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Generalized fractional integral operator in a complex domain

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Abstract. A new fractional integral operator is used to present a generalized class of analytic functions in a complex domain. The method of definition is based on a Hadamard product of analytic function, which is called convolution product. Then we formulate a convolution integral operator acting on the subclass of normalized analytic functions. Consequently, we investigate the suggested convolution operator geometrically. Differential subordination inequalities, taking the starlike formula are given. Some consequences of well known results are illustrated.

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1. Introduction

Many scholars, academic and researchers have applied fractional order integral operators (FOIOs) in real-world situations in a variety of scientific and technological sectors in recent years. It is well known, there are a number of definitions of FOIOs that can be utilized to solve fractional integral equations employing special functions (SFs). Fractional differentiation and integration using the extended Mittag-Leffler kernel were proposed in 2016 [1] and drew interest from a wide range of research sectors. Many features of these differential and integral operators have been noticed in real-world applications, such as crossover behavior (see [28]). These classes of specialized

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functions [8, 15] have newly become crucial in the fields of almost all applied sciences [20], natural science, engineering and computer science (see [2], [9], [24],[25]).

Integrals and the outputs of many different forms of differential equations are examples of special functions. As a consequence, record integrals involve explanations of SFs, which take account of the furthermost fundamental integrals; at the actual slightest, the integral representation of SFs. Because differential operators are important in mathematical sciences and applied mathematics, the theory of SFs is tightly linked to various physics topics [7].

In this note, we investigated the features of the k-Raina function under FOIOs and created several novel images. Via their extended character and utility in the theory of integral operators and a crucial part of computational mathematics, the conclusions produced here involve special classes of analytic functions such as the k-Mittag-Leffler function, S-function and K-function. Our methodology is based on the theory of differential subordination to present a set of differential inequalities type starlikeness in a complex domain.

2. Techniques

Here, we'll proceed over the methods we utilized.

2.1. Geometric approaches

The following concepts can be found in [16]

Definition 2.1. The set $\mathbb{O} := \{\chi \in \mathbb{C} : |\chi| < 1\}$, is the open unit disk in $z-$plane. The analytic functions Σ_1, Σ_2 in $\mathbb{O}$ are under the subordinated inequality $\Sigma_1 \prec \Sigma_2$ or

$$\Sigma_1(\chi) \prec \Sigma_2(\chi), \quad \chi \in \mathbb{O}$$

if for an analytic function $\varsigma, |\varsigma| \leq |\chi| < 1$ holds such that $\Sigma_1(\chi) = \Sigma_2(\varsigma(\chi)), \chi \in \mathbb{O}$.

Definition 2.2. The class of all regular functions given by

$$\sigma(\chi) = \chi + \sum_{n=2}^{\infty} a_n \, \chi^n, \quad \chi \in \mathbb{O}, \quad \sigma(0) = \sigma'(0) - 1 = 0,$$

is denoted by $\aleph$. Moreover, the analytic functions $\sigma_1, \sigma_2 \in \aleph$ are convoluted ($\sigma_1 * \sigma_2$) if they have the Hadamard product [22]

$$(\sigma_1 * \sigma_2)(\chi) = \left(\chi + \sum_{n=2}^{\infty} a_n \, \chi^n\right) * \left(\chi + \sum_{n=2}^{\infty} b_n \, \chi^n\right) = \chi + \sum_{n=2}^{\infty} a_n \, b_n \, \chi^n.$$

Definition 2.3. Define the following class of regular functions

$$\mathcal{P} := \{p : p(\chi) = 1 + \ell_1\chi + \ell_2\chi^2 + ..., \chi \in \mathbb{O}, \Re(p(\chi)) > 0, p(0) = 1\}.$$

Special sub-classes of $\mathcal{P}$ are the starlike subclass of functions $\sigma \in \aleph$ satisfying the functional

$$S_\sigma(\chi) = \frac{\chi\sigma'(\chi)}{\sigma(\chi)};$$

and the convex subclass of functions $\sigma \in \aleph$ having the functional

$$K_\sigma(\chi) = 1 + \frac{\chi\sigma''(\chi)}{\sigma'(\chi)}.$$

2.2. Raina's function

Let's start with the Raina's function (RAF), which is a familiar special feature.

