of $\mathcal{A}$, $Ran(\mathcal{A})$ is used to represent the range set of $\mathcal{A}$, and $Gr(\mathcal{A})$ is the graph of $\mathcal{A}$ whereas $\mathcal{A}^{-1}$ is the inverse of $\mathcal{A}$. Δ is a non-negative real number. The terminology 'fixed points', we denote by 'FP_s'.The Proximal point algorithm is denoted by 'PPA'. It is important to note that the 'set of all fixed points'is denoted by '$\mathcal{F}_{(\Upsilon)}$'. Furthermore, ∇ is used to represent the 'vector differential operator'.

3. Preliminaries

In this section, we discuss key definitions and lemmas that are necessary in order to make this article self-contained.
Throughout the paper, we denote the closed ball with the center at $\mathfrak{a}$ and radius r by $\mathcal{CB}_r[\mathfrak{a}]$ and is defined as

$$\mathcal{CB}_r[\mathfrak{a}] = \{\mathfrak{b} \in \mathfrak{B} : ||\mathfrak{a} - \mathfrak{b}|| \leq r\}.$$

Also, $\mathfrak{B}$ is said to be uniformly convex if for $0 < \epsilon \leq 2$, $||\mathfrak{a}|| \leq 1$, $||\mathfrak{b}|| \leq 1$ and $||\mathfrak{a} - \mathfrak{b}|| \geq \epsilon$ imply $\exists\ \mu = \mu(\epsilon) > 0 \ni$

$$\frac{1}{2}||\mathfrak{a} + \mathfrak{b}|| \leq 1 - \mu.$$

Lemma 3.1. [21] *Let $\mathfrak{m} > 1$ and $r_1 > 0$ be two fixed numbers. Then, $\mathfrak{B}_s$ is uniformly convex iff $\exists$ a convex and strictly increasing function $\Upsilon : [0,\infty) \to [0,\infty)$ with $\Upsilon(0) = 0 \ni$*

$$||\mathfrak{c}\mathfrak{a} + (1-\mathfrak{c})\mathfrak{b}||^{\mathfrak{m}} \leq \mathfrak{c}||\mathfrak{a}||^{\mathfrak{m}} + (1-\mathfrak{c})||\mathfrak{b}||^{\mathfrak{m}} - \mathfrak{c}(1-\mathfrak{c})\Upsilon(||\mathfrak{a} - \mathfrak{b}||),$$

$\forall\ \mathfrak{a}, \mathfrak{b} \in \mathfrak{B}_{\mathfrak{m}} > [0]$ *and* $\mathfrak{c} \in [0,1]$.

For $\mathcal{H}_s$, we have

$$||\mathfrak{c}\mathfrak{a} + (1-\mathfrak{c})\mathfrak{b}||^2 \leq \mathfrak{c}||x||^2 + (1-\mathfrak{c})||y||^2 - \mathfrak{c}(1-\mathfrak{c})||\mathfrak{a} - \mathfrak{b}||,$$

$\forall\ \mathfrak{a}, \mathfrak{b} \in \mathcal{H}_s$ and $\mathfrak{c} \in [0,1]$.

Definition 3.2. A mapping $\Upsilon : \mathfrak{B}_s \to \mathfrak{B}$ has the demiclosed property at $\mathfrak{b} \in \mathfrak{B}$ if

$$\{\mathfrak{a} \in \mathfrak{B}_s, \mathfrak{a}_\eta \to \mathfrak{a} \text{ and } \Upsilon\mathfrak{a}_\eta \to \mathfrak{b} \implies \mathfrak{a} \in \mathfrak{B}_s \text{ and } \Upsilon\mathfrak{a} = \mathfrak{b}\}.$$

Lemma 3.3. [4]*Let $\mathfrak{B}_s$ be a nonempty,closed and convex subset of a uniformly convex Banach space $\mathfrak{B}$.If $\Upsilon : \mathfrak{B}_s \to \mathfrak{B}$ is nonexpansive mappings then $I - \Upsilon$ has the demiclosed property with respect to* 0.

The collection of points of $\mathfrak{B}_s$, unaltered by Υ is defined as follows:

$$F_{(\Upsilon)} = \{\mathfrak{a} \in \mathfrak{B}_s : \Upsilon\mathfrak{a} = \mathfrak{a}\}.$$

For a constant $L \in [0,\infty)$, the mapping Υ is called $L-$Lipschitz if

$$||\Upsilon\mathfrak{a} - \Upsilon\mathfrak{b}|| \leq L||\mathfrak{a} - \mathfrak{B}||,$$

$\forall\ \mathfrak{a}, \mathfrak{b} \in \mathfrak{B}_s$. Every $1-$Lipschitz is called $QNEM$.

A retract of $\mathfrak{B}$ is a subset $\mathfrak{B}_s$ of a Banach space $\mathfrak{B}$ that has a continuous mapping $\mathcal{Q}_{\mathfrak{B}_s}$ from $\mathfrak{B}$ to $\mathfrak{B}_s$ such that $\mathcal{Q}_{\mathfrak{B}_s}(\mathfrak{a}) = \mathfrak{a}$ for any $\mathfrak{a} \in \mathfrak{B}_s$. A $\mathcal{Q}_{\mathfrak{B}_s}$ like this is known as $\mathfrak{B}$ onto $\mathfrak{B}_s$ retraction.
If $\mathcal{Q}_{\mathfrak{B}_s}(\mathcal{Q}_{\mathfrak{B}_s}(\mathfrak{a} + \mathfrak{c}(\mathfrak{a} - \mathcal{Q}_{\mathfrak{B}_s}(\mathfrak{a})))) = \mathcal{Q}_{\mathfrak{B}_s}(\mathfrak{a})$, $\forall\ \mathfrak{a} \in \mathfrak{B}$ and $\mathfrak{c} \geq 0$, a retraction $\mathcal{Q}_{\mathfrak{B}_s}$ is said to be sunny. $\mathfrak{B}_{\mathfrak{s}}$ is a sunny nonexpansive retract of $\mathfrak{B}$ if a sunny retraction $\mathcal{Q}_{\mathfrak{B}_s}$ is also nonexpansive. Let $\mathfrak{B}$ be reflexive and strictly convex Banach space. Let $\mathcal{P}_{\mathfrak{B}_{\mathfrak{s}}} : \mathfrak{B} \to \mathfrak{B}_{\mathfrak{s}}$ be a projection. Also, $\mathcal{P}_{\mathfrak{B}_{\mathfrak{s}}}(\mathfrak{a})$ is in $\mathfrak{B}_s$ with the property

$$||\mathfrak{a} - \mathcal{P}_{\mathfrak{B}_{\mathfrak{s}}}(\mathfrak{a})|| = \{\inf ||\mathfrak{a} - \mathfrak{u}|| : \mathfrak{u} \in \mathfrak{B}_s\}.$$

for $\mathfrak{a} \in \mathfrak{B}$.

It is also well comprehended that $\mathcal{P}_{\mathcal{H}'_s}(\mathfrak{a}) \in \mathcal{H}_s$ and

$$\langle \mathfrak{a} - \mathcal{P}_{\mathcal{H}'_s}(\mathfrak{a}),\ \mathcal{P}_{\mathcal{H}'_s}(\mathfrak{a}) - \mathfrak{b}\rangle \geq 0,$$

$\forall\ \mathfrak{a} \in \mathcal{H}_s$, $\mathfrak{b} \in \mathcal{H}'_s$.
Sunny nonexpansive retractions work in the same way in $\mathfrak{B}$ as projections do in $\mathcal{H}_s$. If a subset $\mathcal{H}'_s \neq \emptyset$ of $\mathcal{H}$ is closed and convex, then $\exists$ a unique sunny nonexpansive retraction from $\mathfrak{B}_s$ to $\mathcal{H}'_s$.

