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Invariant regions and global existence of uniqueness weak solutions for tridiagonal reaction-diffusion systems

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Abstract. In this paper we study the existence of uniqueness global weak solutions for $m \times m$ reaction-diffusion systems for which two main properties hold: the positivity of the weak solutions and the total mass of the components are preserved with time. Moreover we suppose that the non-linearities have critical growth with respect to the gradient. The technique we use here in order to prove global existence is in the same spirit of the method developed by Boccardo, Murat, and Puel for a single equation.

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1. Introduction

In [26, 27], the authors obtained a global existence of solutions for the coupled semilinear reaction-diffusion system with diagonal by order 2, and m, triangular, and full matrix of diffusion coefficients. By combining the compact semigroup methods and some L^1 estimates, we show that global solutions exist for a large class of the term of reaction. In the works [8, 14], we find new developed methods based on truncation functions, fixed point theorems and compactness, etc to prove establish the existence of global solutions.

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In the present work we consider the problem

$$\begin{cases} \frac{\partial U}{\partial t} - A_m \Delta U = F\left(t, x, U, \nabla U\right) & \text{on }]0, +\infty[\times \Omega, \\ U = 0 \quad \text{or} \quad \frac{\partial U}{\partial \eta} = 0 & \text{on }]0, +\infty[\times \partial\Omega, \\ U\left(0, x\right) = U_0\left(x\right) & \text{on } \Omega, \end{cases} \tag{1.1}$$

by using a technique based on L^1-estimate we establish a global existence result of the solution.
We consider the m-equations of reaction-diffusion system (1.1), with $m \geq 2$, where Ω is an open bounded domain of class C^1 in $\mathbb{R}^n$, the vectors U, F, U_0 and the matrix A_m are defined as:

$$\begin{cases} U = (u_1, \ldots, u_m)^T = ((u_s)_{s=1}^m)^T, \\ \nabla U = (\nabla u_1, \ldots, \nabla u_m)^T = ((\nabla u_s)_{s=1}^m)^T, \\ F = (F_1, \ldots, F_m)^T = ((F_s)_{s=1}^m)^T, \\ U_0 = (u_1^0, \ldots, u_m^0)^T = ((u_s^0)_{s=1}^m)^T. \end{cases}$$

$$A_m = \begin{pmatrix} a_1 & b_1 & 0 & \cdots & 0 \\ c_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & c_2 & a_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{m-1} \\ 0 & \cdots & 0 & c_{m-1} & a_m \end{pmatrix}. \tag{1.2}$$

The nonlinearities F_s, $1 \leq s \leq m$, have critical growth with respect to $|\nabla U|$, and the constants $(a_i)_{i=1}^m$, $(b_i)_{i=1}^{m-1}$ et $(c_i)_{i=1}^{m-1}$ are supposed to be strictly positive and satisfy the condition

$$\cos^2\left(\frac{\pi}{m+1}\right) < \frac{a_i a_{i+1}}{\left(b_i + c_i\right)^2} \tag{1.3}$$

which reflects the parabolicity of the system and implies at the same time that the diffusion matrix is positive defnite. That means the eigenvalues $(\lambda_i)_{i=1}^m$, $(\lambda_1 > \lambda_2 > \ldots > \lambda_m)$, of A_m are positive.
Note that $\frac{\partial}{\partial \eta}$ denotes the outward normal derivative on boundary $\partial\Omega$.
The initial data are assumed to be in the regions:

$$\sum\nolimits_{\mathfrak{S},\mathfrak{Z}} = \left\{ U_0 \in \mathbb{R}^m \ : \ \begin{cases} w_z^0 = \langle V_z, U_0 \rangle \leq 0 & \text{if } z \in \mathfrak{Z} \\ w_s^0 = \langle V_s, U_0 \rangle \geq 0 & \text{if } s \in \mathfrak{S} \end{cases} \right\}, \tag{1.4}$$

where

$$\mathfrak{S} \cap \mathfrak{Z} = \phi, \ \mathfrak{S} \cup \mathfrak{Z} = \{1, 2, \ldots, m\}.$$

The notation $\langle .,. \rangle$ denotes the inner product in $\mathbb{R}^m$ and $V_s = (v_{s1}, \ldots, v_{sm})^T$ the eigenvector of the diffusion matrix A_m associated with the eigenvalue $(\lambda_s)_{s=1}^m$. Hence, we can see that there are 2^m regions.
This work represents a generalization to the parabolic case study did in the elliptic case (see [7]) for these systems of arbitrary order. This passage in parabolic case, needs new approaches and also technical difficulties to be overcome. We will explain in detail here.

We found a good idea to present our work as follows: we start initially with an introduction that presents the state of the art of the area studied and some recall the main results obtained previously. This will highlight the contribution of our work and its originality. In the second section we give the definition of the notion of solution used here. We then present the main results of this work. In the last section, we give the proof of global existence and uniqueness of our reaction-diffusion system. This is done in three steps: in the first we truncate the system, the latter we give suitable estimates on the approximate solutions and in the last step we show the convergence of the approximating system by using the technics introduced by Boccardo *et al.* [13] and Dall'Aglio and Orsina [15].