Definition 2.4. In [21], the definition of RAF

$$\rho^{\mu}_{\alpha,\beta}(\chi) = \sum_{n=0}^{\infty} \frac{\mu(n)}{\Gamma(\alpha n + \beta)} \chi^n, \quad \chi \in \mathbb{O}.$$

$$\Big(\alpha, \quad \beta \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \quad \mu := \{\mu(0), \mu(1), ..., \mu(n)\}, \mu(j) \in \mathbb{C} \forall j = 0, ..., n\Big)$$

Remark 2.5.

- $n \geq 0, \mu(n) = 1 \Rightarrow \rho_{\alpha,\beta}(\chi) = \sum_{n=0}^{\infty} \frac{\chi^n}{\Gamma(\alpha n + \beta)}$, the Mittag-Leffler function.
- $\alpha = \beta = 1, \mu(n) = \frac{(v)_n (w)_n}{(u)_n} \Rightarrow {}_2G_1(v, w; u; \chi) = \sum_{n=0}^{\infty} \frac{(v)_n (w)_n}{(u)_n} \frac{\chi^n}{\Gamma(n+1)}$, the hypergeometric function.
- $\mu(n) = \frac{1}{n!} \frac{(w_1)_n ... (w_{k_1})_n}{(u_1)_n ... (u_{k_2})_n} \Rightarrow \mathbb{M}^{k_1,k_2}_{\alpha,\beta}(\chi) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n + \beta)} \frac{(w_1)_n ... (w_{k_1})_n}{(u_1)_n ... (u_{k_2})_n} \frac{\chi^n}{n!}$ the $\mathbb{M}-$series [27].
- $\mu(n) = \frac{(v)_n}{n!} \frac{(w_1)_n ... (w_{k_1})_n}{(u_1)_n ... (u_{k_2})_n} \Rightarrow \mathbb{K}^{k_1,k_2,v}_{\alpha,\beta}(\chi) = \sum_{n=0}^{\infty} \frac{(v)_n}{\Gamma(\alpha n + \beta)} \frac{(w_1)_n ... (w_{k_1})_n}{(u_1)_n ... (u_{k_2})_n} \frac{\chi^n}{n!}$ the $\mathbb{K}-$function [26].

2.3. Complex Raina's FOIOs

The Raina's FOIO is defined for analytic function $f(z), z \in \mathbb{C}$ in a complex domain containing the origin ($\mathbb{O}$) by the formula

$$I^{\mu,\tau}_{\alpha,\beta} f(\chi) = \int_0^{\chi} (\chi - z)^{\beta-1} \rho^{\mu}_{\alpha,\beta}[\tau(\chi - z)^{\alpha}] f(z) dz \tag{2.1}$$

$$\Big(\Re(\alpha) > 0, \quad \Re(\beta) > 0, \quad \chi, \ z \in \mathbb{C}, \quad \tau \in \mathbb{R}\Big).$$

Note that the integral $I^{\mu,\tau}_{\alpha,\beta} f(\chi)$ involves the well known Riemann-Liouville integral operator, when $\tau = 0$ and $\mu(0) = 1$

$$I_\beta f(\chi) = \frac{1}{\Gamma(\beta)} \int_0^{\chi} (\chi - z)^{\beta-1} f(z) dz \quad \Re(\beta) > 0$$

whenever the function $f(\chi)$ is analytic in simply-connected region of the complex z-plane.

In general, we have the integrals

$$\int_0^{\chi} (\chi - z)^{\beta-1}\rho^{\mu}_{\alpha,\beta}[\tau(\chi-z)^{\alpha}]dz = \chi^{\beta}\rho^{\mu}_{\alpha,\beta+1}[\tau(\chi-z)^{\alpha}]$$

$$\int_0^{\chi} (\chi - z)^{\beta}\rho^{\mu}_{\alpha,\beta+1}[\tau(\chi-z)^{\alpha}]dz = \chi^{\beta+1}\rho^{\mu}_{\alpha,\beta+2}[\tau(\chi-z)^{\alpha}]$$

$$\vdots$$

$$\begin{aligned}\int_0^{\chi} (\chi - z)^{\beta+m}\rho^{\mu}_{\alpha,\beta+1+m}[\tau(\chi-z)^{\alpha}]dz &= \chi^{\beta+m+1}\rho^{\mu}_{\alpha,\beta+m+2}[\tau(\chi-z)^{\alpha}]\\ &= \chi^{\beta+m+1}\sum_{n=0}^{\infty}\frac{\mu(n)}{\Gamma(\alpha n+\beta+m+2)}[\tau(\chi-z)^{\alpha}]^{n}.\end{aligned}$$

