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A polynomial algorithm for some instances of NP-complete problems

Marius Costandin and Bogdan Gavrea

Abstract. In this paper, given a fixed reference point and a fixed intersection of finitely many equal radii balls, we consider the problem of finding a point in the said set which is the most distant, under Euclidean distance, to the said reference point. This proble is NP-complete in the general setting. We give sufficient conditions for the existence of an algorithm of polynomial complexity which can solve the problem, in a particular setting. Our algorithm requires that any point in the said intersection to be no closer to the given reference point than the radius of the intersecting balls. Checking this requirement is a convex optimization problem hence one can decide if running the proposed algorithm enjoys the presented theoretical guarantees. We also consider the problem where a fixed initial reference point and a fixed polytope are given and we want to find the farthest point in the polytope to the given reference point. For this problem we give sufficient conditions in which the solution can be found by solving a linear program. Both these problems are known to be NP-complete in the general setup, i.e the existence of an algorithm which solves any of the above problem without restrictions on the given reference point and search set is undecided so far.

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1. Introduction

In this paper we begin by presenting a novel framework for asserting the feasibility of the intersection of convex sets. Our approach is to synthesize the information

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in the given convex sets in a non-smooth convex function whose unconstrained minimizer can be used to assert the feasibility of the intersection. This problem is known in the literature as the so called "convex feasibility problem". Classic algorithms for this problem exist and can be found in [16], [18], [17], [19], while more novel approaches are found here [5], [13]. Our approach to this problem is the presentation of a simple and elegant criterion for asserting the feasibility of the intersection of two convex sets. Unlike (some of) the references above, we do not focus on the convex minimization problem itself, but on the formation of the convex function to be minimized and on the interpretation of the resulting minimizer.

Next we extend the presented method to a particular case of mathematical programming: the assertion of the inclusion of an intersection of equal radii balls in another, bigger, ball. We are able to give meaningful results under some requirements regarding the distance between the center of the bigger ball and the the intersection of the balls.

We will use throughout the paper the symbol $d(\cdot, \times)$ where $\cdot$ can be a point and $\times$ can be a point or a convex set of points, to designate the Euclidean distance between $\cdot$ and $\times$. For a vector $u \in \mathbb{R}^n$, $u = (u_1, ..., u_n)^T$ and $r > 0$, we denote by $\mathcal{B}(u, r)$ the open ball centered at u and of radius r and we denote by

$$\bar{\mathcal{B}}(u, r) = \{x \in \mathbb{R}^n | \|x - u\| \le r\}$$

the closed ball centered in u and of radius r. We also denote by $\|u\|$, $\|u\|^2 = u^T u$, the Euclidean norm of the vector u.

Finally, for a function $f : \mathbb{R}^n \to \mathbb{R}$ we denote by

$$f^+(x) = \max\{f(x), 0\} \qquad f^-(x) = \min\{f(x), 0\} \tag{1.1}$$

Note that $f(x) = f^+(x) + f^-(x)$.

1.1. Convex domains of interest

Let $x \in \mathbb{R}^n$, $n, m \in \mathbb{N}_+$ and let $g_k : \mathbb{R}^n \to \mathbb{R}$ be convex functions for $k \in \{1, \dots, m\}$. We define the convex sets:

$$S_k = \left\{ x \in \mathbb{R}^n \middle| g_k(x) \le 0 \right\}$$

and we are interested if the set

$$\mathcal{S} = \bigcap_{k=1}^{m} S_k \tag{1.2}$$

is empty or not.

For this we define the following function $\tilde{G}(x) : \mathbb{R}^n \to \mathbb{R}$:

$$\widetilde{G}(x) = \sum_{k=1}^{m} g_k^+(x)$$

2. Main results

In this section we present a novel feasibility criteria for the finite intersection of certain convex sets. One classic method from the literature for solving this problem is the method of alternating projections, [4], [13], for finding a feasible solution in the intersection of convex sets. Below, we give a projection-free method for solving set intersection problems. Our approach reformulates the feasibility problem as a non-smooth convex minimization problem.

2.1. Convex feasibility

The following result is a characterization of the set $\mathcal{S}$ in terms of a global minimum of $\widetilde{G}(x)$.