2. Eigenvalues and eigenvectors of the diffusion matrix

The usual norms in the spaces $L^1(\Omega)$, $L^\infty(\Omega)$ and $C(\overline{\Omega})$ are denoted respectively by:

$$\|u\|_1 = \int_\Omega |u(x)|\,dx,$$

$$\|u\|_\infty = ess\sup_{x\in\Omega} |u(x)| \quad \text{and} \quad \|u\|_{C(\overline{\Omega})} = \max_{x\in\overline{\Omega}} |u(x)|.$$

For any initial data in $C(\overline{\Omega})$ or $L^\infty(\Omega)$ local existence and uniqueness of solutions to the initial values problem (1.1) follow from the basic existence theory for abstract semilinear differential equations (see Friedman [16], Henry [17], Pazzy [28]).

Our aim in this section is to get a three term reccurence relation of characteristic polynomial of matrix A of dimension $m \times m$ in terms of matrices of dimensions $(m-1)\times(m-1)$ and $(m-2)\times(m-2)$ so the eigenvectors of this matrix. The solutions of characteristic polynomial $\det(A_m - \lambda I_m) = 0$ are λ which represent eigenvalues of the matrix A_m. We denote the characteristic polynomial of A_m, A_{m-1}, A_{m-2} by $\phi_m(\lambda)$, $\phi_{m-1}(\lambda)$, $\phi_{m-2}(\lambda)$ respectively.

Lemma 2.1 (See [22]**).** *Let A_m be the tridiagonal matrix defined in* (1.2)*, the eigenvalues of A_m are distinct and interlace strictly with eigenvalues of A_{m-1} for $m \geq 2$. Where*

$$\phi_0(\lambda) = 1,\ \phi_1(\lambda) = a_1 - \lambda,\ \phi_m(\lambda) = (\lambda - a_m)\,\phi_{m-1}(\lambda) - b_{m-1}c_{m-1}\phi_{m-2}(\lambda). \tag{2.1}$$

Lemma 2.2 (See Andelic and Fonseca [9]**).** *Let A_m be the real symmetric tridiagonal matrix definied in* (1.2)*, with diagonal entries positive.*
If

$$\cos^2\left(\frac{\pi}{m+1}\right) < \frac{a_i a_{i+1}}{(b_i + c_i)^2},\ \textit{for}\quad i = 1, \ldots, m-1,$$

then A_m is positive definite.

We remark that general characterization in terms of the eigenvalues, i.e. A_m is positive definite if and only if all its eigenvalues are positive.

Lemma 2.3. *Let λ_s for $s = 1, \dots, m$ be the eigenvalues of the tridiagonal matrix A_m. Then the eigenvectors $V_s = (v_{s1}, \dots, v_{sm})^T$ associated to λ_s for $s = 1, \dots, m$ are given by the following expressions*

$$\begin{cases} v_{sm} = 1, \\ v_{s(m-1)} = \frac{\lambda_s - a_m}{c_{m-1}}, \\ v_{s(\ell-1)} = -\frac{b_\ell v_{s(\ell+1)} + (a_\ell - \lambda_s) v_{s\ell}}{c_{\ell-1}}, \quad \ell = 2, \dots, m-1. \end{cases} \tag{2.2}$$

Proof. Recall that the diffusion matrix is positive definite, hence its eigenvalues are necessarily positive. The eigenvectors of the diffusion matrix associated with the eigenvalues λ_s are defind as $V_s = (v_{s1}, v_{s2}, \dots, v_{sm})^T$. For an eigenpair (λ_s, V_s), the components in $A_m V = \lambda V$ are

$$\begin{cases} a_1 v_1 + b_1 v_2 = \lambda v_1 \\ c_{\ell-1} v_{\ell-1} + a_\ell v_\ell + b_\ell v_{\ell+1} = \lambda v_\ell, \quad (2 \le \ell \le m-1) \\ c_{m-1} v_{m-1} + a_m v_m = \lambda v_m \end{cases}$$

if $v_m = 0$, the assumption $b_i \neq 0$, $c_i \neq 0$ for all $i = 1, \dots, m-1$ we said that all v_{si} are zero. We can therefore take $v_m = 1$ and $(v_1, v_2, \dots, v_{m-1})$ is a solution of upper triangular system

$$\begin{cases} c_{\ell-1} v_{\ell-1} + (a_\ell - \lambda) v_\ell + b_\ell v_{\ell+1} = 0 \quad (2 \le \ell \le m-2) \\ c_{m-2} v_{m-2} + (a_{m-1} - \lambda) v_{m-1} = -b_{m-1} \\ c_{m-1} v_{m-1} = \lambda - a_m \end{cases}$$

the solution of this system is given by

$$\begin{cases} v_{m-1} = \frac{\lambda - a_m}{c_{m-1}}, \\ v_{\ell-1} = -\frac{b_\ell v_{\ell+1} + (a_\ell - \lambda) v_\ell}{c_{\ell-1}}, \ (\ell = 2, \dots, m-1). \end{cases}$$

□

3. Diagonalizing system (1.1)

Usually to construct an invariant regions for systems such (1.1) we make a linear change of variables u_i to obtain a new equivalent system with diagonal diffusion matrix for which standard techniques can be applied to deduce global existence (see [1, 2, 3, 4, 5, 21]).