Moreover, we have the integral

$$\begin{aligned}I^{\mu,\tau}_{\alpha,\beta,m}(\chi) &:= \int_0^1 (\chi)^{\beta+m+\alpha-1}\rho^{\mu}_{\alpha,\beta+1+m}[\tau(\chi)^{\alpha}\chi^{1-\alpha}]d\chi\\ &= \chi^{\alpha+\beta+m}\sum_{n=0}^{\infty}\frac{\tau^{n}\mu(n)}{\Gamma(\alpha n+\beta+m+2)}\chi^{n}.\end{aligned}$$

To normalize the above integral, we define the functional integral formula, as follows:

$$\begin{aligned}\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}(\chi) &:= \left(\frac{\Gamma(\alpha+\beta+m+2)}{\tau\mu(1)}\right)\left(\frac{I^{\mu,\tau}_{\alpha,\beta,m}(\chi)}{\chi^{\alpha+\beta+m}} - \frac{\mu(0)}{\Gamma(\beta+m+2)}\right) \qquad (2.2)\\ &= \sum_{n=0}^{\infty}\left(\frac{\Gamma(\alpha+\beta+m+2)}{\tau\mu(1)}\right)\left(\frac{\tau^{n}\mu(n)}{\Gamma(\alpha n+\beta+m+2)}\right)\chi^{n}\\ &= \chi + \sum_{n=2}^{\infty}\left(\frac{\Gamma(\alpha+\beta+m+2)}{\tau\mu(1)}\right)\left(\frac{\tau^{n}\mu(n)}{\Gamma(\alpha n+\beta+m+2)}\right)\chi^{n},\end{aligned}$$

where $\tau \neq 0$, $\mu(1) \neq 0$, $m \in \mathbb{Z}$. It is clear that $\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}(\chi) \in \aleph$.

2.4. Convoluted fractional operator

We continue to define the convolution operator using the Hadamard product combining the suggested integral $\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}(\chi)$ with the function $\sigma \in \aleph$. The main integral convoluted operator in this effort is given, as follows:

$$\begin{aligned}&\left(\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m} * \sigma\right)(\chi) \qquad (2.3)\\ &= \left(\chi + \sum_{n=2}^{\infty}\left(\frac{\Gamma(\alpha+\beta+m+2)}{\tau\mu(1)}\right)\left(\frac{\tau^{n}\mu(n)}{\Gamma(\alpha n+\beta+m+2)}\right)\chi^{n}\right) * \left(\chi + \sum_{n=2}^{\infty}a_n\chi^{n}\right)\\ &= \chi + \sum_{n=2}^{\infty}\left(\frac{\Gamma(\alpha+\beta+m+2)\tau^{n-1}\mu(n)}{\mu(1)\Gamma(\alpha n+\beta+m+2)}\right)a_n\chi^{n}.\end{aligned}$$

Obviously, the convolution integral operator $\left(\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m} * \sigma\right) \in \aleph$.

Remark 2.6. By assuming the factor $\mu(n)$ for any coefficient formulas, we obtain all the convoluted operators, differential operators (like the Sàlàgean differential operator [23] and its generalizations [3]), fractional differential operators (Caputo and its generalizations, ABC-differential operator), integral operators (like the Sàlàgean integral operator [23]), fractional differential and integral operators [11], symmetric differential and integral operators [10], mixed fractional operators in the open unit disk [13], convoluted operator (such as the Carlson and Shaffer convoluted operator [5], Ruscheweyh convoluted operator [22], Noor operator [17] and Attiya operator [4]), special series (like $\mathbb{M}-$series, $\mathbb{S}-$series [29] and the Borel distribution [30]), mixed differential operators and all special functions in the litterateurs including the quantum calculus [12, 18].

Example 2.7.