Lemma 2.1. *Let*

$$x^\star \in \underset{x\in\mathbb{R}^n}{\operatorname{argmin}}\, \widetilde{G}(x). \tag{2.1}$$

Then the following are equivalent:

1. *The set $\mathcal{S}$ is not empty, i.e $\exists\, x_0 \in \mathbb{R}^n$ such that*

$$g_k(x_0) \leq 0 \qquad \forall k \in \{1,\dots,m\}$$

2. *The point $x^\star$ defined by (2.1) satisfies*

$$g_k(x^\star) \leq 0 \qquad \forall k \in \{1,\dots,m\}$$

Proof. The part $2 \Rightarrow 1$ follows immediately from $g_k(x^\star) \leq 0$ for all $k \in \{1,\dots,m\}$ which implies $x^\star \in \mathcal{S}$ and therefore $\mathcal{S} \neq \emptyset$. To prove $1 \Rightarrow 2$, let x_0 such that $g_k(x_0) \leq 0$ for all $k \in \{1,\dots,m\}$ and assume that $\exists k$ such that $g_k(x^\star) > 0$. This implies

$$0 = \widetilde{G}(x_0) < \widetilde{G}(x^\star)$$

which contradicts the fact that $x^\star$ is a global minimum of $\widetilde{G}$. □

Remark 2.2. The simple result above shows that the feasibility of the intersection of m convex sets (sub-level sets of convex functions) can be asserted by examining the global minimum of a non-smooth convex function.

Encouraged by the simplicity of the above result we propose a somewhat similar approach to study the following problem: assert if a fixed intersection of finitely many equal radii balls is included in another given ball.

2.2. Test for the inclusion of an intersection of balls into another ball

We want to solve the following non-convex optimization problem:

$$\begin{array}{ll} \max & \|x-c\|^2 \\ \text{s.t} & \|x-c_k\|^2 \leq R^2, \quad \forall k \in \{1,\dots,m\}, \end{array} \tag{2.2}$$

where $c_k, c \in \mathbb{R}^n$ and $R \in \mathbb{R}$, $R > 0$. Problem (2.2) is equivalent to finding a point in the intersection of the balls centered at c_k and of radius R which is the farthest away from the point c. Please note that for any polytope one can choose c_k and R in such a way that the intersection of the balls provide an approximation of the polytope.

Although we will not expand this approximation here, this is the main reason for considering problem (2.2).

It is obviously a quadratically constrained quadratic maximization problem. Algorithms for such, or similar problem, have been proposed in the literature, see [7], [12], [15], [1]. These treat a similar problem, i.e optimizing a quadratic function with box constraints. The *S-procedure*, [10], is a well known algorithm for solving programs with quadratic objective and quadratic constraints. However, the presented problem is fundamentally different to the problems which the S-procedure can solve in polynomial time. That is, we are interested if an intersection of more balls is included in another ball, whereas the S-procedure can be used for testing ellipsoid containment, i.e to assert if an ellipsoid is included in another. The S-procedure cannot be used to assert if an intersection of ellipsoids is included in another ellipsoid. Also, the presented problem is fundamentally different to the sphere/ellipsoid packing problem, as we are not interested in finding the maximum number of non-overlapping spheres/ellipsoids which can be included in a given sphere/ellipsoid. In our case all the geometrical objects (the balls) are fixed and given. We are just supposed to answer with YES or NO to the question: "is the intersection of the these given balls included in this other ball?". Here is is worth noting the work done in [6] which finds the smallest ball enclosing an intersection of balls. This problem is somewhat similar to ours as one would, in absence of other choices, propose an "approximate" solution to our problem by simply computing the smallest ball enclosing the intersection of balls, then asserting if that is or not included in the bigger ball. Unfortunately, in [6] the number of intersecting balls is required to be strictly smaller than the dimension of the search space. Finally the work presented here [3] treats a slightly more general problem to what we will be discussing in the next section, i.e maximizing a quadratic function over an intersection of half spaces. However, we limit ourselves to analyzing the simpler to understand problem of maximizing the distance to an external point over an intersection of half spaces. The authors of [3] approach is to cover the search space with ellipsoids then to maximize over each to finally obtain an approximation to the initial problem. Unfortunately, covering the search space (or at least its frontier) with small enough ellipsoids (as required by the precision requirements) requires an exponential number of ellipsoids [2], so this approach does not seem to be able to provide a polynomial complexity algorithm for arbitrary small tolerances.