Let $V_s = (v_{s1}, \dots, v_{sm})^T$ be an eigenvector of the matrix A_m associated with its eigenvalue $(\lambda_s)_{s=1}^m$ where $\lambda_1 > \lambda_2 > \dots > \lambda_m$. Multiplying the k^{th} equation of (1.1) by $(-1)^{i_s} V_{sk}$, $i_s = 1, 2$ and $k = 1, \dots, m$, and adding the resulting equations, we get

$$\begin{cases} \frac{\partial W}{\partial t} - diag\left(\lambda_1, \lambda_2, \dots, \lambda_m\right) \Delta W = \Psi\left(t, x, W, \nabla W\right) & \text{on }]0, +\infty[\times \Omega, \\ W = 0 \quad \text{or} \quad \frac{\partial W}{\partial \eta} = 0 & \text{on }]0, +\infty[\times \partial\Omega, \\ W\left(0, x\right) = W_0\left(x\right) & \text{on } \Omega, \end{cases} \tag{3.1}$$

where

$$\begin{cases} W = ((w_s)_{s=1}^m)^T, \quad \nabla W = ((\nabla w_s)_{s=1}^m)^T, \quad w_s = \left\langle (-1)^{i_s} V_s, U \right\rangle, \\ \Psi = ((\Psi_s)_{s=1}^m)^T, \quad \Psi_s = \left\langle (-1)^{i_s} V_s, F \right\rangle, \\ W_0 = ((w_s^0)_{s=1}^m)^T, \qquad w_s^0 = \left\langle (-1)^{i_s} V_s, U_0 \right\rangle, \ m \geq 2. \end{cases}$$

for all $i_s = \{1, 2\}$.

Proposition 3.1. *The system* (3.1) *admits a unique classical solution* W *on* $[0, T_{\max}) \times \Omega$, *where* $T_{\max}\left(\left\|w_1^0\right\|_\infty, \left\|w_2^0\right\|_\infty, \ldots, \left\|w_m^0\right\|_\infty\right)$ *denotes the eventual blow-up time. Furthermore, if* $T_{\max} < +\infty$, *then*

$$\lim_{t \to T_{\max}} \sum_{s=1}^{m} \left\|w_s(t,.)\right\|_\infty = +\infty.$$

Therefore, if there exists a positive constant C *such that*

$$\sum_{s=1}^{m} \left\|w_s(t,.)\right\|_\infty \leq C \quad \textit{for all} \quad t \in [0, T_{\max}),$$

then, $T_{\max} = +\infty$.

4. Statement of the main result

4.1. Assumptions

Let us, now introduce for w_s^0 the hypotheses, for all $1 \leq s \leq m$

(A1) The initial conditions are in $\sum_{\mathfrak{S},3}$, w_s^0, are nonnegative functions in $L^1(\Omega)$.

The following assumptions are also made on the function Ψ defined by:

$$\Psi = ((\Psi_s)_{s=1}^m)^T, \quad \Psi_s = \left\langle (-1)^{i_s} V_s, F \right\rangle, \quad i_s = 1, 2.$$

(A2) Ψ_s are continuously differentiable on $\mathbb{R}_+^m$ and Ψ_s, $s = \overline{1, m}$, are quasi-positives functions which means that, for $s = \overline{1, m}$

$$[w_1 \geq 0, \ldots, w_{s-1} \geq 0, w_{s+1} \geq 0, \ldots, w_m \geq 0],$$

implies

$$\begin{cases} \Psi_s(t, x, w_1, \ldots, w_{s-1}, 0, w_{s+1}, \ldots, w_m, p_1, \ldots, p_{s-1}, 0, p_{s+1}, \ldots, p_m) \geq 0. \\ \quad \text{for all } 1 \leq s \leq m, \ (W, p) \in (\mathbb{R}^+)^m \times \mathbb{R}^{Nm} \text{ and for a.e. } (t, x) \in Q_T \end{cases}$$

These conditions on Ψ guarantee local existence of unique, nonnegative classical solutions on a maximal time interval $[0, T_{\max})$, see Hollis and Morgan [20].

(A3) The inequality

$$\langle S, \Psi(t, x, W, \nabla W) \rangle \leq C_1 (1 + \langle W, 1 \rangle),$$

such that

$$W = (w_1, \ldots, w_m), \ S = (d_1, d_2, \ldots, d_{m-1}, 1),$$

for all $w_s \geq 0$, $s = 1, \ldots, m$ and all constants d_s satisfy $d_s \geq \overline{d}_s$, $s = 1, \ldots, m-1$, where $C_1 \geq 0$ and $\overline{d}_s$ are positive constants sufficiently large.

Under the assumptions (A1)-(A3), the next proposition says that the classical solution of the system (3.1) remains in $\sum_{\mathfrak{S},3}$ for all t in $[0, T_{\max})$.

Proposition 4.1. *Suppose that the assumptions (A1)-(A3) are satisfied. Then for any* W_0 *in* $\sum_{\mathfrak{S},3}$ *the classical solution* W *of the system* (3.1) *on* $[0, T_{\max}) \times \Omega$ *remains in* $\sum_{\mathfrak{S},3}$ *for all* t *in* $[0, T_{\max})$.