- $\tau = 1, m = -2, \mu(n) = \dfrac{1}{n!}\dfrac{\Gamma(\gamma + n\kappa)}{\Gamma(\gamma + \kappa)}$, $\forall n \geq 1$, and $\gamma \in \mathbb{C}$ and $\Re(\kappa) > 0$ then we obtain the convoluted operator in [4]

$$\left(\mathbb{I}^{\mu,1}_{\alpha,\beta,-2} * \sigma\right)(\chi) = \chi + \sum_{n=2}^{\infty} \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha n+\beta)}\frac{\Gamma(\gamma+n\kappa)}{\Gamma(\gamma+\kappa)}\frac{1}{n!}\right) a_n \chi^n.$$

As special cases from the above series, when $\alpha = 0, \gamma = \kappa = 1$, we have $\left(\mathbb{I}^{\mu,1}_{0,\beta,-2} * \sigma\right)(\chi) = \sigma(\chi)$. And for $\alpha = 0$, $\gamma = 2$, $\kappa = 1$, we obtain the operator

$$\left(\mathbb{I}^{\mu,1}_{0,\beta,-2} * \sigma\right)(\chi) = \frac{1}{2}[\sigma(\chi) + \chi\sigma'(\chi)].$$

Moreover, when $\alpha = \gamma = \kappa = 1, \beta = 0, \sigma(\chi) = \dfrac{\chi}{1-\chi}$, we have

$$\left(\mathbb{I}^{\mu,1}_{1,0,-2} * \sigma\right)(\chi) = \chi\, e^{\chi}.$$

Finally, when $\alpha = \gamma = \kappa = 1, \beta = 1, \sigma(\chi) = \dfrac{\chi}{1-\chi}$, we get

$$\left(\mathbb{I}^{\mu,1}_{1,1,-2} * \sigma\right)(\chi) = e^{\chi} - 1.$$

- The Operator (2.3) satisfies the recurrent relation

$$\alpha\chi[\left(\mathbb{I}^{\mu,\tau}_{\alpha,\beta+1,m} * \sigma\right)(\chi)]' = (\alpha+\beta)\left(\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m} * \sigma\right)(\chi) - \beta\left(\mathbb{I}^{\mu,\tau}_{\alpha,\beta+1,m} * \sigma\right)(\chi).$$

Note that, when $\tau = 1, m = -2, \mu(n) = \dfrac{1}{n!}\dfrac{\Gamma(\gamma + n\kappa)}{\Gamma(\gamma + \kappa)}$, $\forall n \geq 1$, and $\gamma \in \mathbb{C}$ and $\Re(\kappa) > 0$ then we have [4]-Lemma 2.1. And under the same set of parameters, with $\sigma(\chi) = \dfrac{\chi}{1-\chi}$, we obtain the result in [28]-Theorem 2.1. Finally, if $\alpha = 1$, we have the equation (1.8) in [6].

In the following section, we illustrate our results concerning the generalized Raina FOIO of a complex variable.

3. Results

In this place, we discuss the sufficient conditions of the Ma-Minda starlike inequality [14]

$$S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi) = \frac{\chi\left(\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma\right)'(\chi)}{\left(\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma\right)(\chi)} \prec \Lambda(\chi).$$

For this purpose, we request the next result [16] (Corollary 3.4h.1 p.135).

Lemma 3.1. *Suppose that λ is analytic and Λ is univalent in $\mathbb{O}$ with $\lambda(0) = \Lambda(0)$, and an analytic function ℓ defined in a domain involving $\Lambda(\mathbb{O})$ and $\Lambda(\mathbb{O})$. If $\chi\Lambda'(\chi)\ell(\Lambda(\chi))$ is starlike, then the relation*

$$\chi\lambda'(\chi)\ell(\lambda(\xi)) \prec \chi\Lambda'(\chi)\ell(\Lambda(\chi))$$

yields $\lambda(\chi) \prec \Lambda(\chi)$ and Λ is the best dominant.

Theorem 3.2. *Take into consideration the following hypotheses:*

(i) $\sigma \in \aleph$, Λ *is univalent in* $\mathbb{O}$;

(ii) $\dfrac{\chi\Lambda'(\chi)}{\Lambda(\chi)(\Lambda(\chi)-1)}$ *is starlike in* $\mathbb{O}$;

(iii) $\dfrac{K_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)-1}{S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)-1} \prec 1 + \dfrac{\chi\Lambda'(\chi)}{\Lambda(\chi)(\Lambda(\chi)-1)}$ *occurs.*

Then

$$S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi) \prec \Lambda(\chi), \quad \chi \in \mathbb{O}$$

and Λ is the best dominant.