Our approach is different to those presented above and focuses on solving a non-smooth minimization problem.

Given $R, r > 0$, we consider the following sets:

$$\begin{aligned}
\mathcal{B}_0 &= \overline{B}(c,r) = \left\{ x \in \mathbb{R}^n \,\middle|\, \|x-c\|^2 \leq r^2 \right\}, \\
\mathcal{B}_k &= \overline{B}(c^k,R) = \left\{ x \in \mathbb{R}^n \,\middle|\, \|x-c_k\| \leq R^2 \right\}, \\
\mathcal{C}_1 &= \bigcap_{k=1}^{m} \mathcal{B}_k, \qquad \mathcal{C}_0 = \mathcal{B}_0
\end{aligned} \tag{2.3}$$

. In order to solve the problem (2.2), we keep R fixed and design a test which can assert if $\mathcal{C}_1 \subseteq \mathcal{C}_0$ for various values of r.

We start by defining the functions $f, f_k : \mathbb{R}^n \to \mathbb{R}$:

$$\begin{aligned} f_k(x) &= \|x - c_k\|^2 - R^2 \\ f(x) &= \|x - c\|^2 - r^2 \end{aligned} \tag{2.4}$$

and the function $G_k : \mathbb{R}^n \to \mathbb{R}$, given by

$$G_k(x) = f_k(x) - f^-(x) + \sum_{i=1, i \neq k}^{m} f_i^+(x)$$

for $k \in \{1, ..., m\}$.

Remark 2.3. It can be seen that G_k is a convex function. First the *"sum"*–term $\sum_{i=1,i\neq k}^{m} f_i^+(x)$ is convex, since each term in the sum is convex. On the other hand, the remaining term of $G_k(x)$, namely $f_k(x) - f^-(x)$, can be written as

$$f_k(x) - f^-(x) = f_k(x) - f(x) + f(x) - f^-(x) = f_k(x) - f(x) + f^+(x)$$

which is convex since it is the sum of the convex function $f^+(x)$ and the affine function $f_k(x) - f(x) = \|x - c_k\|^2 - \|x - c\|^2 - R^2 + r^2 = (c - c_k)^T \cdot (2 \cdot x - c - c_k) - R^2 + r^2$.

We take $G(x)$ to be the maximum of $G_k(x)$, when k ranges from 1 to m. That is,

$$G(x) = \max \left\{ G_k(x) \middle| k \in \{1, \ldots, m\} \right\} = \max_{k=\overline{1,m}} G_k(x)$$

Remark 2.4. We note that, since $G : \mathbb{R}^n \to \mathbb{R}$ is defined as the pointwise maximum of the convex functions $G_k : \mathbb{R}^n \to \mathbb{R}$, it follows that G is convex.

Finally we use $x^\star$, a global minimizer of $G(x)$, i.e.,

$$x^\star \in \underset{x \in \mathbb{R}^n}{\operatorname{argmin}} \quad G(x) \tag{2.5}$$

Before giving our main result, we present a few simple, but usefull lemmas.

Lemma 2.5. *Let $a, b \in \mathbb{R}^n$ and $r > 0$ such that $b \notin B(a, r)$. Then $\forall x \in B(a, r)$ the inequality*

$$(x - b)^T (a - b) > 0$$

holds.

Proof. Using the Euclidean norm properties over $\mathbb{R}^n$, we write

$$\begin{aligned} \|x - a\|^2 &= \|(x - b) + (b - a)\|^2 \\ &= \|x - b\|^2 + \|b - a\|^2 - 2(x - b)^T (a - b). \end{aligned} \tag{2.6}$$

For $x \in B(a, r)$, $b \notin B(a, r)$, we have $\|x - a\|^2 < r^2$ and $\|b - a\|^2 \geq r^2$. Combining these together with $\|x - b\|^2 \geq 0$ in (2.6), leads to $(x - b)^T (a - b) > 0$ and concludes the proof. □