(A4) The total mass of the components $w_1, \ldots, w_m$ is controlled with time, which is ensured by

$$\begin{cases} \sum\limits_{1 \leq s \leq r} \Psi_s(t, x, W, p) \leq 0, \text{ for all } 1 \leq r \leq m \\ \text{for all } (W, p) \in (\mathbb{R}^+)^m \times \mathbb{R}^{Nm} \text{ and a.e. } (t, x) \in Q_T \end{cases}$$

$$\Psi_s :]0, T[\times \Omega \times \mathbb{R}^m \times \mathbb{R}^{mN} \to \mathbb{R} \text{ are measurable} \tag{4.1}$$

$$\Psi_s : \mathbb{R}^m \times \mathbb{R}^{mN} \to \mathbb{R} \text{ are locally Lipschitz continuous} \tag{4.2}$$

namely

$$\begin{aligned} &\sum_{1 \leq s \leq m} \left| \Psi_s(t, x, W, p) - \Psi_s\left(t, x, \hat{W}, \hat{p}\right) \right| \\ \leq \; & K(r) \left(\sum_{1 \leq s \leq m} |w_s - \hat{w}_s| + \sum_{1 \leq s \leq m} \|p_s - \hat{p}_s\| \right) \end{aligned}$$

for a.e. (t, x) and for all $0 \leq |w_s|, |\hat{w}_s|, \|p_s\|, \|\hat{p}_s\| \leq r$.

$$|\Psi_1(t, x, W, \nabla W)| \leq C_1(|w_1|) \left(F_1(t, x) + \|\nabla w_1\|^2 + \sum_{2 \leq j \leq m} \|\nabla w_j\|^{\alpha_j} \right) \tag{4.3}$$

where $C_1 : [0, +\infty) \to [0, +\infty)$ is nondecreasing, $F_1 \in L^1(Q_T)$ and $1 \leq \alpha_j < 2$

$$|\Psi_s(t, x, W, \nabla W)| \leq C_s \left(\sum_{j=1}^{s} |w_j| \right) \left(F_s(t, x) + \sum_{1 \leq j \leq m} \|\nabla w_j\|^2 \right), \; 2 \leq s \leq m \tag{4.4}$$

where $C_s : [0, +\infty) \to [0, +\infty)$ is nondecreasing, $F_s \in L^1(Q_T)$ for all $2 \leq s \leq m$.

Let us know that if the nonlinearities Ψ do not dependent on the gradient (system (3.1) is semilinear), the existence of global positive solutions have been obtained by Hollis *et all* [18], Hollis and Morgan [19] and Martin and Pierre [25]. One can see that in all of these works, the triangular structure, namely hypotheses (A4) plays an important role in the study of semilinear systems. Indeed, if (A4) does not hold, Pierre and Schmitt [29] proved blow up in finite time of the solutions to some semilinear reaction-diffusion systems.

Where $\Psi = (\Psi_1, \Psi_2)$ depends on the gradient, Alaa and Mounir [8] solved the problem where the triangular structure is satisfied and the growth of Ψ_1 and Ψ_2 with respect to $|\nabla w_1|, |\nabla w_2|$ is sub-quadratic. There exists $1 \leq p < 2$, $C : [0,\infty)^2 \to [0,\infty)$ nondecreasing such that

$$|\Psi_1| + |\Psi_2| \leq C(|w_1|, |w_2|)(1 + |\nabla w_1|^p + |\nabla w_2|^p)$$

About the critical growth with respect to the gradient ($p = 2$), we recall that for the case of a single equation ($d_1 = d_2$ and $\Psi_1 = \Psi_2$), existence results have been proved for the elliptic case in [11, 12]. The corresponding parabolic equations have also been studied by many authors; see for instance [6, 13, 15, 24].

5. Statement of the result

First, we have to clarify in which sense we want to solved problem (3.1).
The existence of global unique solutions for the system (3.1) is to equivalence to existence a w_s, $s = \overline{1,m}$, true for the following theorem:

Theorem 5.1. *Suppose that the hypotheses (A1)-(A4) and (4.1)-(4.4) are satisfied, so it exists unique* w_s, $s = \overline{1,m}$ *solution of:*

$$\begin{cases} w_s \in C([0,T]; L^1(\Omega)) \cap L^1\left(0,T; W_0^{1,1}(\Omega)\right), \\ \Psi_s \in L^1(Q_T) \text{ where } Q_T = (0,T) \times \Omega \text{ for all } T > 0, \\ w_s(t) = S_s(t) w_s^0 + \int_0^t S_s(t-\tau)\Psi_s(s,.,W(\tau), \nabla W(\tau))\,d\tau, \\ s = \overline{1,m}, \ \forall t \in [0,T[, \end{cases} \tag{5.1}$$

where $W = (w_1, \ldots, w_m)$, $\nabla W = (\nabla w_1, \ldots, \nabla w_m)$ *and* $S_s(t)$ *are the semigroups of contractions in* $L^1(\Omega)$ *generated by* $\lambda_s \Delta$, $s = \overline{1,m}$.

Example 5.2. For $1 \leq i \leq m$, A typical example where the result of this paper can be applied is

$$\begin{cases} \dfrac{\partial w_i}{\partial t} - d_i \Delta w_i = \sum\limits_{1 \leq j \leq i} a_{ij} \dfrac{w_j}{\sum\limits_{1 \leq k \leq m} w_k} |\nabla w_j|^2 + f_i(t,x) & \text{in } Q_T \\ w_i = 0 & \text{on } \Sigma_T \\ w_i(0,x) = w_{i,0} & \text{in } \Omega \end{cases}$$

Theorem 5.3. *Assume that (A2), (A4) and (4.1)-(4.4) hold. If* $w_s^0 \in L^2(\Omega)$, *for all* $1 \leq s \leq m$, *then there exists a positive global solution* $W = (w_1, \ldots, w_m)$ *of system* (3.1). *Moreover,* $w_1, \ldots, w_m \in L^2\left(0,T; H_0^1(\Omega)\right)$.