Proof. Denotes Ω, as follows:

$$\Omega(\chi) := S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi), \quad \chi \in \mathbb{O}.$$

Thus, a computation implies

$$S_{\Omega}(\chi) = K_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi) - \Omega(\chi).$$

Substituting implies that

$$\frac{K_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)-1}{T_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)-1} = \frac{S_{\Omega}(\chi)+\Omega(\chi)-1}{\Omega(\chi)-1}$$
$$= 1 + \frac{\chi\Omega'(\chi)}{\Omega(\chi)(\Omega(\chi)-1)}.$$

Consequently, we obtain

$$\frac{\chi\Omega'(\chi)}{\Omega(\chi)(\Omega(\chi)-1)} \prec \frac{\chi\Lambda'(\xi)}{\Lambda(\chi)(\Lambda(\chi)-1)}, \quad \chi \in \mathbb{O}.$$

In view of Lemma 3.1, we attain the result. □

Theorem 3.3. *Assume the following hypotheses*

(i) $\sigma \in \aleph, \Lambda$ *is univalent in* $\mathbb{O}$;

(ii) $\dfrac{\chi\Lambda'(\chi)}{\Lambda(\chi)-1}$ *is starlike in* $\mathbb{O}$;

(iii) $S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)\left(\dfrac{K_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\xi)-p}{S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)-1}-1\right)\prec\dfrac{\chi\Lambda'(\chi)}{\Lambda(\chi)-1}$ *holds.*

Then

$$S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)\prec\Lambda(\chi),\quad \chi\in\mathbb{O}$$

and Λ *is the best dominant.*

Proof. Consider the function Ω, as follows:

$$\Omega(\chi):=S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi),\quad \chi\in\mathbb{O}.$$

Accordingly, we have

$$S_\Omega(\chi)+\Omega(\chi)=K_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi).$$

A calculation yields

$$S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)\left(\frac{K_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)-1}{S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)-1}-1\right)=\frac{\chi\Omega'(\chi)}{\Omega(\chi)-1}.$$

Which leads to

$$\frac{\chi\Omega'(\chi)}{\Omega(\chi)-1}\prec\frac{\chi\Lambda'(\chi)}{\Lambda(\chi)-1},\quad \chi\in\mathbb{O}.$$

In virtue of Lemma 3.1, we get the desired outcome. □

Theorem 3.4. *Consider the following assumptions*

(i) $\sigma\in\aleph,\Lambda$ *is univalent in* $\mathbb{O}$;

(ii) S_Λ *IS starlike in* $\mathbb{O}$;

(iii) $K_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)-S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)\prec S_\Lambda(\chi)$ *satisfies.*

Then

$$S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)\prec\Lambda(\chi),\quad \chi\in\mathbb{O}$$

and Λ *is the best dominant.*

Proof. Let Ω as follows:

$$\Omega(\chi):=S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\xi),\quad \chi\in\mathbb{O}.$$

Thus, we get

$$S_\Omega(\chi)+\Omega(\chi)=K_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi).$$

Consequently, we have

$$K_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)-S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)=S_\Omega(\chi).$$

Hence,

$$S_\Omega(\chi)\prec S_\Lambda(\chi),\quad \chi\in\mathbb{O}.$$

Finally, Lemma 3.1 yields the outcome $\Omega(\chi)\prec\Lambda(\chi)$. □

Theorem 3.5. *Use these assumptions:*

(i) $\sigma\in\aleph$, Λ *is univalent in* $\mathbb{O}$;

(ii) $\chi\Lambda'(\chi)$ *is starlike in* $\mathbb{O}$;

(iii) $S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)\left(K_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)-S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)\right)\prec\chi\Lambda'(\chi)$ *occurs.*

Then

$$S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)\prec\Lambda(\chi),\quad \chi\in\mathbb{O}$$

and Λ *is the best dominant.*

Proof. Formulate the function Ω by:

$$\Omega(\chi):=S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi),\quad \chi\in\mathbb{O}.$$

Thus, we have

$$S_{\Omega}(\chi)+\Omega(\chi)=K_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi).$$

Substituting attains

$$S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)\left(K_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)-S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)\right)=\chi\Omega'(\chi).$$