Definition 3.4. [1]Let $\mathfrak{B}$ be a Banach space. For any sequence $\{\mathfrak{a}_\eta\} \to \mathfrak{a} \in \mathfrak{B}$, and $\forall$ $\mathfrak{b} \neq \mathfrak{a}$, we say that $\mathfrak{B}$ satisfies the Opial condition, if the following inequality holds:

$$\lim sup_{\eta\to\infty}||\mathfrak{a}_\eta - \mathfrak{a}|| < \lim sup_{\eta\to\infty}||\mathfrak{a}_\eta - \mathfrak{b}||.$$

It is to be noted that $\lim sup$ can be substituted by $\lim inf$ in this definition and that every Hilbert space satisfies the Opial condition [1]. Let $\emptyset \neq \mathfrak{B}_s \subseteq \mathfrak{B}$, $\Upsilon : \mathfrak{B}_s \to \mathfrak{B}$ a mapping, and $\{\mathfrak{a}_\eta\}$ a sequence in $\mathfrak{B}_s$. If $\lim_{\eta\to\infty} ||\mathfrak{a}_\eta - \Upsilon\mathfrak{a}_\eta|| = 0$, then $\{\mathfrak{a}_\eta\}$ is referred to as a sequence in Υ.
The following proposition is the generalization of Proposition 2.5 [20].

Proposition 3.5. *Let $\Upsilon : \mathfrak{B}_s \to \mathfrak{B}$ be uniformly continuous mapping and $\{\mathfrak{a}_\eta\} \subset \mathfrak{B}_s$ be a sequence of Υ. Then, $\{\mathfrak{b}_\eta\} \subset \mathfrak{B}_s$ is an approximating FP sequence of Υ whenever $\{\mathfrak{b}_\eta\} \in \mathfrak{B}_s \ni \lim_{\eta\to\infty} ||\mathfrak{a}_\eta - \mathfrak{b}_\eta|| = 0$.*

For dual space $\mathfrak{B}^*$ of $\mathfrak{B}$, the symbol $||\cdot||$ denotes the norms of $\mathfrak{B}$ and $\mathfrak{B}^*$. For $\mathfrak{a}^* \in \mathfrak{B}^*$ and $\mathfrak{a} \in \mathfrak{B}$, we use $\langle \mathfrak{a},\ \mathfrak{a}^*\rangle$ instead of $\mathfrak{a}^*(\mathfrak{a})$. The set-valued mapping $J : \mathfrak{B} \to 2^{\mathfrak{B}^*}$ is defined as

$$J(\mathfrak{a}) = \{\mathfrak{a}^* \in \mathfrak{B} : \langle \mathfrak{a},\ \mathfrak{a}^*\rangle = ||\mathfrak{a}||||\mathfrak{a}|| \text{ and } ||\mathfrak{a}^*|| = ||\mathfrak{a}||\}, \quad \mathfrak{a} \in \mathfrak{B},$$

and is known as a normalized duality mapping of $\mathfrak{B}$. For a multi- valued operator $\mathcal{A} : \mathfrak{B} \to 2^{\mathfrak{B}}$, the following are defined as:

$$Dom(\mathcal{A}) = \{\mathfrak{a} \in \mathfrak{B} : \mathcal{A}\mathfrak{a} \neq \emptyset\},$$
$$Ran(\mathcal{A}) = \cup\{\mathcal{A}\mathfrak{u} : \mathfrak{u} \in Dom(\mathcal{A})\},$$

and

$$Gr(\mathcal{A}) = \{(\mathfrak{a}, \mathfrak{b}) \in \mathfrak{B} \times \mathfrak{B} : \mathfrak{a} \in Dom(\mathcal{A}),\ \mathfrak{b} \in \mathcal{A}\mathfrak{a}\}$$

respectively. $\mathcal{A} \subseteq \mathfrak{B} \times \mathfrak{B}$ represents $\mathcal{A} : \mathfrak{B} \to 2^{\mathfrak{B}}$ and the inverse $\mathcal{A}^{-1}$ of $\mathcal{A}$ is as follows:

$$\mathfrak{a} \in \mathcal{A}^{-1}\mathfrak{b} \iff \mathfrak{b} \in \mathcal{A}\mathfrak{a}.$$

If $\forall\ \mathfrak{a}_i \in Dom(\mathcal{A})$ and $\mathfrak{b} \in \mathcal{A}\mathfrak{a}_i$ for $i = 1, 2$, $\exists\ \jmath \in J(\mathfrak{a}_1 - \mathfrak{a}_2) \ni \langle \mathfrak{b}_1 - \mathfrak{b}_2,\ \jmath\rangle \geq 0$, then the operator is known as *accretive.*

An accretive operator is the negation of a dissipative operator. If there is no proper accretive extension of $\mathcal{A}$, it is known as "maximal accretive", and if $Ran(I + \mathcal{A}) = \mathfrak{B}$, where I symbolizes the identity operator on $\mathfrak{B}$. If $\mathcal{A}$ is "$m-$accretive", then it is maximally accretive. For accretive $\mathcal{A}$, the single-valued nonexpansive mapping $\forall\ \Delta > 0$ is

$$J^{\mathcal{A}}_\Delta : Ran(I + \Delta\mathcal{A}) \to Dom(\mathcal{A}), \quad J^{\mathcal{A}}_\Delta = (I + \Delta\mathcal{A})^{-1},$$

and is said to be the *resolvent* of $\mathcal{A}$. The resolvent for an $m-$accretive operator on $\mathfrak{B}$

$$J_{\Delta}^{\mathcal{A}} = (I + \Delta\mathcal{A})^{-1}$$

is a multi-valued nonexpansive mapping whereby the domain is the entire space $\mathfrak{B}$, $\forall\ \Delta > 0$.

Lemma 3.6. [7] *Let $\mathcal{A} : \mathfrak{B} \to 2^{\mathfrak{B}}$ be an $m-$accretive operator. Then $\mathcal{A}$ is the maximal accretive, where $\mathfrak{B}$ is a real Banach space.*

Lemma 3.7. [1] *If $\mathcal{A} : \mathcal{H}_s \to 2^{\mathcal{H}_s}$ is a monotone operator, then $\mathcal{A}$ is the maximal monotone iff $Ran(I + \Delta\mathcal{A}) = \mathcal{H}\ \forall\ \Delta > 0$.*

As a result, if $\mathcal{A} : \mathcal{H}_s \to 2^{\mathcal{H}_s}$ is a maximum monotone operator and $\Delta > 0$, we may define the resolvent of $\mathcal{A}$, $J_{\Delta}^{\mathcal{A}} : \mathcal{H}_s \to: \mathcal{H}_s$, using Lemma 3.7. Also, $J_{\Delta}^{\mathcal{A}}$ satisfies the following inequality

$$||J_{\Delta}^{\mathcal{A}} - \mathfrak{a}J_{\Delta}^{\mathcal{A}}\mathfrak{b}||^2 \leq ||\mathfrak{a} - \mathfrak{b}||^2 - ||(I - J_{\Delta}^{\mathcal{A}})\mathfrak{a} - (I - J_{\Delta}^{\mathcal{A}})\mathfrak{b}||,$$

$\forall\ \mathfrak{a},\ \mathfrak{b} \in \mathcal{H}_s$.