Before giving the proof of this theorem, let us define the following functions.
Given a real positive number k, we set

$$T_k(s) = \max\{-k, \min(k,s)\} \quad \text{and } G_k(s) = s - T_k(s)$$

We remark that

$$\begin{cases} T_k(s) = s & \text{for } 0 \leq s \leq k, \\ T_k(s) = k & \text{for } s > k. \end{cases}$$

Proof of Theorem 5.3.

Approximating scheme. For every function h defined from $\mathbb{R}^+ \times \Omega \times \mathbb{R}^m \times \mathbb{R}^{mN}$ into $\mathbb{R}$, we associate $\hat{\varphi} = \hat{\varphi}(t, x, W, p)$ such that

$$\hat{\varphi} = \begin{cases} \varphi\left(t, x, w_1, \ldots, w_m, p_1, \ldots, p_m\right) & \text{if } w_s \geq 0,\ 1 \leq s \leq m \\ \quad \varphi\left(t, x, w_1, \ldots, w_{s-1}, 0, w_{s+1}, \ldots, w_m,\right. & \\ \quad \left. p_1, \ldots, p_{s-1}, 0, p_{s+1}, \ldots, p_m\right) & \text{if } w_s \leq 0 \text{ and } w_j \geq 0,\ j \neq s \\ \varphi\left(t, x, 0, \ldots, 0, p_1, \ldots, p_m\right) & \text{if } w_s \leq 0,\ 1 \leq s \leq m \end{cases}$$

and consider the system, for $1 \leq s \leq m$

$$\begin{cases} \frac{\partial w_s}{\partial t} - d_s \Delta w_s = \hat{\Psi}_s\left(t, x, W, \nabla W\right) & \text{in }]0, +\infty[\times \Omega, \\ w_s = 0 \quad \text{or} \quad \frac{\partial w_s}{\partial \eta} = 0 & \text{on }]0, +\infty[\times \partial\Omega, \\ w_s\left(0, x\right) = w_s^0\left(x\right) & \text{in } \Omega. \end{cases} \tag{5.2}$$

It is obviously seen, by the structure of $\hat{\Psi}_s$, $1 \leq s \leq m$, that systems (3.1) and (5.2) are equivalent on the set where $w_s \geq 0$, $1 \leq s \leq m$. Consequently, to prove Theorem 5.3, we have to show that problem (5.2) has a weak solution which is positive.

To this end, we define ψ_n a truncation function by $\psi_n \in C_c^\infty(\mathbb{R}), 0 \leq \psi_n \leq 1$, and

$$\psi_n(z) = \begin{cases} 1 & \text{if } |z| \leq n \\ 0 & \text{if } |z| \geq n+1 \end{cases}$$

and the mollification with respect to (t, x) is defined as follows.

Let $\rho \in C_c^\infty\left(\mathbb{R} \times \mathbb{R}^N\right)$ such that

$$supp\rho \subset B(0,1), \quad \int \rho = 1,\ \rho \geq 0 \text{ on } \mathbb{R} \times \mathbb{R}^N$$

and $\rho_n(y) = n^N \rho(ny)$. One can see that

$$\rho_n \in C_c^\infty\left(\mathbb{R} \times \mathbb{R}^N\right),\ supp\rho_n \subset B\left(0, \frac{1}{n}\right),\ \int \rho_n = 1,\ \rho_n \geq 0 \text{ on } \mathbb{R} \times \mathbb{R}^N$$

We also consider nondecreasing sequences $w_{s,0}^n \in C_c^\infty\left(\Omega\right)$ such that

$$w_{s,0}^n \to w_s^0 \text{ in } L^2\left(\Omega\right),\ 1 \leq s \leq m$$

and define for all (t, x, W, p) in $\mathbb{R}^+ \times \Omega \times \mathbb{R}^m \times \mathbb{R}^{mN}$ and $1 \leq s \leq m$,

$$\Psi_{s,n}(t, x, W, p) = \left[\psi_n\left(\sum_{1 \leq j \leq m}\left(|w_j| + \|p_j\|\right)\right)\Psi_s(t, x, W, p)\right] * \rho_n\left(t, x\right).$$

Note that these functions enjoy the same properties as Ψ_s, $1 \leq s \leq m$, moreover they are Hölder continuous with respect to (t, x) and $|\Psi_{s,n}| \leq M_n$, $1 \leq s \leq m$, where M_n is a constant depending only on n (these estimates can be derived from (5.1), the properties of the convolution product, and the fact that $\int \rho_n = 1$).

Let us now consider the truncated system, for $1 \leq s \leq m$

$$\begin{cases} \dfrac{\partial w_{s,n}}{\partial t} - d_s \Delta w_{s,n} = \Psi_{s,n}\left(t, x, W_n, \nabla W_n\right) & \text{in } Q_T \\ w_{s,n} = 0 \quad \text{or} \quad \frac{\partial w_{s,n}}{\partial \eta} = 0 & \text{on } \Sigma_T \\ w_{s,n}\left(0, x\right) = w_{s,0}^n\left(x\right) & \text{in } \Omega \end{cases} \tag{5.3}$$

It is well known that problem (5.3) has a global classical solution (see [23], theorem 7.1, p. 591) for the existence and ([24], Corollary of Theorem 4.9, p. 341) for the regularity of solutions. It remains to show the positivity of the solutions.