Hence,

$$\chi\Omega'(\chi)\prec\chi\Lambda'(\chi),\quad \chi\in\mathbb{O}.$$

By Lemma 3.1, we obtain $\Omega(\chi)\prec\Lambda(\chi)$. □

Theorem 3.6. *Suppose that* Λ *is convex univalent in* $\mathbb{O}$ *satisfying the inequality*

$$S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)\prec\Lambda(\chi),\quad \chi\in\mathbb{O},$$

where $\Lambda(0)=1$. *Then*

$$[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]\prec\chi\exp\left(\int_0^{\chi}\frac{\Lambda(w(\xi))}{\xi}d\xi\right),$$

where w *has the properties* $w(0)=0$ *and* $|w(\chi)|<1$. *In addition, the inequality* $|\chi|:=\rho<1$ *yields*

$$\exp\left(\int_0^1\frac{\Lambda(-\rho)}{\rho}d\rho\right)\le\left|\frac{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma](\chi)}{\chi}\right|\le\exp\left(\int_0^1\frac{\Lambda(\rho)}{\rho}d\rho\right).$$

Proof. A computation implies

$$\frac{\left([\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma](\chi)\right)'}{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma](\chi)}-\frac{1}{\chi}=\frac{\Lambda(w(\chi))-1}{\chi}.$$

Integration yields

$$[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma](\chi)\prec\chi\exp\left(\int_0^{\chi}\frac{\Lambda(w(\xi))}{\xi}d\xi\right),$$

which leads to

$$\frac{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma](\chi)}{\chi}\prec\exp\left(\int_0^{\chi}\frac{\Lambda(w(\xi))}{\xi}d\xi\right).$$

But,

$$\Lambda(-\rho|\chi|)\le\Re\left(\Lambda(w(\chi\rho))\right)\le\Lambda(\rho|\chi|)$$

then, we obtain

$$\int_0^1\frac{\Lambda(-\rho|\chi|)}{\rho}d\rho\le\int_0^1\frac{\Re\left(\Lambda(w(\chi\rho))\right)}{\rho}d\rho\le\int_0^1\frac{\Lambda(\rho|\chi|)}{\rho}d\rho.$$

A combination of the last two relations, we attain

$$\int_0^1 \frac{\Lambda(-\rho|\chi|)}{\rho} d\rho \le \log\left|\frac{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m} * \sigma](\chi)}{\chi}\right| \le \int_0^1 \frac{\Lambda(\rho|\chi|)}{\rho} d\rho.$$

Which imposes

$$\exp\left(\int_0^1 \frac{\Lambda(-\rho)}{\rho} d\rho\right) \le \left|\frac{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m} * \sigma](\chi)}{\chi}\right| \le \exp\left(\int_0^1 \frac{\Lambda(\rho)}{\rho} d\rho\right). \qquad \square$$

Corollary 3.7. *In Theorem 3.6, let*

$$\mu(n) = \frac{1}{n!}\frac{\Gamma(\gamma + n\kappa)}{\Gamma(\gamma+\kappa)},$$

with $\alpha = 0, \gamma = \kappa = 1, m = -2 \Rightarrow \left(\mathbb{I}^{\mu,1}_{0,\beta,-2} * \sigma\right)(\chi) = \sigma(\chi)$.
Consider that Λ *is convex univalent in* $\mathbb{O}$ *with*

$$S_\sigma(\chi) \prec \Lambda(\chi), \quad \xi \in \mathbb{O},$$

where $\Lambda(0) = 1$ *then*

$$\sigma(\chi) \prec \chi \exp\left(\int_0^\chi \frac{\Lambda(w(\xi))}{\xi} d\xi\right),$$

where w *as in Theorem 3.6. In addition, the relation* $|\chi| := \rho < 1$ *gives*

$$\exp\left(\int_0^1 \frac{\Lambda(-\rho)}{\rho} d\rho\right) \le \left|\frac{\sigma(\chi)}{\chi}\right| \le \exp\left(\int_0^1 \frac{\Lambda(\rho)}{\rho} d\rho\right).$$

Lastly, we present a special result when $\Lambda(\chi) := \dfrac{1+\phi\chi}{1+\psi\chi}$, where $-1 \le \psi < \phi \le 1$.

Theorem 3.8. *Consider the generalized FOIO* (2.3).

