Stud. Univ. Babeş-Bolyai Math. 69(2024), No. 2, 351–366
DOI: 10.24193/subbmath.2024.2.07

A $p(x)$-Kirchhoff type problem involving the $p(x)$-Laplacian-like operators with Dirichlet boundary condition

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Abstract. This paper deals with a class of $p(x)$-Kirchhoff type problems involving the $p(x)$-Laplacian-like operators, arising from the capillarity phenomena, depending on two real parameters with Dirichlet boundary conditions. Using a topological degree for a class of demicontinuous operators of generalized (S_+), we prove the existence of weak solutions of this problem. Our results extend and generalize several corresponding results from the existing literature.

Mathematics Subject Classification (2010): 35J60, 35J70, 35D30, 47H11.

Keywords: $p(x)$-Kirchhoff type problems, $p(x)$-Laplacian-like operators, weak solutions, variable exponent Sobolev spaces.

1. Introduction

The study of differential equations and variational problems with nonlinearities and nonstandard $p(x)$-growth conditions or nonstandard $(p(x), q(x))-$ growth conditions have received a lot of attention. Perhaps the impulse for this comes from the new search field that reflects a new type of physical phenomenon is a class of nonlinear problems with variable exponents (see [26]). The motivation for this research comes from the application of similar models in physics to represent the behavior of elasticity [34] and electrorheological fluids (see [30, 32]), which have the ability to modify their mechanical properties when exposed to an electric field (see [3, 4, 7, 11, 15, 27, 28, 29]), specifically the phenomenon of capillarity, which depends on solid-liquid interfacial characteristics as surface tension, contact angle, and solid surface geometry.

Received 23 February 2022; Accepted 18 March 2022.

Let Ω be a bounded domain in $\mathbb{R}^N (N > 1)$ with smooth boundary denoted by $\partial\Omega$, $a \in L^\infty(\Omega)$, $p(x), k(x) \in C_+(\overline{\Omega})$, and let μ and λ be two real parameters.

In this article, we consider a class of $p(x)$-Kirchhoff type problems involving the $p(x)$-Laplacian-like operators, originated from a capillarity phenomena, depending on two real parameters with Dirichlet boundary conditions of the following form:

$$\begin{cases} -\mathcal{M}\Big(\mathcal{C}(u)\Big)\Big(\Delta^{\mathcal{L}}_{p(x)}u - |u|^{p(x)-2}u\Big) + a(x)|u|^{k(x)-2}u \\ \qquad\qquad\qquad\qquad = \mu\, g(x,u) + \lambda\, f(x,u,\nabla u) \quad \text{in } \Omega, \\ u = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where

$$\mathcal{C}(u) := \int_\Omega \frac{|\nabla u|^{p(x)} + \sqrt{1+|\nabla u|^{2p(x)}} + |u|^{p(x)}}{p(x)}\, dx,$$

and

$$\Delta^{\mathcal{L}}_{p(x)}u := \operatorname{div}\Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)$$

is the $p(x)$-Laplacian-like operators, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions that satisfy the assumption of growth, and
$\mathcal{M} : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function.

Problems related to (1.1) have been studied by many scholars, for example, Ni and Serrin [20, 21] considered the following equation

$$-\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = f(u) \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

The operator $-\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big)$ is most often denoted by the specified mean curvature operator and $\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$ is the Kirchhoff stress term.

”Elliptic boundary value problems” involving the mean curvature operator play apivotal role in the mathematical analysis of several physical or geometrical issues, such as capillarity phenomena for incompressible or compressible fluids, mathematical models in physiology or in electrostatics, flux-limited diffusion phenomena, prescribed mean curvature problems for Cartesian surfaces in the Euclidean space: relevant references on these topics include [8, 9, 13, 14].

In the case when $\mathcal{M}\Big(\mathcal{C}(u)\Big) \equiv 1$, $\mu = a = 0$, $\lambda > 0$, f independent of ∇u and without the term $|u|^{p(x)-2}u$, we know that the problem (1.1) has a nontrivial solutions from [31].

For $\mathcal{M}\Big(\mathcal{C}(u)\Big) \equiv 1$, $k(x) = p(x)$, $\mu \geq 0$, $\lambda > 0$, $a \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_\Omega a > 0$ and f independent of ∇u, Afrouzi et al. [5] established some new sufficient conditions underwhich the problem (1.1), under Neumann boundary condition, possesses infinitely many weak solutions. Their discussion is based on a fully variational method and the main tool is a general critical point theorem.

Note that, in the case when

$$\mathcal{C}(u)=\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx,\ \Delta^{\mathcal{L}}_{p(x)}u=\Delta_{p(x)}u=\operatorname{div}\Big(|\nabla u|^{p(x)-2}\nabla u\Big),$$

$\mu=a=0$, $\lambda=1$, f independent of ∇u and without the term $|u|^{p(x)-2}u$, then we obtain the following problem

$$\begin{cases} -\mathcal{M}\Big(\displaystyle\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\Big)\Delta_{p(x)}u=f(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases} \tag{1.3}$$

which is called the $p(x)$-Kirchhoff type problem. In this case, Dai et al. [10], by a direct variational approach, established conditions ensuring the existence and multiplicity of solutions to (1.3). Furthermore, the problem (1.3) is a generalization of the stationary problem of a model introduced by Kirchhoff [17] of the following form:

$$\rho\frac{\partial^2 u}{\partial t^2}-\Big(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L\Big|\frac{\partial u}{\partial x}\Big|^2dx\Big)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.4}$$

where ρ,ρ_0,h,E,L are all constants, which extends the classical D'Alembert's wave equation, by considering the effect of the changing in the length of the string during the vibration.