Lemma 5.4. *Let $w_n = (w_{1,n}, \dots, w_{m,n})$ be a classical solution of (5.3) and suppose that $w^n_{1,0}, \dots, w^n_{m,0} \geq 0$. Then $w_{1,n}, \dots, w_{m,n} \geq 0$.*

Proof. See [8], Lemma 1, p 537. □

5.1. A priori estimates

The hypotheses (A2) and (A4) allowed the following lemma.

Lemma 5.5. *(i) There exists a constant M depending on $\sum\limits_{1\leq j\leq m} \|w_{j,0}\|_{L^1(\Omega)}$ such that*

$$\int_\Omega \left(\sum_{1\leq j\leq m} w_{j,n}(t) \right) \leq M, \text{ for all } t \in [0,T]$$

(ii) There exists a constant R_1 depending on $\sum\limits_{1\leq j\leq m} \|w_{j,0}\|_{L^1(\Omega)}$, such that

$$\sum_{1\leq j\leq m} \int_\Omega |\Psi_{j,n}(t,x,W_n,\nabla W_n)| \leq R_1.$$

(iii) There exists a constant R_2 depending on k and $\sum\limits_{1\leq s\leq m} \left\|w^0_s\right\|_{L^1(\Omega)}$, such that for all $1 \leq j \leq m$

$$\int_{Q_T} |\nabla T_k(w_{j,n})|^2 \leq R_2.$$

(iv) There exists a constant R_3 depending on $\sum\limits_{1\leq j\leq r} \|w_{j,0}\|_{L^2(\Omega)}$ such that for all $2 \leq r \leq m$,

$$\int_{Q_T} \left| \nabla T_k \left(\sum_{1\leq j\leq r} w_{j,n} \right) \right|^2 \leq R_3$$

(v) There exists a constant R_4 depending on $\sum\limits_{1\leq j\leq m} \|w_{j,0}\|_{L^2(\Omega)}$ and $d_1, \dots, d_m$ such that

$$\int_{Q_T} |\Psi_{j,n}(t,x,W_n,\nabla W_n)| \left(\sum_{1\leq r\leq m} (m-r+1) w_{k,n} \right) \leq R_4, \text{ for all } 1 \leq j \leq m.$$

Proof. See Bouarifi *et al.* [14]. □

5.2. Convergence

Our objective is to show that $W_n = (w_{1,n}, \ldots, w_{m,n})$ converges to some $W = (w_1, \ldots, w_m)$ solution of the problem (5.1). The sequences $w_{1,0}^n, \ldots, w_{m,0}^n$ are uniformly bounded in $L^1(\Omega)$ (since they converge in $L^2(\Omega)$), and by Lemma 5.5, the nonlinearities $\Psi_{1,n}, \ldots, \Psi_{m,n}$ are uniformly bounded in $L^1(Q_T)$. Then according to a result in [10] the applications

$$\left(w_{s,0}^n, \Psi_{s,n}\right) \to w_{s,n},\ 1 \le s \le m$$

are compact from $L^1(\Omega) \times L^1(Q_T)$ into $L^1\left(0, T; W_0^{1,1}(\Omega)\right)$.
Therefore, we can extract a subsequence, still denoted by $(w_{1,n}, \ldots, w_{m,n})$, such that

$$\begin{array}{ll} (w_{1,n}, \ldots, w_{m,n}) \to (w_1, \ldots, w_m) & \text{in } L^1\left(0, T; W_0^{1,1}(\Omega)\right) \\ (w_{1,n}, \ldots, w_{m,n}) \to (w_1, \ldots, w_m) & \text{a.e. in } Q_T \\ (\nabla w_{1,n}, \ldots, \nabla w_{m,n}) \to (\nabla w_1, \ldots, \nabla w_m) & \text{a.e. in } Q_T \end{array}$$

Since $\Psi_{1,n}, \ldots, \Psi_{m,n}$ are continuous, we have

$$\Psi_{s,n}(t, x, W_n, \nabla W_n) \to \Psi_s(t, x, W, \nabla W) \qquad \text{a.e. in } Q_T,\ 1 \le s \le m.$$

This is not sufficient to ensure that $(w_1, \ldots, w_m)$ is a solution of (5.1). In fact, we have to prove that the previous convergence are in $L^1(Q_T)$. In view of the Vitali theorem, to show that $\Psi_{s,n}(t, x, W_n, \nabla W_n)$, $1 \le s \le m$, converges to $\Psi_s(t, x, W, \nabla W)$ in $L^1(Q_T)$, is equivalent to proving that $\Psi_{s,n}(t, x, W_n, \nabla W_n)$, $1 \le s \le m$ are equi-integrable in $L^1(Q_T)$

Lemma 5.6. *$\Psi_{s,n}(t, x, W_n, \nabla W_n)$, for all $1 \le s \le m$, are equi-integrable in $L^1(Q_T)$.*

The proof of this lemma requires the following result based on some properties of two time-regularization denoted by w_γ and $w_\sigma (\gamma, \sigma > 0)$ which we define for a function $w \in L^2\left(0, T; H_0^1(\Omega)\right)$ such that $w(0) = w_0 \in L^2(\Omega)$ (for more details see [8]). In the following we will denote by $\omega(\varepsilon)$ a quantity that tends to zero as ε tends to zero, and $\omega^\sigma(\varepsilon)$ a quantity that tends to zero for every fixed σ as ε tends to zero.