For a function $\wp : \mathcal{H}_s \to (\infty, \infty]$,the domain is defined by:

$$dom(\wp) = \{\mathfrak{a} \in \mathcal{H}_s : \wp(\mathfrak{a}) < \infty\}.$$

Lemma 3.8. [3] *Let $\wp \in \Gamma_0(H)$. Then, $\wp$ is maximal monotone.*

4. Main results

The $CR-$iteration approach allows us to compute the common FP_s of two operators. Our objective is to analyze the asymptotic behaviour of our designed algorithm in Banach spaces. Let $\Upsilon_1,\ \Upsilon_2 : \mathfrak{B} \to \mathfrak{B}_s$ be mappings with at least one common FP between Υ_1 and Υ_2. The collection of common FP_s of mappings Υ_2 and Υ_1 is denoted by $\digamma_{(\Upsilon_2,\Upsilon_1)}$.
We now present the $G - CR-$iteration algorithm, which is as follows:

$$\begin{cases} \mathfrak{a}_0 \in \mathfrak{B}_s, \\ \mathfrak{a}_{\eta+1} = \mathcal{Q}_{\mathfrak{B}_s}[(1 - \kappa_\eta^1)\mathfrak{b}_\eta + \kappa_\eta^1\Upsilon_1\mathfrak{b}_\eta], \\ \mathfrak{b}_\eta = \mathcal{Q}_{\mathfrak{B}_s}[(1 - \kappa_\eta^2)\Upsilon_2\mathfrak{a}_\eta + \kappa_\eta^2\Upsilon_1\mathfrak{c}_\eta], \\ \mathfrak{c}_\eta = \mathcal{Q}_{\mathfrak{B}_s}[(1 - \kappa_\eta^3)\mathfrak{a}_\eta + \kappa_\eta^3\Upsilon_2\mathfrak{a}_\eta], \end{cases} \tag{$G-CR$}$$

where the sequences $\{\kappa_\eta^1\}$, $\{\kappa_\eta^2\}$, $\{\kappa_\eta^3\} \in (0,1)$. The sequence $\{\mathfrak{a}_\eta\}$ defined by $G-CR$ is called the generalized $CR-$iteration algorithm for mappings Υ_1 and Υ_2. If $\Upsilon_1 = \Upsilon_2$, then $G - CR$ iterative algorithm is defined as follows:

$$\begin{cases} \mathfrak{a}_0 \in \mathfrak{B}_s, \\ \mathfrak{a}_{\eta+1} = \mathcal{Q}_{\mathfrak{B}_s}[(1 - \kappa_\eta^1)\mathfrak{b}_\eta + \kappa_\eta^1\Upsilon_1\mathfrak{b}_\eta], \\ \mathfrak{b}_\eta = \mathcal{Q}_{\mathfrak{B}_s}[(1 - \kappa_\eta^2)\Upsilon_1\mathfrak{a}_\eta + \kappa_\eta^2\Upsilon_1\mathfrak{c}_\eta], \\ \mathfrak{c}_\eta = \mathcal{Q}_{\mathfrak{B}_s}[(1 - \kappa_\eta^3)\mathfrak{a}_\eta + \kappa_\eta^3\Upsilon_1\mathfrak{a}_\eta], \end{cases}$$

where $\{\kappa_\eta^1\}$. $\{\kappa_\eta^2\}$ and $\{\kappa_\eta^3\}$ are sequences in $(0,1)$. To prove the main results, we start with the following lemma.

Lemma 4.1. *Let $\mathcal{Q}_{\mathfrak{B}_s}$ be the sunny nonexpansive retraction and $\Upsilon_1,\ \Upsilon_2 : \mathfrak{B}_s \to \mathfrak{B}$ be QNEM $\ni F_{(\Upsilon_2,\ \Upsilon_1)} \neq \emptyset$. Let $\{\kappa_\eta^1\}$, $\{\kappa_\eta^2\}$, and $\{\kappa_\eta^3\}$ be sequences of real numbers $\ni$ $0 < \kappa_\eta^1,\ \kappa_\eta^2,\ \kappa_\eta^3 < 1$, $\forall\ \eta \in \mathbb{N} \cup\ \{0\}$. Let the sequence $\{\mathfrak{a}_\eta\}$ be generated from $\mathfrak{a}_0 \in \mathfrak{B}_s$ and be defined by $G-CR$. Then, for each $\sigma \in F_{(\Upsilon_2,\Upsilon_1)}$, $\lim_{\eta\to\infty} ||\mathfrak{a}_\eta - \sigma||$ exists and*

$$\begin{aligned} &||\mathfrak{b}_\eta - \sigma|| \le ||\mathfrak{a}_\eta - \sigma||, \ \ \textit{and} \\ &||\mathfrak{c}_\eta - \sigma|| \le ||\mathfrak{a}_\eta - \sigma||, \quad \forall \eta \in \mathbb{N} \cup\ \{0\}. \end{aligned} \tag{4.1}$$

Proof. Let σ be a common FP of Υ_1 and Υ_2. Then, for $\eta \in \mathbb{N} \cup\ \{0\}$, the following inequalities hold:

$$\begin{aligned} ||\mathfrak{a}_{\eta+1} - \sigma|| &= ||\mathcal{Q}_{\mathfrak{B}_s}[(1-\kappa_\eta^1)\mathfrak{b}_\eta + \kappa_\eta^1 \Upsilon_1 \mathfrak{b}_\eta] - \mathcal{Q}_{\mathfrak{B}_s}[\sigma]|| \\ &\le ||(1-\kappa_\eta^1)(\mathfrak{b}_\eta - \sigma) + \kappa_\eta^1(\Upsilon_1\mathfrak{b}_\eta - \sigma)|| \\ &\le (1-\kappa_\eta^1)||\mathfrak{b}_\eta - \sigma|| + \kappa_\eta^1||\Upsilon_1\mathfrak{b}_\eta - \sigma|| \\ &\le (1-\kappa_\eta^1)||\mathfrak{b}_\eta - \sigma|| + \kappa_\eta^1||\mathfrak{b}_\eta - \sigma|| \\ &= ||\mathfrak{b}_\eta - \sigma||. \end{aligned} \tag{4.2}$$

Also,

$$\begin{aligned} ||\mathfrak{b}_\eta - \sigma|| &= ||\mathcal{Q}_{\mathfrak{B}_s}[(1-\kappa_\eta^2)\Upsilon_2\mathfrak{a}_\eta + \kappa_\eta^2 \Upsilon_1 \mathfrak{c}_\eta] - \mathcal{Q}_{\mathfrak{B}_s}[\sigma]|| \\ &\le ||(1-\kappa_\eta^2)(\Upsilon_2\mathfrak{a}_\eta - \sigma) + \kappa_\eta^2(\Upsilon_1\mathfrak{c}_\eta - \sigma)|| \\ &\le ||(1-\kappa_\eta^2)(\mathfrak{a}_\eta - \sigma) + \kappa_\eta^2(\mathfrak{c}_\eta - \sigma)|| \\ &\le (1-\kappa_\eta^2)||\mathfrak{a}_\eta - \sigma|| + \kappa_\eta^2||\mathfrak{c}_\eta - \sigma||. \end{aligned} \tag{4.3}$$

Similarly,

$$\begin{aligned} ||\mathfrak{c}_\eta - \sigma|| &= ||\mathcal{Q}_{\mathfrak{B}_s}[(1-\kappa_\eta^3)\mathfrak{a}_\eta + \kappa_\eta^3 \Upsilon_2 \mathfrak{a}_\eta] - \mathcal{Q}_{\mathfrak{B}_s}[\sigma]|| \\ &\le ||(1-\kappa_\eta^3)(\mathfrak{a}_\eta - \sigma) + \kappa_\eta^3(\Upsilon_2\mathfrak{a}_\eta - \sigma)|| \\ &\le (1-\kappa_\eta^3)||\mathfrak{a}_\eta - \sigma|| + \kappa_\eta^3||\Upsilon_2\mathfrak{a}_\eta - \sigma|| \\ &\le (1-\kappa_\eta^3)||\mathfrak{a}_\eta - \sigma|| + \kappa_\eta^3||\mathfrak{a}_\eta - \sigma|| \\ &= ||\mathfrak{a}_\eta - \sigma||. \end{aligned} \tag{4.4}$$

Using inequality (4.4) in (4.3), we have

$$||\mathfrak{b}_\eta - \sigma|| \le ||\mathfrak{a}_\eta - \sigma||. \tag{4.5}$$

Hence, the inequality (4.2) results

$$||\mathfrak{a}_{\eta+1} - \sigma|| \le ||\mathfrak{a}_\eta - \sigma||. \tag{4.6}$$

Considering (4.6) and (4.2), we calculate the following result

$$||\mathfrak{a}_{\eta+1} - \sigma|| \le ||\mathfrak{a}_\eta - \sigma|| \le ||\mathfrak{a}_{\eta-1} - \sigma|| \le \ \ldots \ \le ||\mathfrak{a}_0 - \sigma||, \tag{4.7}$$

$\forall\ \eta \in \mathbb{N} \cup\ \{0\}$. Since $\{||\mathfrak{a}_\eta - \sigma||\}$ is monotonically decreasing, it confirms the convergence of $\{||\mathfrak{a}_\eta - \sigma||\}$. □

The convergence behaviour for $QNEMs$ is now studied by the following theorem.