Lapa et al. [19] showed, by using a Fredholm-type result for a couple of nonlinear operators, and the theory of variable exponent Sobolev spaces, the existence of weak solutions for the problem (1.1), under no-flux boundary conditions, in the case when $\mu=a=0$, $\lambda=1$ and f independent of ∇u.

In the present paper, we will generalize these works, by proving the existence of a weak solutions for the problem (1.1). Note that the problem (1.1) has not a variational structure, so the most usual variational methods can not used to study it. To attack it we will employ a topological degree for a class of demicontinuous operators of generalized (S_+) type of [6].

2. Preliminaries

In the analysis of problem (1.1), we will use the theory of the generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$. For convenience, we only recall some basic facts with will be used later, we refer to [12, 18, 22, 25, 23, 24] for more details.

Let Ω be a smooth bounded domain in $\mathbb{R}^N(N>1)$, with a Lipschitz boundary denoted by $\partial\Omega$. Set

$$C_+(\overline{\Omega})=\Big\{p: p\in C(\overline{\Omega}) \text{ such that } p(x)>1 \text{ for any } x\in\overline{\Omega}\Big\}.$$

For each $p\in C_+(\overline{\Omega})$, we define

$$p^+ := \max\Big\{p(x),\ x\in\overline{\Omega}\Big\} \ \text{ and } \ p^- := \min\Big\{p(x),\ x\in\overline{\Omega}\Big\}.$$

For every $p \in C_+(\overline{\Omega})$, we define

$$L^{p(x)}(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \ \text{ is measurable such that } \ \int_\Omega |u(x)|^{p(x)} dx < +\infty \Big\},$$

equipped with the Luxemburg norm

$$|u|_{p(x)} = \inf \Big\{ \lambda > 0 : \rho_{p(x)} \Big(\frac{u}{\lambda} \Big) \leq 1 \Big\},$$

where

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)} dx, \ \forall \ u \in L^{p(x)}(\Omega).$$

Proposition 2.1. [12] *Let (u_n) and $u \in L^{p(x)}(\Omega)$, then*

$$|u|_{p(x)} < 1 \Big(resp. = 1; > 1 \Big) \ \Leftrightarrow \ \rho_{p(x)}(u) < 1 \Big(resp. = 1; > 1 \Big), \tag{2.1}$$

$$|u|_{p(x)} > 1 \ \Rightarrow \ |u|^{p^-}_{p(x)} \leq \rho_{p(x)}(u) \leq |u|^{p^+}_{p(x)}, \tag{2.2}$$

$$|u|_{p(x)} < 1 \ \Rightarrow \ |u|^{p^+}_{p(x)} \leq \rho_{p(x)}(u) \leq |u|^{p^-}_{p(x)}, \tag{2.3}$$

$$\lim_{n\to\infty} |u_n - u|_{p(x)} = 0 \ \Leftrightarrow \ \lim_{n\to\infty} \rho_{p(x)} \Big(u_n - u \Big) = 0. \tag{2.4}$$

Remark 2.2. According to (2.2) and (2.3), we have

$$|u|_{p(x)} \leq \rho_{p(x)}(u) + 1, \tag{2.5}$$

$$\rho_{p(x)}(u) \leq |u|^{p^-}_{p(x)} + |u|^{p^+}_{p(x)}. \tag{2.6}$$

Proposition 2.3. [18] *The space $\Big(L^{p(x)}(\Omega), |\cdot|_{p(x)} \Big)$ is a separable and reflexive Banach spaces.*

Proposition 2.4. [18] *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$ where*

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$$

for all $x \in \Omega$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have the following Hölder-type inequality

$$\Big| \int_\Omega uv \ dx \Big| \leq \Big(\frac{1}{p-} + \frac{1}{p'^-} \Big) |u|_{p(x)} |v|_{p'(x)} \leq 2 |u|_{p(x)} |v|_{p'(x)}. \tag{2.7}$$

Remark 2.5. If p_1, $p_2 \in C_+(\overline{\Omega})$ with $p_1(x) \leq p_2(x)$ for any $x \in \overline{\Omega}$, then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.

Now, let $p \in C_+(\overline{\Omega})$ and we define $W^{1,p(x)}(\Omega)$ as

$$W^{1,p(x)}(\Omega) = \Big\{ u \in L^{p(x)}(\Omega) \text{ such that } |\nabla u| \in L^{p(x)}(\Omega) \Big\},$$

equipped with the norm

$$||u|| = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

We also define $W_0^{1,p(x)}(\Omega)$ as the subspace of $W^{1,p(x)}(\Omega)$, which is the closure of $C_0^\infty(\Omega)$ with respect to the norm $|| \cdot ||$.

Proposition 2.6. [12] *If the exponent* $p(x)$ *satisfies the log-Hölder continuity condition, i.e. there is a constant* $a > 0$ *such that for every* $x,\ y \in \Omega,\ x \neq y$ *with* $|x-y| \leq \frac{1}{2}$ *one has*

$$|p(x)-p(y)| \leq \frac{a}{-\log|x-y|}, \tag{2.8}$$

then we have the Poincaré inequality, i.e. there exists a constant $C > 0$ *depending only on* Ω *and the function* p *such that*

$$|u|_{p(x)} \leq C|\nabla u|_{p(x)},\ \forall\ u \in W_0^{1,p(x)}(\Omega). \tag{2.9}$$

In this paper we will use the following equivalent norm on $W_0^{1,p(x)}(\Omega)$

$$|u|_{1,p(x)} = |\nabla u|_{p(x)},$$

which is equivalent to $||\cdot||$.
Furthermore, we have the compact embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$(see [18]).