Lemma 5.7. *Let (w_n) be a sequence in $L^2\left(0, T; H_0^1(\Omega)\right) \cap C([0, T])$ such that $w_n(0) = w_0^n \in L^2(\Omega)$ and $(w_n)_t = \rho_{1,n} + \rho_{2,n}$ with $\rho_{1,n} \in L^2\left(0, T; H^{-1}(\Omega)\right)$ and $\rho_{2,n} \in L^1(Q_T)$. Moreover assume that w_n converges to w in $L^2(Q_T)$, and w_0^n converges to $w(0)$ in $L^2(\Omega)$.*
Let Υ be a function in $C^1([0, T])$ such that $\Upsilon \ge 0$, $\Upsilon' \le 0, \Upsilon(T) = 0$. Let φ be a Lipschitz increasing function in $C^0(\mathbb{R})$ such that $\varphi(0) = 0$. Then for all $k, \gamma > 0$

$$\begin{aligned} &\left\langle \rho_{1n}, \Upsilon\varphi\left(T_k(w_n) - T_k(w_m)_\gamma\right)\right\rangle + \int_{Q_T} \rho_{2n} \Upsilon\varphi\left(T_k(w_n) - T_k(w_m)_\gamma\right) \\ \ge\ & \omega^{\gamma,n}\left(\frac{1}{m}\right) + \omega^\gamma\left(\frac{1}{n}\right) + \int_\Omega \Upsilon(0)\, \Phi\left(T_k(w) - T_k(w)_\gamma\right)(0)\, dx \\ &- \int_\Omega G_k(w)(0)\, \Upsilon(0)\, \varphi\left(T_k(w) - T_k(w)_\gamma\right)(0)\, dx \end{aligned}$$

where $\Phi(t) = \int_0^t \varphi(s) ds$ and $G_k(s) = s - T_k(s)$

Proof. See [8], Lemma 7, p 544. □

Lemma 5.8. *Suppose that $w_{j,n}, w_j$, $1 \leq j \leq m$, are as above.*

(i) If

$$|\Psi_{1,n}| \leq C_1(|w_{1,n}|)\left(F_1(t,x) + |\nabla w_{1,n}|^2 + \sum_{2\leq j\leq m} |\nabla w_j|^{\alpha_j}\right)$$

where $C_1 : [0,+\infty) \to [0,+\infty)$ is nondecreasing, $F_1 \in L^1(Q_T)$ and $1 \leq \alpha_j < 2$. Then for each fixed k

$$\lim_{n\to\infty} \int_{Q_T} |\nabla T_k(w_{1,n}) - \nabla T_k(w_1)|^2 \chi_{\left[\sum\limits_{1\leq j\leq m} w_{j,n}\leq k\right]} = 0.$$

(ii) If

$$|\Psi_{s,n}(t,x,W,\nabla W)| \leq C_s\left(\sum_{j=1}^{s} |w_j|\right)\left(F_s(t,x) + \sum_{1\leq j\leq m} |\nabla w_j|^2\right),\ 2 \leq s \leq m$$

where $C_s : [0,+\infty) \to [0,+\infty)$ is nondecreasing, $F_s \in L^1(Q_T)$ for all $2 \leq s \leq m$. Then for each fixed k and for all $2 \leq s \leq m$

$$\lim_{n\to\infty} \int_{Q_T} \left|\nabla T_k\left(\sum_{1\leq j\leq s} w_{j,n}\right) - \nabla T_k\left(\sum_{1\leq j\leq s} w_j\right)\right|^2 \chi_{\left[\sum\limits_{1\leq j\leq m} w_{j,n}\leq k\right]} = 0.$$

Proof. (i) This is a direct consequence of the resulting output established in [8, 14] □

Proof of Lemma 5.6. Let A be a measurable subset of Ω, we have

$$\begin{aligned}\int_A |\Psi_{1,n}(t,x,W_n,\nabla W_n)| &= \int_{A\cap[E_n>k]} |\Psi_{1,n}| + \int_{A\cap[E_n\leq k]} |\Psi_{1,n}| \\ &\leq \int_{A\cap[\theta_n>k]} |\Psi_{1,n}| + \int_{A\cap[E_n\leq k]} |\Psi_{1,n}|\end{aligned}$$

with $E_n = \sum\limits_{1\leq j\leq m} w_{j,n}$ and $\theta_n = \sum\limits_{1\leq k\leq m} (m-k+1)w_{k,n}$.