Lemma 2.6. *Let $x \in \mathcal{C}_1$, with $\mathcal{C}_1$ defined by (2.3). Then for $y \in \mathbb{R}^n$ such that $d(y, \mathcal{C}_1) >$ R one has*

$$(x - y)^T (c_k - y) > 0, \quad \forall k \in \{1, \ldots, m\}$$

Proof. For $x \in \mathcal{C}_1$, one has $d(x, c_k) \le R$ and therefore $c_k \in B(x, R)$. From $d(y, \mathcal{C}_1) > R$, it follows that $d(x, y) > R$, hence $y \notin B(x, R)$. Applying 2.5, with $a := x$, $b := y$, $r := R$, and $x := c_k$, one obtains the desired conclusion. $\square$

Lemma 2.7. *Let $z, y, c^1, c^2 \in \mathbb{R}^n$ with $\|y - c_1\| = \|y - c_2\|$. Assume, without loss of generality, that $\|z - c_1\|^2 \ge \|z - c_2\|^2$ then*

$$\|y + t(z - y) - c_1\|^2 \ge \|y + t(z - y) - c_2\|^2, \qquad \forall t \ge 0.$$

Proof. Let

$$h(t) = \|y + t(z - y) - c_1\|^2 - \|y + t(z - y) - c_2\|^2.$$

From the identity above, it can be seen that $h(t)$ is a polynomial of degree at most 1 in t. Since $\|y - c_1\| = \|y - c_2\|$ gives $h(0) = 0$ and $\|z - c_1\| \ge \|z - c_2\|$ gives $h(1) \ge h(0) = 0$, it follows that $h(t)$ is a non-decreasing first order polynomial in t and therefore

$$h(t) \ge 0 = h(0), \ \ \forall t \ge 0,$$

which completes the proof. $\square$

Lemma 2.8. *Let $y, c^1, \dots, c^m \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$ such that $\|v\| = 1$. Let $p \in \{1, ..., m-1\}$ be such that*

$$\|y - c_i\| = \|y - c_j\| > \|y - c_l\| \tag{2.7}$$

for all $i, j \in \{1, \dots, p\}$ and $l \in \{p + 1, \dots, m\}$. Then $\exists k_v \in \{1, ..., p\}$ and $\delta_v > 0$ such that for all $i \in \{1, \dots, m\}$ one has

$$\|y + tv - c_{k_v}\| \ge \|y + tv - c^i\| \qquad \forall t \in (0, \delta_v), \tag{2.8}$$

which is stating that there is a small segment starting at y in the direction of v, such that for all the points on this segment, c^{k_v} remains the furthest away. For the case $p = m$, (2.8) holds without any additional requirements.

Proof. First, we consider the case $p \in \{1, ..., m - 1\}$. We define

$$\rho := \|y - c_1\| = \dots = \|y - c_p\|.$$

Let $\delta > 0$ and $z \in B(y, \delta)$. The triangle inequality gives

$$\begin{aligned} \|z - c_k\| &\ge \|c_k - y\| - \|z - y\|, \\ \|z - c_i\| &\le \|c_i - y\| + \|z - y\|. \end{aligned}$$

Using the above inequalities with arbitrary $k \in \{1, ..., p\}$ and $i \in \{p + 1, ..., m\}$, gives

$$\begin{cases} d(z, c_k) \ge \rho - \delta, \\ d(z, c_i) \le \eta + \delta. \end{cases} \tag{2.9}$$

where $\eta = \|y - c^i\| < \|y - c^k\| = \rho$ Following (2.9), we will pick $\delta > 0$ such that $\rho - \delta > \eta + \delta$. Since (2.7) implies $\rho - \eta > 0$, it follows that any $\delta \in \left(0, \frac{\rho - \eta}{2}\right)$ will satisfy this requirement. Thus, for any $\delta \in \left(0, \frac{\rho - \eta}{2}\right)$ and any $z \in B(y, \delta)$, we have

$$d(z, c_k) > d(z, c_i) \quad \ \forall k \in \{1, \dots, p\}, \ \forall i \in \{p + 1, \dots, m\}. \tag{2.10}$$

Let $\delta_v = \frac{\delta}{2}$, $z = y + \delta_v v$ and $k_v \in \underset{k\in\{1,\ldots,p\}}{\operatorname{argmax}} \|z - c_k\|$. For the points c_k, $k \in \{1, ..., p\}$, we apply Lemma 2.7 to obtain

$$\|y + (t\delta_v)v - c_{k_v}\|^2 \geq \|y + (t\delta_v)v - c_k\|^2,\ \forall t \geq 0,\ \forall k \in \{1, ..., p\}. \tag{2.11}$$