Proposition 2.7. [12, 18] *The spaces* $\left(W_0^{1,p(x)}(\Omega), |\cdot|_{1,p(x)}\right)$ *and* $\left(W_0^{1,p(x)}(\Omega), |\cdot|_{1,p(x)}\right)$ *are separable and reflexive Banach spaces.*

Remark 2.8. The dual space of $W_0^{1,p(x)}(\Omega)$ denoted $W^{-1,p'(x)}(\Omega)$, is equipped with the norm

$$|u|_{-1,p'(x)} = \inf\left\{|u_0|_{p'(x)} + \sum_{i=1}^{N}|u_i|_{p'(x)}\right\},$$

where the infinimum is taken on all possible decompositions $u = u_0 - \text{div}F$ with $u_0 \in L^{p'(x)}(\Omega)$ and $F = (u_1, \ldots, u_N) \in (L^{p'(x)}(\Omega))^N$.

3. A review on the topological degree theory

Now, we give some results and properties from the theory of topological degree. The readers can find more information about the history of this theory in [1, 2, 6, 16].

In what follows, let X be a real separable reflexive Banach space and X^* be its dual space with dual pairing $\langle \cdot, \cdot \rangle$ and given a nonempty subset Ω of X. Strong (weak) convergence is represented by the symbol $\to$ ($\rightharpoonup$).

Definition 3.1. Let Y be real Banach space. A operator $F : \Omega \subset X \to Y$ is said to be:

1. bounded, if it takes any bounded set into a bounded set.
2. demicontinuous, if for any sequence $(u_n) \subset \Omega,\ u_n \to u$ implies that $F(u_n) \rightharpoonup F(u)$.
3. compact, if it is continuous and the image of any bounded set is relatively compact.

Definition 3.2. A mapping $F : \Omega \subset X \to X^*$ is said to be:

1. of type (S_+), if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$ and $\limsup_{n\to\infty}\langle Fu_n, u_n - u\rangle \leq 0$, we have $u_n \to u$.

2. quasimonotone, if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, we have $\limsup_{n\to\infty} \langle Fu_n,\ u_n - u\rangle \geq 0$.

Definition 3.3. Let $T : \Omega_1 \subset X \to X^*$ be a bounded operator such that $\Omega \subset \Omega_1$. For any operator $F : \Omega \subset X \to X$, we say that

1. F of type $(S_+)_T$, if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$ and $\limsup_{n\to\infty} \langle Fu_n,\ y_n - y\rangle \leq 0$, we have $u_n \to u$.
2. F has the property $(QM)_T$, if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$, we have $\limsup_{n\to\infty} \langle Fu_n, y - y_n\rangle \geq 0$.

In the sequel, we consider the following classes of operators:

$$\mathcal{F}_1(\Omega) := \Big\{F : \Omega \to X^* : F \text{ is bounded, demicontinuous and of type } (S_+)\Big\},$$

$$\mathcal{F}_{T,B}(\Omega) := \Big\{F : \Omega \to X : F \text{ is bounded, demicontinuous and of type } (S_+)_T\Big\},$$

$$\mathcal{F}_T(\Omega) := \Big\{F : \Omega \to X : F \text{ is demicontinuous and of type } (S_+)_T\Big\},$$

for any $\Omega \subset D(F)$, where $D(F)$ denotes the domain of F, and any $T \in \mathcal{F}_1(\Omega)$. Now, let $\mathcal{O}$ be the collection of all bounded open sets in X and we define

$$\mathcal{F}(X) := \Big\{F \in \mathcal{F}_T(\overline{E}) :\ E \in \mathcal{O},\ \mathrm{T} \in \mathcal{F}_1(\overline{\mathrm{E}})\Big\},$$

where, $\mathrm{T} \in \mathcal{F}_1(\overline{\mathrm{E}})$ is called an essential inner map to F.

Lemma 3.4. [16, Lemma 2.3] *Let $T \in \mathcal{F}_1(\overline{E})$ be continuous and $S : D(S) \subset X^* \to X$ be demicontinuous such that $T(\overline{E}) \subset D(S)$, where E is a bounded open set in a real reflexive Banach space X. Then the following statements are true:*

1. *If S is quasimonotone, then $I + S \circ T \in \mathcal{F}_T(\overline{E})$, where I denotes the identity operator.*
2. *If S is of type (S_+), then $S \circ T \in \mathcal{F}_T(\overline{E})$.*

Definition 3.5. Suppose that E is bounded open subset of a real reflexive Banach space X, $T \in \mathcal{F}_1(\overline{E})$ is continuous and $F, S \in \mathcal{F}_T(\overline{E})$. The affine homotopy $\mathcal{H} : [0,1] \times \overline{E} \to X$ defined by

$$\mathcal{H}(t,u) := (1-t)Fu + tSu, \quad \text{for all} \quad (t,u) \in [0,1] \times \overline{E}$$

is called an admissible affine homotopy with the common continuous essential inner map T.

Remark 3.6. [16, Lemma 2.5] The above affine homotopy is of type $(S_+)_T$.

Next, as in [16] we give the topological degree for the type $\mathcal{F}(X)$.