Theorem 4.2. *Let $\emptyset \neq \mathfrak{B}_s \subseteq \mathfrak{B}$, with $\mathcal{Q}_{\mathfrak{B}_s}$ as the sunny nonexpansive retraction. Let $\Upsilon_1,\ \Upsilon_2 : \mathfrak{B}_s \to \mathfrak{B}$ be $QNEMs \ni \digamma_{(\Upsilon_1,\Upsilon_2)} \neq \emptyset$. Let the real sequences $\{\kappa_\eta^1\}$, $\{\kappa_\eta^2\}$ and $\{\kappa_\eta^3\} \ni 0 < a \leq \kappa_\eta^1 \leq \overline{a} < 1$, $0 < b \leq \kappa_\eta^2 \leq \overline{b} < 1$ and $0 < c \leq \kappa_\eta^3 \leq \overline{c} < 1$ $\forall$ $\eta \in \mathbb{N} \cup \ \{0\}$. Let $\mathfrak{a}_0 \in \mathfrak{B}_s$ and $\mathcal{P}_{\digamma_{(\Upsilon_1,\Upsilon_2)}}(\mathfrak{a}_0) = \mathfrak{a}^*$. Let $\{\mathfrak{a}_\eta\}$ be the sequence defined by $(G-CR)$. Then, we have*

1. *$\{\mathfrak{a}_\eta\}$ is in a closed convex bounded set $\mathcal{CB}_r[\mathfrak{a}^*] \cap \mathfrak{B}_s$, where $r \in (0,\infty) \ni ||\mathfrak{a}_0 - \mathfrak{a}^*|| \leq r$.*

2. *If Υ be uniformly continuous, then*
$$\lim_{\eta\to\infty} ||\mathfrak{a}_\eta - \Upsilon_1\mathfrak{a}_\eta|| = 0 \ \textit{and} \ \lim_{\eta\to\infty} ||\mathfrak{a}_\eta - \Upsilon_2\mathfrak{a}_\eta|| = 0,$$
then $\wp_c : [0,\infty) \to [0,\infty)$, $\wp(0) = 0$, where error bounds are as follows-
$$\underline{a}(1-\overline{a})\sum_{i=0}^{\eta} \wp_c(||\mathfrak{b}_i - \Upsilon_1\mathfrak{b}_i||) \leq ||\mathfrak{a}_0 - \mathfrak{a}^*||^2 - ||\mathfrak{a}_{\eta+1} - \mathfrak{a}^*||^2, \tag{4.8}$$
$$\underline{b}(1-\overline{b})\sum_{i=0}^{\eta} \wp_c(||\Upsilon_2\mathfrak{a}_i - \Upsilon_1\mathfrak{c}_i||) \leq ||\mathfrak{a}_0 - \mathfrak{a}^*||^2 - ||\mathfrak{a}_{\eta+1} - \mathfrak{a}^*||^2 - \sum_{i=0}^{\eta} \kappa_i^1(1-\kappa_i^1)\wp_c(||\mathfrak{b}_i - \Upsilon_1\mathfrak{b}_i||), \tag{4.9}$$
$$\underline{bc}(1-\overline{c})\sum_{i=0}^{\eta} \wp_c(||\mathfrak{a}_i - \Upsilon_2\mathfrak{a}_i||) \leq ||\mathfrak{a}_0 - \mathfrak{a}^*||^2 - ||\mathfrak{a}_{\eta+1} - \mathfrak{a}^*||^2 - \sum_{i=0}^{\eta} \kappa_i^2(1-\kappa_i^2)\wp_c(||\Upsilon_2\mathfrak{a}_i - \Upsilon_1\mathfrak{c}_i||) - \sum_{i=0}^{\eta} \kappa_i^1(1-\kappa_i^1)\wp_c(||\mathfrak{b}_i - \Upsilon_1\mathfrak{b}_i||), \tag{4.10}$$
$\forall\ \eta \in \mathbb{N} \cup \ \{0\}$.

3. *If $I - \Upsilon_2$ and $I - \Upsilon_1$ are demiclosed at 0 and $\mathfrak{B}$ satisfies the Opial condition, then $\{\mathfrak{a}_\eta\} \to \ell$ where $\ell \in \digamma_{(\Upsilon_2,\Upsilon_1)} \cap \mathcal{CB}_r[\mathfrak{a}^*]$, where the convergence is weak.*

Proof. (1) Let $\mathfrak{a}^* \in \digamma_{(\Upsilon_2,\Upsilon_1)}$. From inequality (4.7) the following holds for all $\eta \in \mathbb{N} \cup \{0\}$.
$$||\mathfrak{a}_{\eta+1} - \mathfrak{a}^*|| \leq ||\mathfrak{a}_\eta - \mathfrak{a}^*|| \leq ||\mathfrak{a}_{\eta-1} - \mathfrak{a}^*|| \leq \ \ldots \ \leq ||\mathfrak{a}_0 - \mathfrak{a}^*||.$$
Hence, $\{\mathfrak{a}_\eta\} \in \mathcal{CB}_r[\mathfrak{a}^*] \cap \mathfrak{B}_s$.

(2) Let Υ_2 be uniformly continuous. By Lemma 4.1, we have that $\{\mathfrak{a}_\eta\}, \{\mathfrak{b}_\eta\}$ and $\{\mathfrak{c}_\eta\} \in \mathcal{CB}_r[\mathfrak{a}^*] \cap \mathfrak{B}_s$, and hence, from inequality (4.1), we have
$$||\Upsilon_2\mathfrak{a}_\eta - \mathfrak{a}^*|| \leq r, \ \ ||\Upsilon_1\mathfrak{a}_\eta - \mathfrak{a}^*|| \leq r, \ \ ||\Upsilon_1\mathfrak{b}_\eta - \mathfrak{a}^*|| \leq r \ \text{ and } \ ||\Upsilon_1\mathfrak{c}_\eta - \mathfrak{a}^*|| \leq r,$$

$\forall\ \eta \in \mathbb{N} \cup\ \{0\}$.