On the other hand, for the points c_i, $i \in \{p+1, ..., m\}$ we let $z := y + tv$ in (2.10) which gives

$$\|y + tv - c_{k_v}\|^2 > \|y + tv - c_i\|^2\ \ \forall i \in \{p+1, ..., m\},\ \forall t \in (0, \delta_v). \tag{2.12}$$

Combining (2.11) and (2.12) leads to the desired conclusion (2.8). For the case $p = m$, (2.8) follows immediately. □

The following theorem represents our **main result**. This is a localization result for $x^\star$ using the balls intersection denoted by $\mathcal{C}_1$ and the *"outside"* ball denoted $\mathcal{C}_0$.

Theorem 2.9. *For $R, \mathcal{C}, \mathcal{C}_1$ defined by (2.3), if $d(C, \mathcal{C}_1) > R$ then*

$$\mathcal{C}_1 \setminus \operatorname{int}(\mathcal{C}_0) \neq \emptyset \iff x^\star \in \mathcal{C}_1 \setminus \operatorname{int}(\mathcal{C}_0) \tag{2.13}$$

where $x^\star$ is defined by (2.5)

Proof. Clearly the implication $x^\star \in \mathcal{C}_1 \setminus \operatorname{int}(\mathcal{C}_0) \Rightarrow \mathcal{C}_1 \setminus \operatorname{int}(\mathcal{C}_0) \neq \emptyset$ is trivial. We now assume that $\mathcal{C}_1 \setminus \operatorname{int}(\mathcal{C}_0) \neq \emptyset$ and first show that in such a case $x^\star \in \mathcal{C}_1$.

Indeed, for $x \notin \mathcal{C}_1$ $(= \bigcap_{k=1}^m \mathcal{B}_k)$ i.e. it is not in the intersection of congruent balls, follows that $\|x - c_k\| > R^2$ for some $k \in \{1, ..., m\}$ or equivalently $f_k(x) > 0$ for some $k \in \{1, ..., m\}$. From the definitions of f^- and f_i^+, we have $-f^-(x) \geq 0$ and $f_i^+(x) \geq 0$. Combining this with $f_k(x) > 0$, leads to the fact that for $x \notin \mathcal{C}_1$ we have $G_k(x) > 0$, hence $G(x) = \max_{k\in\{1,\ldots,m\}} G_k(x) > 0$ as well. On the other hand if $x \in \mathcal{C}_1 \setminus \operatorname{int}(\mathcal{C}_0)$, we have $-f^-(x) = 0$, $f_k(x) \leq 0$, $\forall k \in \{1, ..., m\}$, implying $G(x) \leq 0$ therefore $x^\star$, a minimizer of G, is not outside of $\mathcal{C}_1$ since there are "better" points in $\mathcal{C}_1$.

From the observations above, it follows that $x^\star \in \mathcal{C}_1$. Next, we will show that $x^\star \notin \operatorname{int}(\mathcal{C}_1 \cap \mathcal{C}_0)$, leading to the desired conclusion. Let $y \in \operatorname{int}(\mathcal{C}_1 \cap \mathcal{C}_0)$. It follows that there exists $\delta_y > 0$ such that $B(y, \delta_y) \subseteq \operatorname{int}(\mathcal{C}_1 \cap \mathcal{C}_0)$. We can assume without loss of generality that $\exists p \in \{1, ..., m-1\}$ such that

$$\|y - c_1\| = ... = \|y - c_p\| > \|y - c_l\|,\ \forall l \in \{p+1, ..., m\}.$$

This implies

$$G(y) = G_1(y) = ... = G_p(y).$$

From Lemma 2.8 follows that $\forall\ v \in \mathbb{R}^n$ with $\|v\| = 1$, $\exists k_v \in \{1, ..., p\}$ and $\delta_v > 0$ such that

$$G(y + tv) = G_{k_v}(y + tv) \qquad \forall t \in [0, \delta_v). \tag{2.14}$$

Let $\delta := \min\{\delta_y, \delta_v\}$, $v = \frac{y-c}{\|y-c\|}$ and $z = y + \frac{\delta}{2}$. Clearly $z \in \operatorname{int}(\mathcal{C}_1 \cap \mathcal{C}_0)$. Let $h(t) := G(y + tv)$, $\forall t \in [0, \delta_v)$. From (2.14), it follows that $h(t) = G_{k_v}(y + tv)$, or equivalently

$$h(t) = r^2 - \|y - c + tv\|^2 + \|y - c_{k_v} + tv\|^2 - R^2,\ \forall t \in [0, \delta_v). \tag{2.15}$$