Theorem 3.7. *Let*

$$M = \Big\{(F, E, h) : E \in \mathcal{O},\ T \in \mathcal{F}_1(\overline{E}),\ F \in \mathcal{F}_{T,B}(\overline{E}),\ h \notin F(\partial E)\Big\},$$

then, there exists a unique degree function $d : M \longrightarrow \mathbb{Z}$ that satisfies the following properties:

1. (Normalization) *For any* $h \in E$*, we have* $d(I, E, h) = 1$.
2. (Homotopy invariance) *If* $\mathcal{H} : [0,1] \times \overline{E} \to X$ *is a bounded admissible affine homotopy with a common continuous essential inner map and* $h : [0,1] \to X$ *is a continuous path in* X *such that* $h(t) \notin \mathcal{H}(t, \partial E)$ *for all* $t \in [0,1]$*, then*

$$d(\mathcal{H}(t,\cdot), E, h(t)) = const \text{ for all } t \in [0,1].$$

3. (Existence) *If* $d(F, E, h) \neq 0$*, then the equation* $Fu = h$ *has a solution in* E.

Definition 3.8. [16, Definition 3.3] The above degree is defined as follows:

$$d(F, E, h) := d_B(F|_{\overline{E}_0}, E_0, h),$$

where d_B is the Berkovits degree [6] *and* E_0 *is any open subset of* E *with* $F^{-1}(h) \subset E_0$ *and* F *is bounded on* $\overline{E}_0$.

4. Existence of weak solution

In this section, we will discuss the existence of weak solutions of (1.1).

We assume that $\Omega \subset \mathbb{R}^N (N > 1)$ is a bounded domain with a Lipschitz boundary $\partial\Omega$, $p \in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition (2.8), $a \in L^\infty(\Omega)$, $k \in C_+(\overline{\Omega})$ with $1 < k^- \leq k(x) \leq k^+ < p^-$, $\mathcal{M} : \mathbb{R}^+ \to \mathbb{R}^+$, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are functions such that:

(A_1). f is a Carathéodory function.

(A_2). There exists $\varrho > 0$ and $\gamma \in L^{p'(x)}(\Omega)$ such that

$$|f(x,\zeta,\xi)| \leq \varrho(\gamma(x) + |\zeta|^{q(x)-1} + |\xi|^{q(x)-1}).$$

(A_3). g is a Carathéodory function.

(A_4). There are $\sigma > 0$ and $\nu \in L^{p'(x)}(\Omega)$ such that

$$|g(x,\zeta)| \leq \sigma(\nu(x) + |\zeta|^{s(x)-1}),$$

for a.e. $x \in \Omega$ and all $(\zeta, \xi) \in \mathbb{R} \times \mathbb{R}^N$, where $q, s \in C_+(\overline{\Omega})$ with $1 < q^- \leq q(x) \leq q^+ < p^-$ and $1 < s^- \leq s(x) \leq s^+ < p^-$.

(M_0). $\mathcal{M} : [0, +\infty) \to (m_0, +\infty)$ is a continuous and increasing function with $m_0 > 0$.

Remark 4.1. • Note that, for all $u, v \in W_0^{1,p(x)}(\Omega)$

$$\mathcal{M}\Big(\mathcal{C}(u)\Big) \int_\Omega \Big(\Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla v + |u|^{p(x)-2}u\, v\Big)dx$$

is well defined (see [19]).

- $a(x)|u|^{k(x)-2}u$, $\mu\, g(x,u)$ and $\lambda\, f(x,u,\nabla u)$ are belongs to $L^{p'(x)}(\Omega)$ under $u \in W_0^{1,p(x)}(\Omega)$, the assumptions (A_2) and (A_4) and the given hypotheses about the exponents p, k, q and s because:

$$\gamma \in L^{p'(x)}(\Omega),\ \nu \in L^{p'(x)}(\Omega),\ r(x) = (q(x)-1)p'(x) \in C_+(\overline{\Omega})$$

with $r(x) < p(x)$, $\beta(x) = (k(x)-1)p'(x) \in C_+(\overline{\Omega})$ with $\beta(x) < p(x)$ and

$$\kappa(x) = (s(x)-1)p'(x) \in C_+(\overline{\Omega}) \text{ with } \kappa(x) < p(x).$$

Then, by Remark 2.5 we can conclude that

$$L^{p(x)} \hookrightarrow L^{r(x)},\ L^{p(x)} \hookrightarrow L^{\beta(x)} \text{ and } L^{p(x)} \hookrightarrow L^{\kappa(x)}.$$

Hence, since $v \in L^{p(x)}(\Omega)$, we have

$$\Big(-a(x)|u|^{k(x)-2}u + \mu\, g(x,u) + \lambda\, f(x,u,\nabla u)\Big)v \in L^1(\Omega).$$

This implies that, the integral

$$\int_\Omega \Big(-a(x)|u|^{k(x)-2}u + \mu\, g(x,u) + \lambda\, f(x,u,\nabla u)\Big)v dx$$

exists.

Then, we shall use the definition of weak solution for problem (1.1) in the following sense:

Definition 4.2. We say that a function $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of (1.1), if for any $v \in W_0^{1,p(x)}(\Omega)$, it satisfies the following:

$$\begin{aligned}\mathcal{M}\Big(\mathcal{C}(u)\Big)\int_\Omega \bigg(\Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla v + |u|^{p(x)-2}u\, v\bigg)dx \\ = \int_\Omega \Big(-a(x)|u|^{k(x)-2}u + \mu\, g(x,u) + \lambda\, f(x,u,\nabla u)\Big)v dx.\end{aligned}$$

Before giving our main result we first give two results that will be used later.

Lemma 4.3. *If (M_0) holds, then the operator $\mathcal{T} : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ defined by*

$$\langle \mathcal{T}u, v\rangle = \mathcal{M}\Big(\mathcal{C}(u)\Big)\int_\Omega \bigg(\Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla v + |u|^{p(x)-2}u\, v\bigg)dx,$$

is continuous, bounded, strictly monotone and is of type (S_+).