Let $\wp_c$ be the function as defined in Lemma 1 for $\mathfrak{m} = 2$ and $r_1 = r$. Benefiting from inequality (4.1) as well, we have

$$\begin{aligned}
||\mathfrak{a}_{\eta+1} - \mathfrak{a}^*||^2 &= ||\mathcal{Q}_{\mathfrak{B}_s}[(1-\kappa^1_\eta)\mathfrak{b}_\eta + \kappa^1_\eta \Upsilon_1 \mathfrak{b}_\eta] - \mathcal{Q}_{\mathfrak{B}_s}[\mathfrak{a}^*]||^2 \\
&\leq ||(1-\kappa^1_\eta)(\mathfrak{b}_\eta - \mathfrak{a}^*) + \kappa^1_\eta(\Upsilon_1\mathfrak{b}_\eta - \mathfrak{a}^*)||^2 \\
&\leq (1-\kappa^1_\eta)||\mathfrak{b}_\eta - \mathfrak{a}^*||^2 + \kappa^1_\eta||\Upsilon_1\mathfrak{b}_\eta - \mathfrak{a}^*||^2 - \kappa^1_\eta(1-\kappa^1_\eta)\wp_c(||\mathfrak{b}_\eta - \Upsilon_1\mathfrak{b}_\eta||) \\
&\leq (1-\kappa^1_\eta)||\mathfrak{b}_\eta - \mathfrak{a}^*||^2 + \kappa^1_\eta||\mathfrak{b}_\eta - \mathfrak{a}^*||^2 - \kappa^1_\eta(1-\kappa^1_\eta)\wp_c(||\mathfrak{b}_\eta - \Upsilon_1\mathfrak{b}_\eta||) \\
&= ||\mathfrak{b}_\eta - \mathfrak{a}^*||^2 - \kappa^1_\eta(1-\kappa^1_\eta)\wp_c(||\mathfrak{b}_\eta - \Upsilon_1\mathfrak{b}_\eta|| \qquad (4.11) \\
&\leq ||\mathfrak{a}_\eta - \mathfrak{a}^*||^2 - \kappa^1_\eta(1-\kappa^1_\eta)\wp_c(||\mathfrak{b}_\eta - \Upsilon_1\mathfrak{b}_\eta||,
\end{aligned}$$

$\forall\ \eta \in \mathbb{N} \cup\ \{0\}$. By the bounds of sequence $\{\kappa^1_\eta\}$, we have

$$\kappa^1_\eta(1-\kappa^1_\eta)\wp_c(||\mathfrak{b}_\eta - \Upsilon_1\mathfrak{b}_\eta||) \leq ||\mathfrak{a}_\eta - \mathfrak{a}^*||^2 - ||\mathfrak{a}_{\eta+1} - \mathfrak{a}^*||^2.q$$

Observe that

$$\underline{a}(1-\overline{a})\sum_{\eta=0}^{\infty}\wp_c(||\mathfrak{b}_\eta - \Upsilon_1\mathfrak{b}_\eta||) \leq ||\mathfrak{a}_0 - \mathfrak{a}^*|| < \infty.$$

We obtain that $\lim_{\eta\to\infty}||\mathfrak{b}_\eta - \Upsilon_1\mathfrak{b}_\eta|| = 0$. Using $(G-CR)$, we have

$$\begin{aligned}
||\mathfrak{b}_\eta - \mathfrak{a}^*||^2 &= ||\mathcal{Q}_{\mathfrak{B}_s}[(1-\kappa^2_\eta)\Upsilon_2\mathfrak{a}_\eta + \kappa^2_\eta\Upsilon_1\mathfrak{c}_\eta] - \mathcal{Q}_{\mathfrak{B}_s}[\mathfrak{a}^*]||^2 \\
&\leq ||(1-\kappa^2_\eta)(\Upsilon_2\mathfrak{a}_\eta - \mathfrak{a}^*) + \kappa^2_\eta(\Upsilon_1\mathfrak{c}_\eta - \mathfrak{a}^*)||^2 \\
&\leq (1-\kappa^2_\eta)||\Upsilon_2\mathfrak{a}_\eta - \mathfrak{a}^*||^2 + \kappa^2_\eta||\Upsilon_1\mathfrak{c}_\eta - \mathfrak{a}^*||^2 - \kappa^2_\eta(1-\kappa^2_\eta)\wp_c(||\Upsilon_2\mathfrak{a}_\eta - \Upsilon_1\mathfrak{c}_\eta||). \\
&\leq (1-\kappa^2_\eta)||\mathfrak{a}_\eta - \mathfrak{a}^*||^2 + \kappa^2_\eta||\mathfrak{c}_\eta - \mathfrak{a}^*||^2 - \kappa^2_\eta(1-\kappa^2_\eta)\wp_c(||\Upsilon_2\mathfrak{a}_\eta - \Upsilon_1\mathfrak{c}_\eta||) \\
&\leq ||\mathfrak{a}_\eta - \mathfrak{a}^*||^2 - \kappa^2_\eta(1-\kappa^2_\eta)\wp_c(||\Upsilon_2\mathfrak{a}_\eta - \Upsilon_1\mathfrak{c}_\eta||). \qquad (4.12)
\end{aligned}$$

Using inequality(4.11), we have

$$\begin{aligned}
&||\mathfrak{a}_{\eta+1} - \mathfrak{a}^*||^2 \\
&\leq \left[||\mathfrak{a}_\eta - \mathfrak{a}^*||^2 - \kappa^2_\eta(1-\kappa^2_\eta)\wp_c(||\Upsilon_2\mathfrak{a}_\eta - \Upsilon_1\mathfrak{c}_\eta||)\right] - \kappa^1_\eta(1-\kappa^1_\eta)\wp_c(||\mathfrak{b}_\eta - \Upsilon_1\mathfrak{b}_\eta||) \\
&\leq \left[||\mathfrak{a}_\eta - \mathfrak{a}^*||^2 - \kappa^1_\eta\kappa^2_\eta(1-\kappa^2_\eta)\wp_c(||\Upsilon_2\mathfrak{a}_\eta - \Upsilon_1\mathfrak{c}_\eta||)\right] - \kappa^1_\eta(1-\kappa^1_\eta)\wp_c(||\mathfrak{b}_\eta - \Upsilon_1\mathfrak{b}_\eta||).
\end{aligned}$$

Noticeably $\underline{a}\ \underline{b}(1-\overline{b}) \leq \kappa^1_\eta\kappa^2_\eta(1-\kappa^2_\eta)\ \forall\ \eta \in \mathbb{N} \cup\ \{0\}$. We obtain that

$$\begin{aligned}
\underline{a}\ \underline{b}\sum_{i=0}^{\eta}\wp_c(||\Upsilon_2\mathfrak{a}_i - \Upsilon_1\mathfrak{c}_i||) \leq ||\mathfrak{a}_0 - \mathfrak{a}^*||^2 - ||\mathfrak{a}_{\eta+1} - \mathfrak{a}^*||^2 \\
- \sum_{i=0}^{\eta}\kappa^1_\eta(1-\kappa^1_\eta)\wp_c(||\mathfrak{b}_i - \Upsilon_1\mathfrak{b}_i||).
\end{aligned}$$

Now, we have

$$\underline{a}\ \underline{b}\sum_{\eta=0}^{\infty}\wp_c(||\Upsilon_2\mathfrak{a}_\eta-\Upsilon_1\mathfrak{c}_\eta||)\leq||\mathfrak{a}_0-\mathfrak{a}^*||^2<\infty.$$

It results in that

$$\lim_{\eta\to\infty}||\Upsilon_2\mathfrak{a}_\eta-\Upsilon_1\mathfrak{c}_\eta||=0.$$

Using the inequality (4.12), we have

$$\begin{aligned}||\mathfrak{b}_\eta-\mathfrak{a}^*||&\leq(1-\kappa_\eta^2)||\mathfrak{a}_\eta-\mathfrak{a}^*||^2+\kappa_\eta^2\Big[||(1-\kappa_\eta^3)(\mathfrak{a}_\eta-\mathfrak{a}^*)-\kappa_\eta^3(\Upsilon_2\mathfrak{a}_\eta-\mathfrak{a}^*)||^2\Big]\\&\quad-\kappa_\eta^2(1-\kappa_\eta^2)\wp_c(||\Upsilon_2\mathfrak{a}_\eta-\Upsilon_1\mathfrak{c}_\eta||)\\&\leq(1-\kappa_\eta^2)||\mathfrak{a}_\eta-\mathfrak{a}^*||^2+\kappa_\eta^2\Big[(1-\kappa_\eta^3)||\mathfrak{a}_\eta-\mathfrak{a}^*||^2+\kappa_\eta^3||\Upsilon_2\mathfrak{a}_\eta-\mathfrak{a}^*)||^2\\&\quad-\kappa_\eta^3(1-\kappa_\eta^3)\wp_c(||\mathfrak{a}_\eta-\Upsilon_2\mathfrak{a}_\eta||)\Big]-\kappa_\eta^2(1-\kappa_\eta^2)\wp_c(||\Upsilon_2\mathfrak{a}_\eta-\Upsilon_1\mathfrak{c}_\eta||).\\&\leq||\mathfrak{a}_\eta-\mathfrak{a}^*||-\kappa_\eta^2\kappa_\eta^3(1-\kappa_\eta^2)\wp_c(||\mathfrak{a}_\eta-\Upsilon_2\mathfrak{a}_\eta||)-\kappa_\eta^2(1-\kappa_\eta^2)(||\Upsilon_2\mathfrak{a}_\eta-\Upsilon_1\mathfrak{c}_\eta||),\end{aligned}$$