Differentiating (2.15) with respect to t gives

$$\begin{aligned} h'(t) &= -(y - c + tv)^T\ v + (y - c_{k_v} + tv)^T v \\ &= -(c_{k_v} - c)^T \frac{y - c}{\|y - c\|}. \end{aligned} \tag{2.16}$$

Since $d(c, \mathcal{C}_1) > R$, it follows from Lemma 2.6 and (2.16) that $h'(t) < 0$, $\forall t \in [0, \delta_v)$ implying that $h(t)$ is strictly decreasing. Therefore $h(0) > h(\frac{\delta}{2})$, which is equivalent to $G(z) < G(y)$, for $z = y + \frac{\delta}{2}v \in \text{int}(\mathcal{C}_0 \cap \mathcal{C}_1)$. It follows that $x^\star \in \underset{x \in \mathbb{R}^n}{\text{argmin}}\, G(x) \notin \text{int}(\mathcal{C}_1 \cap \mathcal{C}_0)$. Since $\mathcal{C}_1$ can be partitioned as

$$\mathcal{C}_1 = \mathcal{C}_1 \setminus \mathcal{C}_0 \cup \text{int}(\mathcal{C}_1 \cap \mathcal{C}_0) \cup \partial(\mathcal{C}_1 \cap \mathcal{C}_0)$$

and we showed that $x^\star \in \mathcal{C}_1$, $x^\star \notin \text{int}(\mathcal{C}_1 \cap \mathcal{C}_0)$, we have

$$\begin{aligned} x^\star &\in \mathcal{C}_1 \setminus \mathcal{C}_0 \cup \partial(\mathcal{C}_1 \cap \mathcal{C}_0) \\ &\subseteq \mathcal{C}_1 \setminus \mathcal{C}_0 \cup \partial\mathcal{C}_0, \end{aligned}$$

implying that $x^\star \in \mathcal{C}_1 \setminus \text{int}(\mathcal{C}_0)$. This concludes our proof. □

2.3. Complexity Analysis

Theorem 2.9 allows one to solve (2.2) if $d(c, \mathcal{C}_1) > R$. Indeed, let $x_0 \in \mathcal{C}_1$ (this can be found initially by the use of Section 2.1 assuming that $\mathcal{C}_1 \neq \emptyset$). Then one can show that $\mathcal{C}_1 \subseteq B(x_0, 2R)$. Let $\underline{r} = R$ and $\bar{r} = 2R + \|x_0 - c\|$. It is obvious that $\mathcal{C}_1 \setminus B(c, \underline{r}) \neq \emptyset$ and $\mathcal{C}_1 \setminus B(c, \bar{r}) = \emptyset$.

We can now search for $r^\star \in [\underline{r}, \bar{r}]$ such that $\mathcal{C}_1 \setminus B(c, r^\star - \epsilon) \neq \emptyset$ and $\mathcal{C}_1 \setminus B(c, r^\star + \epsilon) = \emptyset$ for some arbitrarily fixed precision $\epsilon > 0$, using Theorem 2.9 and the bisection algorithm.

From the computational complexity point of view, each bisection step involves the application of Theorem 2.9 for some $r \in [\underline{r}, \bar{r}]$. For this, one has to solve (2.5) to find $x^\star$. Once $x^\star$ is found, asserting its membership to $\mathcal{C}_1 \setminus B(c, r)$ involves computing $m+1$ distances in $\mathbb{R}^n$, that is $(m+1)n$ flops (for the square of the distances) and comparing them to some real numbers, hence another $m + 1$ flops. Finally the computational complexity analysis for each step is completed by analyzing the cost of finding $x^\star$. This basically involves an unconstrained minimization of a continuous, non-differentiable convex function. The starting point can be considered x_0 and the search radius can be taken $2R$. There are various algorithms (of sub-gradient, [9] or ellipsoid type, [18]) which are known to have polynomial deterministic worst case complexity for such a problem. Let Λ (a polynomial in $n, m, \log(R), -\log(\epsilon)$) denote the number of floating point operations required to solve (2.5). Then solving (2.2) requires

$$\mathcal{O}\left((\Lambda + (m+1) \cdot n) \cdot \log_2 \left(\frac{R + \|X_0 - C\|}{\epsilon} \right) \right),$$

where $\epsilon > 0$ is the precision used to find $r^\star$.