Proof. Let us consider the following functional:

$$\mathcal{J}(u) := \widehat{\mathcal{M}}\Big(\mathcal{C}(u)\Big), \text{ where } \widehat{\mathcal{M}}(s) = \int_0^s \mathcal{M}(\tau)\mathrm{d}\tau,$$

such that $\mathcal{M}(\tau)$ satisfies the assumption (M_0).
From [19], it is obvious that $\mathcal{J}$ is a continuously Gâteaux differentiable function whose Gâteaux derivative at the point $u \in W_0^{1,p(x)}(\Omega)$ is the functional $\mathcal{T}(u) := \mathcal{J}'(u) \in W^{-1,p'(x)}(\Omega)$ given by

$$\langle \mathcal{T}u, v\rangle = \mathcal{M}\Big(\mathcal{C}(u)\Big)\int_\Omega \bigg(\Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla v + |u|^{p(x)-2}u\, v\bigg)dx,$$

for all $u, v \in W_0^{1,p(x)}(\Omega)$ where $\langle\cdot,\cdot\rangle$ means the duality pairing between $W^{-1,p'(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$.

Hence, by using the similar argument as in the Theorem 3.1. of [19] and in the Proposition 3.1. of [31], we conclude that $\mathcal{T}$ is continuous, bounded, strictly monotone and is of type (S_+). $\square$

Proposition 4.4. *Assume that the assumptions* $(A_1)-(A_4)$ *hold, then the operator*

$$\mathcal{S}: W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$$
$$\langle \mathcal{S}u, v\rangle = -\int_\Omega \Big(-a(x)|u|^{k(x)-2}u + \mu\, g(x,u) + \lambda\, f(x,u,\nabla u)\Big) v dx,$$

for all $u, v \in W_0^{1,p(x)}(\Omega)$, *is compact.*

Proof. In order to prove this proposition, we proceed in four steps.
Step 1: Let $\Psi_1 : W_0^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ be an operator defined by

$$\Psi_1 u(x) := -\mu\, g(x,u).$$

In this step, we prove that the operator Ψ_1 is bounded and continuous.
First, let $u \in W_0^{1,p(x)}(\Omega)$, bearing (A_4) in mind and using (2.5) and (2.6), we infer

$$\begin{aligned}
|\Psi_1 u|_{p'(x)} &\le \rho_{p'(x)}(\Psi_1 u) + 1 \\
&= \int_\Omega |\mu\, g(x,u(x))|^{p'(x)} dx + 1 \\
&= \int_\Omega |\mu|^{p'(x)} |g(x,u(x)|^{p'(x)} dx + 1 \\
&\le \Big(|\mu|^{p'^-} + |\mu|^{p'^+}\Big) \int_\Omega |\sigma\Big(\nu(x) + |u|^{s(x)-1}\Big)|^{p'(x)} dx + 1 \\
&\le const\Big(|\mu|^{p'^-} + |\mu|^{p'^+}\Big) \int_\Omega \Big(|\nu(x)|^{p'(x)} + |u|^{\kappa(x)}\Big) dx + 1 \\
&\le const\Big(|\mu|^{p'^-} + |\mu|^{p'^+}\Big)\Big(\rho_{p'(x)}(\nu) + \rho_{\kappa(x)}(u)\Big) + 1 \\
&\le const\Big(|\nu|_{p(x)}^{p'^+} + |u|_{\kappa(x)}^{\kappa^+} + |u|_{\kappa(x)}^{\kappa^-}\Big) + 1.
\end{aligned}$$

Then, we deduce from (2.9) and $L^{p(x)} \hookrightarrow L^{\kappa(x)}$, that

$$|\Psi_1 u|_{p'(x)} \le const\Big(|\nu|_{p(x)}^{p'^+} + |u|_{1,p(x)}^{\kappa^+} + |u|_{1,p(x)}^{\kappa^-}\Big) + 1,$$

that means Ψ_1 is bounded on $W_0^{1,p(x)}(\Omega)$.
Second, we show that the operator Ψ_1 is continuous. To this purpose let $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$. We need to show that $\Psi_1 u_n \to \Psi_1 u$ in $L^{p'(x)}(\Omega)$. We will apply the Lebesgue's theorem.
Note that if $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$, then $u_n \to u$ in $L^{p(x)}(\Omega)$. Hence there exist a subsequence (u_k) of (u_n) and ϕ in $L^{p(x)}(\Omega)$ such that

$$u_k(x) \to u(x) \ \text{ and } \ |u_k(x)| \le \phi(x), \tag{4.1}$$

for a.e. $x \in \Omega$ and all $k \in \mathbb{N}$.
Hence, from (A_2) and (4.1), we have

$$|g(x,u_k(x))| \le \sigma(\nu(x) + |\phi(x)|^{s(x)-1}),$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.
On the other hand, thanks to (A_3) and (4.1), we get, as $k \longrightarrow \infty$

$$g(x, u_k(x)) \to g(x, u(x)) \text{ a.e. } x \in \Omega.$$

Seeing that

$$\nu + |\phi|^{s(x)-1} \in L^{p'(x)}(\Omega) \text{ and } \rho_{p'(x)}(\Psi_1 u_k - \Psi_1 u) = \int_\Omega |g(x, u_k(x)) - g(x, u(x))|^{p'(x)} dx,$$

then, from the Lebesgue's theorem and the equivalence (2.4), we have

$$\Psi_1 u_k \to \Psi_1 u \text{ in } L^{p'(x)}(\Omega),$$

and consequently

$$\Psi_1 u_n \to \Psi_1 u \text{ in } L^{p'(x)}(\Omega),$$

that is, Ψ_1 is continuous.
Step 2: We define the operator $\Psi_2 : W_0^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ by