$\forall\ \eta\in\mathbb{N}\cup\ \{0\}$. On the other hand, from inequality (4.11), we have

$$\begin{aligned}&||\mathfrak{a}_{\eta+1}-\mathfrak{a}^*||\\&\quad=||\mathfrak{b}_\eta-\mathfrak{a}^*||^2-\kappa_\eta^1(1-\kappa_\eta^1)\wp_c(||\mathfrak{b}_\eta-\Upsilon_1\mathfrak{b}_\eta||\\&\quad=\Big[||\mathfrak{a}_\eta-\mathfrak{a}^*||-\kappa_\eta^2\kappa_\eta^3(1-\kappa_\eta^2)\wp_c(||\mathfrak{a}_\eta-\Upsilon_2\mathfrak{c}_\eta||)-\kappa_\eta^2(1-\kappa_\eta^2)(||\Upsilon_2\mathfrak{a}_\eta-\Upsilon_1\mathfrak{c}_\eta||)\Big]\\&\quad-\kappa_\eta^1(1-\kappa_\eta^1)\wp_c(||\mathfrak{b}_\eta-\Upsilon_1\mathfrak{b}_\eta||.\end{aligned}$$

Therefore, $\underline{b}\ \underline{c}(1-\overline{c})\leq\mathfrak{b}_\eta\mathfrak{c}_\eta(1-\mathfrak{c}_\eta)$, $\forall\ \eta\in\mathbb{N}\cup\ \{0\}$. Noticeably

$$\begin{aligned}\underline{b}\ \underline{c}(1-\overline{c})\sum_{i=0}^{\eta}\wp_c(\mathfrak{a}_i-\Upsilon_2\mathfrak{c}_i)\leq||\mathfrak{a}_0-\mathfrak{a}^*||^2-||\mathfrak{a}_{\eta+1}-\mathfrak{a}^*||^2\\-\sum_{i=0}^{\eta}\kappa_i^2(1-\kappa_i^2)\wp_c(||\Upsilon_2\mathfrak{a}_\eta-\Upsilon_1\mathfrak{c}_\eta||)\\-\sum_{i=0}^{\eta}\kappa_i^1(1-\kappa_i^1)\wp_c(||\mathfrak{b}_i-\Upsilon_1\mathfrak{b}_i||),\end{aligned}$$

which follows that $\lim_{\eta\to\infty}||\mathfrak{a}_\eta-\Upsilon_2\mathfrak{a}_\eta||\to0$. Note that

$$\begin{aligned}||\mathfrak{c}_\eta-\mathfrak{a}_\eta||&=||\mathcal{Q}_{\mathfrak{B}_s}[(1-\kappa_\eta^3)\mathfrak{a}_\eta+\kappa_\eta^3\Upsilon_2\mathfrak{a}_\eta]-\mathcal{Q}_{\mathfrak{B}_s}[\mathfrak{a}^*]||\\&=||\Upsilon_2\mathfrak{a}_\eta-\mathfrak{a}_\eta||\to0\text{ as }\eta\to\infty.\end{aligned}$$

It is given that Υ_2 is uniformly continuous, so using Proposition (3.5)

$$\lim_{\eta\to\infty}||\mathfrak{c}_\eta-\Upsilon_2\mathfrak{c}_\eta||=0.$$

Therefore, from

$$\lim_{\eta\to\infty}||\Upsilon_2\mathfrak{a}_\eta-\Upsilon_1\mathfrak{c}_\eta||=0,$$

we have

$$||\mathfrak{a}_\eta - \Upsilon_2\mathfrak{a}_\eta|| = 0.$$

(3) Let $\mathfrak{B}$ satisfies the Opial condition and Υ_1 and Υ_2 with CFP ω, where $\omega \in \mathcal{CB}_r[\mathfrak{a}^*] \cap \mathfrak{B}_s$. Lemma 4.1 results that $\lim_{\eta\to\infty} ||\mathfrak{a}_\eta - \omega||$ exists. Let $\exists$ $\{\mathfrak{a}_{\eta_p}\}$ and $\{\mathfrak{a}_{\theta_q}\}$ convergent to two distinct points ω_1 and ω_2 in $\mathcal{CB}_r \cap \mathfrak{B}_s$, respectively. Since both $I - \Upsilon_1$ and $I - \Upsilon_2$ are demiclosed at 0, we have

$$\Upsilon_1\omega_1 = \Upsilon_2\omega_1 = \omega$$

and

$$\Upsilon_1\omega_2 = \Upsilon_2\omega_2 = \omega.$$

Furthermore, the Opial condition results

$$\lim_{\eta\to\infty} ||\mathfrak{a}_\eta - \omega_1|| = \lim_{p\to\infty} ||\mathfrak{a}_{\eta_p} - \omega_1|| < \lim_{q\to\infty} ||\mathfrak{a}_{\theta_q} - \omega_2|| = \lim_{\eta\to\infty} ||\mathfrak{a}_\eta - \omega_2||.$$

In similar manner, we have

$$\lim_{\eta\to\infty} ||\mathfrak{a}_\eta - \omega_2|| < \lim_{\eta\to\infty} ||\mathfrak{a}_\eta - \omega_1||,$$

which is a contradiction. Hence, $\omega_1 = \omega_2$, which confirms the existence of the convergent sequence $\{\mathfrak{a}_\eta\}$ which converges weakly to $\omega \in \mathcal{F}_{(\Upsilon_1,\Upsilon_2)} \cap \mathcal{CB}_r[\mathfrak{a}^*]$. □

Also, if any nonexpansive mapping is uniformly continuous, we may deduce a convergence theorem for estimating the common FP_s of two nonexpansive mappings from Theorem 4.2 and Lemma 3.3.

Theorem 4.3. *Let $\emptyset \neq \mathfrak{B}_s \subseteq \mathfrak{B}$ with $\mathcal{Q}_{\mathfrak{B}_s}$ as the sunny nonexpansive retraction. Let $\Upsilon_1, \Upsilon_2 : \mathfrak{B}_s \to \mathfrak{B}$ be nonexpansive mappings such that $\mathcal{F}_{(\Upsilon_1,\Upsilon_2)} \neq \emptyset$. Let the real sequences $\{\kappa_\eta^1\}$, $\{\kappa_\eta^2\}$ and $\{\kappa_\eta^3\} \ni 0 < a \leq \kappa_\eta^1 \leq \overline{a} < 1$, $0 < b \leq \kappa_\eta^2 \leq \overline{b} < 1$ and $0 < c \leq \kappa_\eta^3 \leq \overline{c} < 1 \;\forall\; \eta \in \mathbb{N} \cup \{0\}$. Let $\mathfrak{a}_0 \in \mathfrak{B}_s$ and $\mathcal{P}_{\mathcal{F}_{(\Upsilon_1,\Upsilon_2)}}(\mathfrak{a}_0) = \mathfrak{a}^*$. Let $\{\mathfrak{a}_\eta\}$ be the sequence defined by $(G - CR)$. Then, we have*