(i) *If the following subordination holds:*

$$\left(S_{S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m} * \sigma]}(\chi)}(\chi) - 1\right)\left[S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m} * \sigma](\chi)}\right]^{-1} + 1 \prec \frac{(\phi-\psi)\chi(2+\phi\chi)}{(1+\phi\chi)^2},$$

then

$$S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m} * \sigma](\chi)} \prec \frac{1+\phi\chi}{1+\psi\chi}.$$

$$\Big(-1 \le \psi < \phi \le 0, \quad \chi \in \mathbb{O}\Big)$$

Moreover,

$$\frac{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m} * \sigma](\chi)}{\chi} \prec (1+\psi\chi)^{\frac{\phi-\psi}{\psi}}, \quad \psi \ne 0.$$

(ii) *If the next inequality occurs*

$$S_{S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m} * \sigma]}(\chi)}(\chi) + 1 \prec \frac{1+\phi\chi}{1+\psi\chi}$$

then

$$S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m} * \sigma](\chi)} \prec (1+\psi\chi)^{\frac{\phi-\psi}{\psi}}.$$

$$\Big(|\frac{\phi-\psi}{\psi}\pm 1|\le 1, \quad \chi\in\mathbb{O}\Big)$$

(iii) *If the following relation exists*

$$S_{S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)}(\chi)+1\prec 1+\phi\chi$$

then

$$S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma](\chi)}\prec e^{\phi\chi}.$$

$$\Big(\psi=0, \quad |\phi|<\pi, \quad \chi\in\mathbb{O}\Big)$$

In addition,

$$\frac{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma](\chi)}{\chi}\prec e^{\phi\chi}.$$

All the above results are the best dominant.

Proof. Clearly, based on the definition of the functional S, we have

$$S_{S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)}(\chi)=\frac{\chi\left(S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma](\chi)}\right)'}{S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma](\chi)}},$$

where $S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(0)=1$. Now let

$$\Omega(\chi):=S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi).$$

Then a calculation implies

$$\begin{aligned}1-\frac{1}{\Omega(\chi)}+\frac{\chi\Omega'(\chi)}{\Omega^2(\chi)}&=1-\frac{1}{S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)}+\frac{\chi[S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)]'}{[S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)]^2}\\&=\left(S_{S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)}(\chi)-1\right)[S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma](\chi)}]^{-1}+1\\&\prec\frac{(\phi-\psi)\chi(2+\phi\chi)}{(1+\phi\chi)^2}.\end{aligned}$$

Then in view of [19]-Lemma 3, we have the outcome in (i). Since we have

$$S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma](\chi)}\prec\frac{1+\phi\chi}{1+\psi\chi}.$$

$$\Big(-1\le\psi<\phi\le 0, \quad \chi\in\mathbb{O}\Big)$$

Then the second inequality comes from [19]-Theorem 2

$$\frac{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma](\chi)}{\chi}\prec(1+\psi\chi)^{\frac{\phi-\psi}{\psi}}, \quad \psi\ne 0.$$

We aim to prove (ii). Since

$$\begin{aligned} 1+\frac{\chi\Omega'(\chi)}{\Omega(\chi)} &= 1+\frac{\chi[S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)]'}{S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)} \\ &= S_{S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)}(\chi)+1 \\ &\prec \frac{1+\phi\chi}{1+\psi\chi}. \end{aligned}$$

Then in view of [19]-Lemma 4(i), we obtain the result in (ii).
A computation yields

$$\begin{aligned} 1+\frac{\chi\Omega'(\chi)}{\Omega(\chi)} &= 1+\frac{\chi[S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)]'}{S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)} \\ &= S_{S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)}(\chi)+1 \\ &\prec 1+\phi\chi. \end{aligned}$$

Then in view of [19]-Lemma 4(ii), we get the result in (iii). Since

$$S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma](\chi)} \prec e^{\phi\chi},$$

$$\Big(\psi=0, \quad |\phi|<\pi, \quad \chi\in\mathbb{O}\Big)$$

we attain the second part by using [19]-Theorem 2

$$\frac{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma](\chi)}{\chi} \prec e^{\phi\chi}.$$

□

Corollary 3.9. [19] *In Theorem 3.8, let*

$$\mu(n)=\frac{1}{n!}\frac{\Gamma(\gamma+n\kappa)}{\Gamma(\gamma+\kappa)},$$

with $\alpha=0,\gamma=\kappa=1,m=-2\Rightarrow\left(\mathbb{I}^{\mu,1}_{0,\beta,-2}*\sigma\right)(\chi)=\sigma(\chi).$