$$\Psi_2 u(x) := a(x)|u(x)|^{k(x)-2} u(x).$$

We will prove that Ψ_2 is bounded and continuous.
It is clear that Ψ_2 is continuous. Next we show that Ψ_2 is bounded.
Let $u \in W_0^{1,p(x)}(\Omega)$ and using (2.5) and (2.6), we obtain

$$\begin{aligned}
|\Psi_2 u|_{p'(x)} &\leq \rho_{p'(x)}(\Psi_2 u) + 1 \\
&= \int_\Omega |a(x)|u|^{k(x)-2} u|^{p'(x)} dx + 1 \\
&= \int_\Omega |a(x)|^{p'(x)} |u|^{(k(x)-1)p'(x)} dx + 1 \\
&\leq ||a||^{p'}_{L^\infty(\Omega)} \int_\Omega |u|^{\beta(x)} dx + 1 \\
&= ||a||^{p'}_{L^\infty(\Omega)} \rho_{\beta(x)}(u) + 1 \\
&\leq ||a||^{p'}_{L^\infty(\Omega)} \Big(|u|^{\beta^-}_{\beta(x)} + |u|^{\beta^+}_{\beta(x)} \Big) + 1.
\end{aligned}$$

Hence, we deduce from $L^{p(x)} \hookrightarrow L^{\beta(x)}$ and (2.9) that

$$|\Psi_2 u|_{p'(x)} \leq const \Big(|u|^{\beta^-}_{1,p(x)} + |u|^{\beta^+}_{1,p(x)} \Big) + 1,$$

and consequently, Ψ_2 is bounded on $W_0^{1,p(x)}(\Omega)$.
Step 3: Let us define the operator $\Psi_3 : W_0^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ by

$$\Psi_3 u(x) := -\lambda\, f(x, u(x), \nabla u(x)).$$

We will show that Ψ_3 is bounded and continuous.
Let $u \in W_0^{1,p(x)}(\Omega)$. According to (A_2) and the inequalities (2.5) and (2.6), we obtain

$$\begin{aligned}
|\Psi_3 u|_{p'(x)} &\le \rho_{p'(x)}(\Psi_3 u) + 1 \\
&= \int_\Omega |\lambda f(x, u(x), \nabla u(x))|^{p'(x)} dx + 1 \\
&= \int_\Omega |\lambda|^{p'(x)} |f(x, u(x), \nabla u(x))|^{p'(x)} dx + 1 \\
&\le \Big(|\lambda|^{p'^-} + |\lambda|^{p'^+}\Big) \int_\Omega |\varrho\Big(\gamma(x) + |u|^{q(x)-1} + |\nabla u|^{q(x)-1}\Big)|^{p'(x)} dx + 1 \\
&\le const\Big(|\lambda|^{p'^-} + |\lambda|^{p'^+}\Big) \int_\Omega \Big(|\gamma(x)|^{p'(x)} + |u|^{r(x)} + |\nabla u|^{r(x)}\Big) dx + 1 \\
&\le const\Big(|\lambda|^{p'^-} + |\lambda|^{p'^+}\Big)\Big(\rho_{p'(x)}(\gamma) + \rho_{r(x)}(u) + \rho_{r(x)}(\nabla u)\Big) + 1 \\
&\le const\Big(|\gamma|_{p(x)}^{p'^+} + |u|_{r(x)}^{r^+} + |u|_{r(x)}^{r^-} + |\nabla u|_{r(x)}^{r^+} + |\nabla u|_{r(x)}^{r^-}\Big) + 1.
\end{aligned}$$

Taking into account that $L^{p(x)} \hookrightarrow L^{r(x)}$ and (2.9), we have then

$$|\Psi_3 u|_{p'(x)} \le const\Big(|\gamma|_{p(x)}^{p'^+} + |u|_{1,p(x)}^{r^+} + |u|_{1,p(x)}^{r^-}\Big) + 1,$$

and consequently Ψ_3 is bounded on $W_0^{1,p(x)}(\Omega)$.
It remains to show that Ψ_3 is continuous. Let $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$, we need to show that $\Psi_3 u_n \to \Psi_3 u$ in $L^{p'(x)}(\Omega)$. We will apply the Lebesgue's theorem.
Note that if $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$, then $u_n \to u$ in $L^{p(x)}(\Omega)$ and $\nabla u_n \to \nabla u$ in $(L^{p(x)}(\Omega))^N$. Hence, there exist a subsequence (u_k) and ϕ in $L^{p(x)}(\Omega)$ and ψ in $(L^{p(x)}(\Omega))^N$ such that

$$u_k(x) \to u(x) \ \text{ and } \ \nabla u_k(x) \to \nabla u(x), \tag{4.2}$$

$$|u_k(x)| \le \phi(x) \ \text{ and } \ |\nabla u_k(x)| \le |\psi(x)|, \tag{4.3}$$

for a.e. $x \in \Omega$ and all $k \in \mathbb{N}$.
Hence, thanks to (A_1) and (4.2), we get, as $k \longrightarrow \infty$

$$f(x, u_k(x), \nabla u_k(x)) \to f(x, u(x), \nabla u(x)) \ \text{ a.e. } \ x \in \Omega.$$