Thanks to (iii) (Lemma 5.5), we obtain $\forall\varepsilon > 0$, $\exists k_0$ such that if $k \geq k_0$ then for all n

$$\begin{aligned}&\int_{A\cap[E_n>k]} |\Psi_{1,n}(t,x,W_n,\nabla W_n)| \\ \leq\ & \frac{1}{k}\int_{[E_n>k]} k|\Psi_{1,n}| \leq \frac{1}{k}\int_{Q_T} E_n|\Psi_{1,n}| \leq \frac{1}{k}\int_{Q_T} \theta_n|\Psi_{1,n}| \leq \frac{\varepsilon}{m+2}\end{aligned}$$

Hypothesis (4.3) implies that for all $k > k_0$

$$\begin{aligned}
&\int_A \left|\Psi_{1,n}\left(t,x,W_n,\nabla W_n\right)\right| \\
\leq\ & \frac{\varepsilon}{m+2} + C_1(k)\left(\int_A F_1\left(t,x\right) + \int_{A\cap[E_n\leq k]} \left|\nabla w_{1,n}\right|^2\right) \\
& +C_1(k)\sum_{2\leq j\leq m}\left(\int_{A\cap[E_n\leq k]} \left|\nabla w_{j,n}\right|^{\alpha_j}\right) \\
\leq\ & \frac{\varepsilon}{m+2} + C_1(k)\left(\int_A F_1\left(t,x\right) + \int_{A\cap[E_n\leq k]} \left|\nabla T_k\left(w_{1,n}\right)\right|^2\right) \\
& \quad +C_1(k)\sum_{2\leq j\leq m}\left(\int_{A\cap[E_n\leq k]} \left|\nabla T_k\left(w_{j,n}\right)\right|^{\alpha_j}\right)
\end{aligned}$$

Using Hölder's inequality for $1 \leq \alpha_j < 2$ and (iii) (Lemma 5.5), we obtain

$$\begin{aligned}
C_1(k)\int_{A\cap[E_n\leq k]} \left|\nabla T_k\left(w_{j,n}\right)\right|^{\alpha_j} &\leq C_1(k)\left(\int_A \left|\nabla T_k\left(w_{j,n}\right)\right|^2\right)^{\frac{\alpha_j}{2}} |A|^{\frac{2-\alpha_j}{2}} \\
&\leq C_1(k) R_2^{\frac{\alpha_j}{2}} |A|^{\frac{2-\alpha_j}{2}} \leq \frac{\varepsilon}{m+2}
\end{aligned}$$

Whenever $|A| \leq \varrho_j$, with $\varrho_j = \left(\frac{\varepsilon}{m+2} C_1^{-1}(k) R_2^{-\frac{\alpha_j}{2}}\right)^{\frac{2}{2-\alpha_j}}$, $2 \leq j \leq m$ To deal with the second integral we write

$$\int_{A\cap[E_n\leq k]} \left|\nabla T_k\left(w_{1,n}\right)\right|^2 \leq 2\int_{A\cap[E_n\leq k]} \left|\nabla T_k\left(w_{1,n}\right) - \nabla T_k\left(w_1\right)\right|^2 + 2\int_A \left|\nabla T_k\left(w_1\right)\right|^2$$

According to (iii) (Lemma 5.5), $\left|\nabla T_k\left(w_{1,n}\right) - \nabla T_k\left(w_1\right)\right|^2 \chi_{[E_n\leq k]}$ is equi-integrable in $L^1\left(\Omega\right)$ since it converges strongly to 0 in $L^1\left(\Omega\right)$. So, there exists ϱ_{m+1} such that if $|A| \leq \varrho_{m+1}$, then

$$2C_1(k)\int_{A\cap[E_n\leq k]} \left|\nabla T_k\left(w_{1,n}\right) - \nabla T_k\left(w_1\right)\right|^2 \leq \frac{\varepsilon}{m+2}$$

On the other hand $F_1, \left|\nabla T_k\left(w_1\right)\right|^2 \in L^1\left(\Omega\right)$, therefore there exists ϱ_{m+2} such that

$$C_1(k)\left(2\int_A \left|\nabla T_k\left(w_1\right)\right|^2 + \int_A F_1\left(t,x\right)\right) \leq \frac{\varepsilon}{m+2}$$

whenever $|A| \leq \varrho_{m+2}$. Choose $\varrho_0 = \inf\left\{\varrho_j,\ 2 \leq j \leq m+2\right\}$, If $|A| \leq \varrho_0$ we obtain

$$\int_A \left|\Psi_{1,n}\left(x,W_n,\nabla W_n\right)\right| \leq \varepsilon.$$

Similarly, we get for all $2 \leq s \leq m$

$$\int_A \left|\Psi_{s,n}\right| \leq \frac{\varepsilon}{m+2}$$

$$+C_s(k)\left(\int_A F_s(t,x)+\int_{A\cap[E_n\le k]}\left(6\left|\nabla w_1\right|^2+6\left|\nabla T_k(w_{1,n})-\nabla T_k(w_1)\right|^2\right)\right)$$

$$+8C_s(k)\sum_{2\le r\le m}\left(\sum_{A\cap[E_n\le k]}\left|\nabla T_k\left(\sum_{1\le j\le r}w_j\right)\right|^2\right)$$

$$+8C_s(k)\sum_{2\le r\le m}\left(\sum_{A\cap[E_n\le k]}\left|\nabla T_k\left(\sum_{1\le j\le r}w_{j,n}\right)-\nabla T_k\left(\sum_{1\le j\le r}w_j\right)\right|^2\right)$$

Arguing in the same way as before, we obtain the required result. □

Then $(w_1,\ldots,w_m)$ verify (3.1) consequently $(w_1,\ldots,w_m)$ is the solution of (1.1).

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