3. Results regarding polytopes

In this section we tackle a similar problem as in the previous section but instead of considering a finite intersection of balls, we will consider a polytope $\mathcal{P}$ (i.e a finite intersection of half-spaces) and find a vertex that is the farthest away from a point of the form $c+\alpha d$ with $c, d \in \mathbb{R}^n$, for all sufficiently large values of the scalar α. Without any restrictions on α this is also known to be an NP-hard problem, i.e maximizing the distance to a point over a polytope, but under certain restrictions, we prove that this problem can be reduced to a linear program over the polytope $\mathcal{P}$.

Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $\mathcal{P} = \{x \in \mathbb{R}^n | \ Ax + b \leq 0\}$ a given polytope (closed, bounded polyhedral set). Let $b = [b_1, ..., b_m]^T$ and $A_{(i,:)}$, $i = \overline{1,m}$ denote the rows of the matrix A, viewed as column vectors, i.e., $A^T = \left[A_{(1,:)}, ..., A_{(m,:)}\right]$. In what follows, we give several results related to polytopes.

Theorem 3.1. *Let $c, d \in \mathbb{R}^n$. Then there exists $\alpha_0 \in \mathbb{R}^*_+$ such that if v^* is a vertex of $\mathcal{P}$, with $v^* \in argmax_{x\in\mathcal{P}} \ \ \|c + \alpha_0 d - x\|^2$, then*

$$v^* \in \underset{x\in\mathcal{P}}{argmax} \ \ \|c + \alpha d - x\|^2 \tag{3.1}$$

for all $\alpha \geq \alpha_0$.

Proof. Since we are maximizing a continuous function over the compact subset $\mathcal{P}$ of $\mathbb{R}^n$, the maximum is attained for any value of α. For an arbitrarly selected α, writing (3.1) as a minimization problem, leads to a concave quadratic program (QP), which is known to attain its minimum in a vertex of the polytope $\mathcal{P}$ (see for example [14]). If $v_1, ..., v_p$ are the vertices of the polytope $\mathcal{P}$, it then follows that $\forall\alpha$, $\exists i_\alpha \in \{1, ..., p\}$ such that

$$v_{i_\alpha} \in \underset{x\in\mathcal{P}}{\operatorname{argmax}} \ \ \|c + \alpha d - x\|^2.$$

Let $\underline{\alpha} > 0$ and $\overline{\alpha} > \underline{\alpha}$ be such that

$$v_{i_{\underline{\alpha}}} \in \underset{x\in\mathcal{P}}{\operatorname{argmax}} \ \ \|c + \underline{\alpha} d - x\|^2 \text{ and } v_{i_{\underline{\alpha}}} \notin \underset{x\in\mathcal{P}}{\operatorname{argmax}} \ \ \|c + \overline{\alpha} d - x\|^2. \tag{3.2}$$

If (3.2) does not hold, then the conclusion automatically follows, i.e $\nexists\ \overline{\alpha} > \underline{\alpha}$ such that $v_{i_{\underline{\alpha}}} \notin \operatorname{argmax}_{x\in\mathcal{P}} \ \ \|c + \overline{\alpha} d - x\|^2$, hence simply take $\alpha_0 = \underline{\alpha}$ and $v^\star = v_{i_{\underline{\alpha}}}$. Otherwise, if (3.2) holds, then we show that

$$v_{i_{\underline{\alpha}}} \notin \underset{x\in\mathcal{P}}{\operatorname{argmax}} \ \ \|c + \alpha d - x\|^2, \ \forall\alpha, \ \alpha \geq \overline{\alpha}, \tag{3.3}$$

i.e., $\forall\alpha$, $\alpha \geq \overline{\alpha}$, $v_{i_{\underline{\alpha}}}$ is not the vertex furthest away from $c + \alpha d$. To see this, let $i_{\overline{\alpha}} \in \{1, ..., p\} \setminus i_{\underline{\alpha}}$ be such that

$$v_{i_{\overline{\alpha}}} \in \underset{x\in\mathcal{P}}{\operatorname{argmax}} \ \ \|c + \overline{\alpha} d - x\|^2.$$