On the other hand, from (A_2) and (4.3), we can deduce the estimate

$$|f(x, u_k(x), \nabla u_k(x))| \le \varrho(\gamma(x) + |\phi(x)|^{q(x)-1} + |\psi(x)|^{q(x)-1}),$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.
Seeing that

$$\gamma + |\phi|^{q(x)-1} + |\psi(x)|^{q(x)-1} \in L^{p'(x)}(\Omega),$$

and taking into account the equality

$$\rho_{p'(x)}(\Psi_3 u_k - \Psi_3 u) = \int_\Omega |f(x, u_k(x), \nabla u_k(x)) - f(x, u(x), \nabla u(x))|^{p'(x)} dx,$$

then, we conclude from the Lebesgue's theorem and (2.4) that

$$\Psi_3 u_k \to \Psi_3 u \ \text{ in } \ L^{p'(x)}(\Omega),$$

and consequently

$$\Psi_3 u_n \to \Psi_3 u \ \text{ in } \ L^{p'(x)}(\Omega),$$

and then Ψ_3 is continuous.

Step 4: Let $I^* : L^{p'(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ be the adjoint operator of the operator $I : W_0^{1,p(x)}(\Omega) \to L^{p(x)}(\Omega)$.

We then define

$$I^* \circ \Psi_1 : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega),$$

$$I^* \circ \Psi_2 : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega),$$

and

$$I^* \circ \Psi_3 : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega).$$

On another side, taking into account that I is compact, then I^* is compact. Thus, the compositions $I^* \circ \Psi_1$, $I^* \circ \Psi_2$ and $I^* \circ \Psi_3$ are compact, that means

$$\mathcal{S} = I^* \circ \Psi_1 + I^* \circ \Psi_2 + I^* \circ \Psi_3$$

is compact. With this last step the proof of Proposition 4.4 is completed. □

We are now in the position to give the existence result of weak solution for (1.1).

Theorem 4.5. *Assume that $(A_1)-(A_4)$ and (M_0) hold, then the problem (1.1) admits at least one weak solution u in $W_0^{1,p(x)}(\Omega)$.*

Proof. We will reduce the problem (1.1) to a new one governed by a Hammerstein equation, and we will apply the theory of topological degree introduced in Section 3. For all $u, v \in W_0^{1,p(x)}(\Omega)$, we define the operators $\mathcal{T}$ and $\mathcal{S}$, as defined in Lemma 4.3 and Proposition 4.4 respectively,

$$\mathcal{T} : W_0^{1,p(x)}(\Omega) \longrightarrow W^{-1,p'(x)}(\Omega)$$
$$\langle \mathcal{T}u, v\rangle = \mathcal{M}\Big(\mathcal{C}(u)\Big)\int_\Omega \Big(\Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla v + |u|^{p(x)-2}u\, v\Big)dx,$$

and

$$\mathcal{S} : W_0^{1,p(x)}(\Omega) \longrightarrow W^{-1,p'(x)}(\Omega)$$
$$\langle \mathcal{S}u, v\rangle = -\int_\Omega \Big(-a(x)|u|^{k(x)-2}u + \mu\, g(x,u) + \lambda\, f(x,u,\nabla u)\Big)v dx.$$

Consequently, the problem (1.1) is equivalent to the equation

$$\mathcal{T}u + \mathcal{S}u = 0, \quad u \in W_0^{1,p(x)}(\Omega). \tag{4.4}$$

Taking into account that, by Lemma 4.3, the operator $\mathcal{T}$ is a continuous, bounded, strictly monotone and of type (S_+), then, by [33, Theorem 26 A], the inverse operator

$$\mathcal{L} := \mathcal{T}^{-1} : W^{-1,p'(x)}(\Omega) \to W_0^{1,p(x)}(\Omega),$$

is also bounded, continuous, strictly monotone and of type (S_+).

On another side, according to Proposition 4.4, we have that the operator $\mathcal{S}$ is bounded, continuous and quasimonotone.

Consequently, following Zeidler's terminology [33], the equation (4.4) is equivalent to the following abstract Hammerstein equation

$$u = \mathcal{L}v \quad \text{and} \quad v + \mathcal{S} \circ \mathcal{L}v = 0, \quad u \in W_0^{1,p(x)}(\Omega) \text{ and } v \in W^{-1,p'(x)}(\Omega). \tag{4.5}$$

Seeing that (4.4) is equivalent to (4.5), then to solve (4.4) it is thus enough to solve (4.5). In order to solve (4.5), we will apply the Berkovits topological degree introduced in Section 3.

First, let us set

$$\mathcal{B} := \Big\{ v \in W^{-1,p'(x)}(\Omega) : \exists\, t \in [0,1] \ \text{ such that } \ v + t\mathcal{S} \circ \mathcal{L}v = 0 \Big\}.$$

Next, we show that $\mathcal{B}$ is bounded in $\in W^{-1,p'(x)}(\Omega)$.

Let us put $u := \mathcal{L}v$ for all $v \in \mathcal{B}$.

Taking into account that $|\mathcal{L}v|_{1,p(x)} = |\nabla u|_{p(x)}$, then we have the following two cases:

First case: If $|\nabla u|_{p(x)} \leq 1$, then $|\mathcal{L}v|_{1,p(x)} \leq 1$, that means $\Big\{ \mathcal{L}v : v \in \mathcal{B} \Big\}$ is bounded.