1. *$\{\mathfrak{a}_\eta\}$ is in a closed convex bounded set $\mathcal{CB}_r[\mathfrak{a}^*] \cap \mathfrak{B}_s$, where*
$$r \in (0, \infty) \ni ||\mathfrak{a}_0 - \mathfrak{a}^*|| \leq r.$$
2. *$\lim_{\eta\to\infty} ||\mathfrak{a}_\eta - \Upsilon_1\mathfrak{a}_\eta|| = 0$ and $\lim_{\eta\to\infty} ||\mathfrak{a}_\eta - \Upsilon_2\mathfrak{a}_\eta|| = 0$ with the same error bounds (2) defined in Theorem 4.2.*
3. *If $I - \Upsilon_2$ and $I - \Upsilon_1$ are demiclosed at 0 and $\mathfrak{B}$ satisfies the Opial condition, then $\{\mathfrak{a}_\eta\}$ is convergent to an element of $\mathcal{F}_{(\Upsilon_2,\Upsilon_1)} \cap \mathcal{CB}_r[\mathfrak{a}^*]$, where the convergence is weak convergence.*

We may restate condition (3) of Theorem 4.3 as if $\mathfrak{B}$ meets the Opial condition, $\{\mathfrak{a}_\eta\}$ weakly converges to an element of $\mathcal{F}_{(\Upsilon_1,\Upsilon_2)}$, if $\mathcal{P}_{\mathcal{F}_{(\Upsilon_1,\Upsilon_2)}}$ cannot be determined. Therefore we can define the following:

Corollary 4.4. *Let Υ_1, $\Upsilon_2 : \mathcal{H}_{\mathfrak{s}_*} \to \mathcal{H}_{\mathfrak{s}_*}$ be nonexpansive mappings such that $\mathcal{F}_{(\Upsilon_1,\Upsilon_2)} \neq \emptyset$. Let the real sequences $\{\kappa_\eta^1\}$, $\{\kappa_\eta^2\}$ and $\{\kappa_\eta^3\} \ni 0 < a \leq \kappa_\eta^1 \leq \overline{a} < 1$,*

$0 < b \leq \kappa_\eta^2 \leq \bar{b} < 1$ *and* $0 < c \leq \kappa_\eta^3 \leq \bar{c} < 1 \; \forall \; \eta \in \mathbb{N} \cup \; \{0\}$. *Let the sequence* $\{\mathfrak{a}_\eta\}$ *is defined as follows*

$$\begin{cases} \mathfrak{a}_0 \in \mathfrak{B}_s, \\ \mathfrak{a}_{\eta+1} = (1-\kappa_\eta^1)\mathfrak{b}_\eta + \kappa_\eta^1 \Upsilon_1 \mathfrak{b}_\eta, \\ \mathfrak{b}_\eta = (1-\kappa_\eta^2)\Upsilon_2 \mathfrak{a}_\eta + \kappa_\eta^2 \Upsilon_1 \mathfrak{c}_\eta, \\ \mathfrak{c}_\eta = (1-\kappa_{\eta^3})\mathfrak{a}_\eta + \kappa_{\eta^3} \Upsilon_2 \mathfrak{a}_\eta, \qquad \eta \in \mathbb{N} \cup 0. \end{cases} \tag{$CR-PPA$}$$

Then the sequence $\{\mathfrak{a}_\eta\}$ *is convergent weakly to an element of* $F_{(\Upsilon_1,\Upsilon_2)}$.

5. Application

It is important to note that various problems based on signal processing and machine learning can be expressed in accordance with the following manner.

Problem 1. For an $m-$accretive operator $\mathcal{A} : \mathfrak{B} \to 2^{\mathfrak{B}}$, find an element that satisfies

$$\mathfrak{a} \in \mathfrak{B} \text{ such that } 0 \in \mathcal{A}\mathfrak{a}. \tag{5.1}$$

PPA, introduced by Martinet (see [15], [14]) and generalized by Rockafellar ([17], [18]) is one of the popular methods to solve this problem. Also, Rockafellar [17] studied the weak convergence of the PPA, namely:

$$\mathfrak{a}_{\eta+1} = J_{\Delta_\eta}^{\mathcal{A}} \mathfrak{a}_\eta, \text{ for all } \eta \in \mathbb{N} \cup 0, \tag{5.2}$$

for the solution to Problem 5.1 and $\mathfrak{a}_0 \in \mathfrak{B}$. The weak and strong convergences of the sequence $\{x_\eta\}$ defined by equation (5.2) have been extensively studied in various ambient spaces e.g. Hilbert and Banach spaces (see [23], [22], [24], [25] and the references therein). The general form of Problem 1 is as follows:

Problem 2. Let the mappings $\mathcal{A}$, $\mathcal{A}_1 : \mathfrak{B} \to 2^{\mathfrak{B}}$ be $m-$accretive operators. Find an element

$$\mathfrak{a} \in \mathfrak{B} \ni \; 0 \in \mathcal{A}\mathfrak{a} \cap \mathcal{A}_1\mathfrak{a}, \tag{5.3}$$

when $\mathcal{A}$ and $\mathcal{A}_1$ are two maximal monotonic operators in a $\mathcal{H}_s$.

We are now eligible to utilize our observations, which are primarily focused on accretive operators' common zeros. We name $(G-CR)$ an iteration - based proximal point algorithm when $\Upsilon_1 = J_\Delta^{\mathcal{A}}$ and $\Upsilon_2 = J_\Delta^{\mathcal{A}_1}$. In a more generalized context, we now analyze its convergence to solve Problem 2.

Theorem 5.1. *Let* $\emptyset \neq \mathfrak{B}_s$ *be Opial condition. Let* $\mathcal{A} : Dom(\mathcal{A}) \subseteq \mathfrak{B}_s \to 2^{\mathfrak{B}}$ *and* $\mathcal{A}_1 : Dom(\mathcal{A}_1) \subseteq \mathfrak{B}_s \to 2^{\mathfrak{B}}$ *be accretive operators* $\ni Dom(\mathcal{A}) \subseteq \mathfrak{B}_s \subseteq \cap_{\lambda>0} Ran(I+\lambda\mathcal{A}), Dom(\mathcal{A}_1) \subseteq \mathfrak{B}_s \subseteq \cap_{\lambda>0} Ran(I+\lambda\mathcal{A}_1)$ *and* $\mathcal{A}^{-1}(0) \cap \mathcal{A}_1^{-1}(0) \neq \emptyset$. *Let* $\{\kappa_\eta^1\}$, $\{\kappa_\eta^2\}$, *and* $\{\kappa_\eta^3\}$ *be sequences of real numbers* $\ni 0 < a \leq \kappa_\eta^1 < \bar{a} < 1$, $b \leq \kappa_\eta^2 < \bar{b}$, *where* b, $\bar{b} \in (0,1)$ *and* $c \leq \kappa_\eta^3 < \bar{c}$, c, $\bar{c} \in (0,1) \; \forall \; \eta \in \mathbb{N} \cup 0$. *Let* $\Delta > 0$,

$\mathfrak{a}_0 \in \mathfrak{B}_s$ *and* $\mathcal{P}_{\mathcal{A}^{-1}(0)\cap\mathcal{A}_1 0^{-1}}(\mathfrak{a}_0) = \mathfrak{a}^*$. *Let the sequence* $\{\mathfrak{a}_\eta\}$ *be defined as follows:*

$$\begin{cases} \mathfrak{a}_0 \in \mathfrak{B}_s, \\ \mathfrak{a}_{\eta+1} = (1-\kappa_\eta^1)\mathfrak{b}_\eta + \kappa_\eta^1 J_\Delta^{\mathcal{A}} \mathfrak{b}_\eta, \\ \mathfrak{b}_\eta = (1-\kappa_\eta^2) J_\Delta^{\mathcal{A}_1} \mathfrak{a}_\eta + \kappa_\eta^2 J_\Delta^{\mathcal{A}} \mathfrak{c}_\eta, \\ \mathfrak{c}_\eta = (1-\kappa_\eta^3)\mathfrak{a}_\eta + \kappa_\eta^3 J_\Delta^{\mathcal{A}_1} \mathfrak{a}_\eta, \end{cases}$$