(i) *If the following subordination holds:*

$$\left(S_{S_{\sigma(\chi)}}(\chi)-1\right)[S_{\sigma(\chi)}]^{-1}+1\prec\frac{(\phi-\psi)\chi(2+\phi\chi)}{(1+\phi\chi)^2},$$

then

$$S_{\sigma(\chi)}\prec\frac{1+\phi\chi}{1+\psi\chi}.$$

$$\Big(-1\le\psi<\phi\le 0, \quad \chi\in\mathbb{O}\Big)$$

Moreover,

$$\frac{\sigma(\chi)}{\chi}\prec(1+\psi\chi)^{\frac{\phi-\psi}{\psi}}, \quad \psi\neq 0.$$

(ii) *If the next inequality occurs*

$$S_{S_{\sigma(\chi)}}(\chi)+1 \prec \frac{1+\phi\chi}{1+\psi\chi}$$

then

$$S_{\sigma(\chi)} \prec (1+\psi\chi)^{\frac{\phi-\psi}{\psi}}.$$

$$\Big(|\frac{\phi-\psi}{\psi}\pm 1|\leq 1, \quad \chi\in\mathbb{O}\Big)$$

(iii) *If the following relation exists*

$$S_{S_{\sigma(\chi)}}(\chi)+1 \prec 1+\phi\chi$$

then

$$S_{\sigma(\chi)} \prec e^{\phi\chi}.$$

$$\Big(\psi=0, \quad |\phi|<\pi, \quad \chi\in\mathbb{O}\Big)$$

In addition,

$$\frac{\sigma(\chi)}{\chi} \prec e^{\phi\chi}.$$

All the above results are the best dominant.

Example 3.10. Under the assumptions of Corollary 3.9, we have the following examples (see Fig. 1):

- For $\phi = 1-2a, a\in[0,1), \psi=-1$, we get

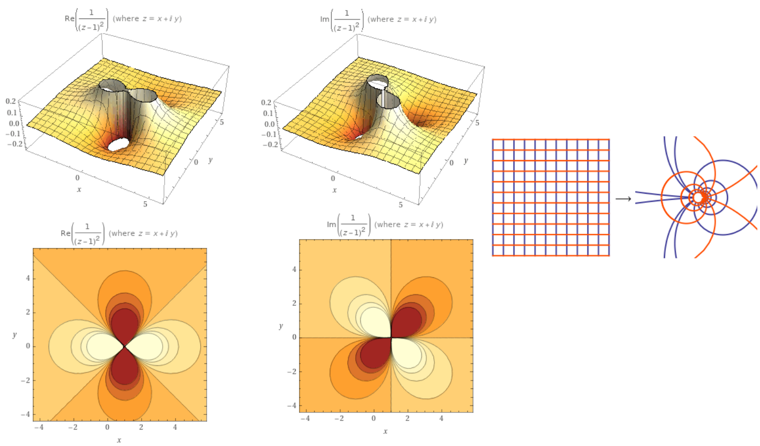
$$\frac{\sigma(\chi)}{\chi} \prec \frac{1}{(1-\chi)^{2(1-a)}}.$$

- For $a=0$, we obtain

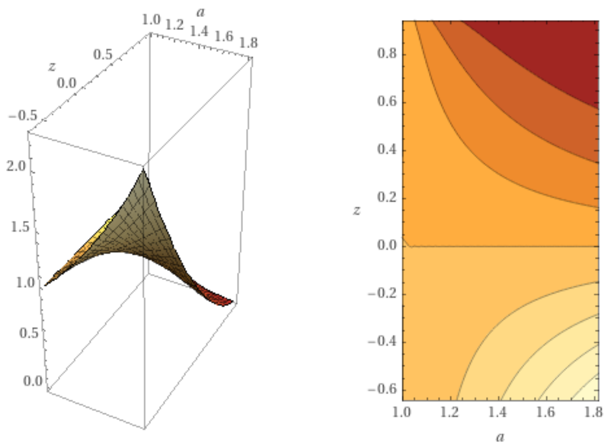
$$\frac{\sigma(\chi)}{\chi} \prec \frac{1}{(1-\chi)^{2}}.$$

4. Conclusion

The Generalized fractional integral operator is formulated using the Raina's function. The suggested FOIO is a generalization of many operators and series. We formulated the FOIO in classes of starlike functions and explored the sufficient conditions for these classes. Many recent results are conformed as special cases. We suggest to include it in different other classes of analytic functions, for the next step of research.



(A) Plotting of $\frac{1}{(1-\chi)^2}, a = 0$



(B) Plotting of $\frac{1}{(1-\chi)^{2(1-a)}}, a \neq 0$

FIGURE 1. Functions in Example 3.10

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