Second case: If $|\nabla u|_{p(x)} > 1$, then, we deduce from (2.2), (A_2) and (A_4), the inequalities (2.7) and (2.6) and the Young's inequality that

$$\begin{aligned}
&|\mathcal{L}v|_{1,p(x)}^{p^-} \\
&\leq \rho_{p(x)}(\nabla u) \\
&\leq \langle \mathcal{T}u,\ u \rangle \\
&= \langle v,\ \mathcal{L}v \rangle \\
&= -t\langle \mathcal{S} \circ \mathcal{L}v,\ \mathcal{L}v \rangle \\
&= t \int_\Omega \Big(- a(x)|u|^{k(x)-2}u + \mu\, g(x,u) + \lambda\, f(x,u,\nabla u) \Big) u dx \\
&\leq t \max(||a||_{L^\infty(\Omega)}, \sigma|\mu|, \varrho|\lambda|) \Big(\rho_{k(x)}(u) + \int_\Omega |\nu(x)u(x)| dx + \int_\Omega |\gamma(x)u(x)| dx \\
&\qquad\qquad + \rho_{s(x)}(u) + \rho_{q(x)}(u) + \int_\Omega |\nabla u|^{q(x)-1}|u| dx \Big) \\
&\leq \mathit{const} \Big(|u|_{k(x)}^{k^-} + |u|_{k(x)}^{k^+} + |\nu|_{p'(x)}|u|_{p(x)} + |\gamma|_{p'(x)}|u|_{p(x)} + |u|_{s(x)}^{s^+} + |u|_{s(x)}^{s^-} \\
&\qquad\qquad + |u|_{q(x)}^{q^+} + |u|_{q(x)}^{q^-} + \frac{1}{q'-}\rho_{q(x)}(\nabla u) + \frac{1}{q-}\rho_{q(x)}(u) \Big) \\
&\leq \mathit{const} \Big(|u|_{k(x)}^{k^-} + |u|_{k(x)}^{k^+} + |u|_{p(x)} + |u|_{s(x)}^{s^+} + |u|_{s(x)}^{s^-} + |u|_{q(x)}^{q^+} + |u|_{q(x)}^{q^-} + |\nabla u|_{q(x)}^{q^+} \Big),
\end{aligned}$$

then, according to $L^{p(x)} \hookrightarrow L^{k(x)}$, $L^{p(x)} \hookrightarrow L^{s(x)}$ and $L^{p(x)} \hookrightarrow L^{q(x)}$, we get

$$|\mathcal{L}v|_{1,p(x)}^{p^-} \leq \mathit{const} \Big(|\mathcal{L}v|_{1,p(x)}^{k^+} + |\mathcal{L}v|_{1,p(x)} + |\mathcal{L}v|_{1,p(x)}^{s^+} + |\mathcal{L}v|_{1,p(x)}^{q^+} \Big),$$

what implies that $\Big\{ \mathcal{L}v : v \in \mathcal{B} \Big\}$ is bounded.

On the other hand, we have that the operator is $\mathcal{S}$ is bounded, then $\mathcal{S} \circ \mathcal{L}v$ is bounded. Thus, thanks to (4.5), we have that $\mathcal{B}$ is bounded in $W^{-1,p'(x)}(\Omega)$.

However, $\exists\ a > 0$ such that

$$|v|_{-1,p'(x)} < a \quad \text{for all} \quad v \in \mathcal{B},$$

which leads to

$$v + t\mathcal{S} \circ \mathcal{L}v \neq 0, \quad v \in \partial\mathcal{B}_a(0) \quad \text{and} \quad t \in [0,1],$$

where $\mathcal{B}_a(0)$ is the ball of center 0 and radius a in $W^{-1,p'(x)}(\Omega)$.
Moreover, by Lemma 3.4, we conclude that

$$I + \mathcal{S} \circ \mathcal{L} \in \mathcal{F}_{\mathcal{L}}(\overline{\mathcal{B}_a(0)}) \quad \text{and} \quad I = \mathcal{T} \circ \mathcal{L} \in \mathcal{F}_{\mathcal{L}}(\overline{\mathcal{B}_a(0)}).$$

On another side, taking into account that I, $\mathcal{S}$ and $\mathcal{L}$ are bounded, then $I + \mathcal{S} \circ \mathcal{L}$ is bounded. Hence, we infer that

$$I + \mathcal{S} \circ \mathcal{L} \in \mathcal{F}_{\mathcal{L},B}(\overline{\mathcal{B}_a(0)}) \quad \text{and} \quad I = \mathcal{T} \circ \mathcal{L} \in \mathcal{F}_{\mathcal{L},B}(\overline{\mathcal{B}_a(0)}).$$

Now, we define the homotopy $\mathcal{H} : [0,1] \times \overline{\mathcal{B}_a(0)} \to W^{-1,p'(x)}(\Omega)$ by

$$\mathcal{H}(t,\vartheta) := \vartheta + t\mathcal{S} \circ \mathcal{L}\vartheta.$$

Applying the homotopy invariance and normalization property of the degree d seen in Theorem 3.7, we have

$$d(I + \mathcal{S} \circ \mathcal{L}, \mathcal{B}_a(0), 0) = d(I, \mathcal{B}_a(0),\ 0) = 1 \neq 0.$$

Since $d\big(I + \mathcal{S} \circ \mathcal{L}, \mathcal{B}_a(0), 0\big) \neq 0$, then by the existence property of the degree d stated in Theorem 3.7, we conclude that there exists $\vartheta \in \mathcal{B}_a(0)$ which verifies

$$\big(I + \mathcal{S} \circ \mathcal{L}\big)(\vartheta) = 0 \Leftrightarrow \vartheta + \mathcal{S} \circ \mathcal{L}\vartheta = 0 \Leftrightarrow \mathcal{T} \circ \mathcal{L}\vartheta + \mathcal{S} \circ \mathcal{L}\vartheta = 0.$$

Hence, we conclude that $u = \mathcal{L}v$ is a weak solutions of (1.1). The proof is completed. □

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