Then, we have

1. $\{\mathfrak{a}_\eta\}$ *is in a closed convex bounded set* $\mathcal{CB}_r[\mathfrak{a}^*] \cap \mathfrak{B}_s$, *where*
$$r \in (0,\infty) \ni ||\mathfrak{a}_0 - \mathfrak{a}^*|| \le r.$$
2. $\lim_{\eta\to\infty} ||\mathfrak{a}_\eta - J_\Delta^{\mathcal{A}}\mathfrak{a}_\eta|| = 0$ *and* $\lim_{\eta\to\infty} ||\mathfrak{a}_\eta - J_\Delta^{\mathcal{A}_1}\mathfrak{a}_\eta|| = 0$ *with the same error bounds (2) defined in Theorem 4.2 where* $\Upsilon_1 = J_\Delta^{\mathcal{A}}$ *and* $\Upsilon_2 = J_\Delta^{\mathcal{A}_1}$.
3. $\{\mathfrak{a}_\eta\}$ *is convergent to an element of* $\mathcal{A}^{-1}(0)\cap\mathcal{A}_1^{-1}(0)\cap\mathcal{CB}[\mathfrak{a}^*]$ *and the convergence is weak convergence.*

Proof. As $Dom(\mathcal{A}) \subseteq \mathfrak{B}_s \subseteq \cap_{\lambda>0} Ran(I+\lambda\mathcal{A})$, it is to note that $J_\Delta^{\mathcal{A}} : \mathfrak{B}_s \to \mathfrak{B}_s$ is nonexpansive. Also, $J_\Delta^{\mathcal{A}} : \mathfrak{B}_s \to \mathfrak{B}_s$ is nonexpansive. Also, $Dom(\mathcal{A}) \cap Dom(\mathcal{B}) \subseteq \mathfrak{B}_s$, hence we have

$$\begin{aligned} \mathfrak{a} \in \mathcal{A}^{-1}(0)\mathcal{A}_1^{-1}(0) &\implies \mathfrak{a} \in Dom(\mathcal{A}) \cap Dom(\mathcal{A}_1) \text{ with } 0 \in \mathcal{A}\mathfrak{a} \text{ and } 0 \in \mathcal{A}_1\mathfrak{a} \\ &\implies \mathfrak{a} \in \mathfrak{B}_s \text{ with } J_\Delta^{\mathcal{A}}\mathfrak{a} = \mathfrak{a} \text{ and } J_\Delta^{\mathcal{A}_1}\mathfrak{a} = \mathfrak{a} \\ &\implies \mathfrak{a} \in F_{(J_\Delta^{\mathcal{A}}, J_\Delta^{\mathcal{A}_1})}. \end{aligned} \tag{5.4}$$

Substitute $\Upsilon_1 = J_\Delta^{\mathcal{A}}$ and $\Upsilon_2 = J_\Delta^{\mathcal{A}_1}$. As a result, Theorem 5.1 refers to the proof from Theorem 4.3.

Example 5.2. For the problem given below, find the element which satisfies

$$\alpha \in \overset{\circ}{\mathcal{J}} := \partial\mathcal{A}^{-1}(0) \cap \partial\mathcal{A}_1^{-1}(0),$$

where $\mathcal{A}, \mathcal{A}_1 : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are defined as follows:

$$\mathcal{A}(\mathfrak{a}) = \frac{1}{2}\langle \nabla_f(\mathfrak{a}),\mathfrak{a})\rangle + \langle \mathfrak{a},\beta\rangle.$$

Also

$$\mathcal{A}_1(\mathfrak{a}) = \frac{1}{2}\langle \nabla_g(\mathfrak{a}),\mathfrak{a})\rangle + \langle \mathfrak{a},\gamma\rangle$$

$\forall\ \mathfrak{a} \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}$ and

$$\nabla_f = \begin{pmatrix} 1 & 2 & -3 \\ 1 & 2 & -3 \\ -1 & -1 & 3 \end{pmatrix}$$

and

$$\nabla_g = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$\beta = (2,6,8)$ and $\gamma = (2,6,0)$. Here, it easy to conclude that the functions ∇_f and ∇_g are convex and continuous as well on $\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ with $del\nabla_f(\cdot) = \mathcal{A}(\cdot)+\beta$, $del\nabla_g(\cdot) = \mathcal{A}_1(\cdot)+\gamma$ and

$$\mathring{\mathcal{J}} = \{\mathfrak{a},\mathfrak{b},\mathfrak{c} : \mathfrak{a}+\mathfrak{b} = 8,\ \mathfrak{c} = 0\}.$$

Let us define a sequence $\{\mathfrak{a}_\eta,\mathfrak{b}_\eta,\mathfrak{c}_\eta\}$ with initial value $\{\mathfrak{a}_0,\mathfrak{b}_0,\mathfrak{c}_0\}$ as follows:

$$\begin{cases}\mathfrak{a}_0\in\mathfrak{B}_s,\\ (\mathfrak{a}^1_{\eta+1},\mathfrak{b}^1_{\eta+1},\mathfrak{c}^1_{\eta+1}) = (1-\kappa^1_\eta)(\mathfrak{b}^1_\eta,\mathfrak{b}^2_\eta,\mathfrak{b}^3_\eta)+\kappa^1_\eta\Upsilon_1(\mathfrak{b}^1_\eta,\mathfrak{b}^2_\eta,\mathfrak{b}^3_\eta),\\ (\mathfrak{b}^1_\eta,\mathfrak{b}^2_\eta,\mathfrak{b}^3_\eta) = (1-\kappa^2_\eta)\Upsilon_2(\mathfrak{a}^1_\eta,\mathfrak{b}^1_\eta,\mathfrak{c}^1_\eta)+\kappa^2_\eta\Upsilon_1(\mathfrak{c}^1_\eta,\mathfrak{c}^2_\eta,\mathfrak{c}^2_\eta),\\ (\mathfrak{c}^1_\eta,\mathfrak{c}^2_\eta,\mathfrak{c}^2_\eta) = (1-\kappa^3_\eta)(\mathfrak{a}^1_\eta,\mathfrak{b}^1_\eta,\mathfrak{c}^1_\eta)+\kappa^3_\eta\Upsilon_2(\mathfrak{a}^1_\eta,\mathfrak{b}^1_\eta,\mathfrak{c}^1_\eta),\end{cases} \tag{$\mathcal{E}$}$$

where $\Upsilon_1 = (I+del\nabla_f)^{-1}$ and $\Upsilon_2 = (I+del\nabla_g)^{-1}$, $0<\kappa^1_\eta,\kappa^2_\eta,\kappa^3_\eta<1$. Using initial value as $(\mathfrak{a}_0,\mathfrak{b}_0,\mathfrak{c}_0)$, $\forall\ \mathfrak{a}_0,\mathfrak{b}_0,\mathfrak{c}_0\in\mathbb{R}$ in Theorem 4.2, we can find the solution for distinct values of $(\mathfrak{a}_0,\mathfrak{b}_0,\mathfrak{c}_0)$.

Conclusion. Inspired by two well-known concepts, $CR-$iterative algorithm by Chug et al. [5] and common zero of two accretive operators by Kim & Tuyen [10], in this analysis we have introduced the Generalized $G-CR$ iteration algorithm and analyzed its convergence behaviour to find CFP_s for nonself $QNEMs$ in convex Banach spaces. In order to understand the work, application of the the same is also analyzed.

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