Clearly, we have

$$\|c + \overline{\alpha} d - v_{i_{\underline{\alpha}}}\| < \|c + \overline{\alpha} d - v_{i_{\overline{\alpha}}}\|. \tag{3.4}$$

We define

$$f(\alpha) = \|c + \alpha d - v_{i_{\underline{\alpha}}}\|^2 - \|c + \alpha d - v_{i_{\overline{\alpha}}}\|^2.$$

From (3.2) and (3.4), it follows that

$$f(\underline{\alpha}) \geq 0 \text{ and } f(\overline{\alpha}) < 0,$$

which together with $\underline{\alpha} < \overline{\alpha}$ and the fact that f is affine, implies that f is a strictly decreasing function of α. This leads to $f(\alpha) < f(\overline{\alpha})$, $\forall \alpha > \overline{\alpha}$, which implies (3.3).

To finish the proof, assume that the conclusion of the theorem does not hold. This is to say that for any $\alpha_0 > 0$, there exists $\alpha_1 > \alpha_0$ such that

$$v_{i_{\alpha_0}} \in \operatorname*{argmax}_{x\in\mathcal{P}} \ \|c+\alpha_0 d - x\|^2 \text{ and } v_{i_{\alpha_0}} \notin \operatorname*{argmax}_{x\in\mathcal{P}} \ \|c+\alpha_1 d - x\|^2.$$

According to what we have shown above, $v_{i_{\alpha_0}}$ will never be the furthest point away from $c + \alpha d$ for any $\alpha \geq \alpha_1$. We can repeat this reasoning now with α_0 replaced by α_1 and i_{α_0} replaced by $i_{\alpha_1} \in \{1, .., p\} \setminus i_{\alpha_0}$. After $p - 1$ such repetitions, we are exhausting all the vertices from the solution set, which is a contradiction to the fact that the problem attains its maximum in a vertex for any value of α. □

The next result shows that the point $v^\star$ of Theorem 3.1 can be found as the solution of a linear program (LP), whenever the solution set of this LP is a singleton.

Theorem 3.2. *Let $\mathcal{P}$ be a polytope, $c \in \mathbb{R}^n$ and $d \in \mathbb{R}^n$ with $\|d\| = 1$ such that*

$$x^\star = \operatorname*{argmin}_{x\in\mathcal{P}} \ d^T x$$

is unique. Let α_0 and $v^\star$ be given by Theorem 3.1, i.e.,

$$v^\star = \operatorname*{argmax}_{x\in\mathcal{P}} \ \|c+\alpha d - x\|^2 = \operatorname*{argmax}_{x\in\mathcal{P}} \ \|c+\alpha_0 d - x\|^2 \qquad \forall \alpha \geq \alpha_0$$

Then $v^\star = x^\star$.

Proof. To show that $v^\star = x^\star$, it is enough to prove that

$$(v^\star)^T d \leq x^T d, \forall x \in \mathcal{P}. \tag{3.5}$$

Now assume, that (3.5) does not hold. It follows that there exists $\widetilde{x} \in \mathcal{P}$, such that $\widetilde{x}^T d < (v^\star)^T d$. Define $f(\alpha) = \|c + \alpha d - v^\star\|^2 - \|c + \alpha d - \widetilde{x}\|^2$. A simple calculation leads to

$$f'(\alpha) = (\widetilde{x} - v^\star)^T d < 0,$$

implying that the linear function $f(\alpha)$ is decreasing and therefore $\lim\limits_{\alpha\to\infty} f(\alpha) = -\infty$. The latter implies that there exists $\alpha_1 > 0$, such that $f(\alpha) < 0$, $\forall \alpha \geq \alpha_1$ or equivalently $\|c + \alpha d - v^\star\|^2 < \|c + \alpha d - \widetilde{x}\|$, $\forall \alpha > \alpha_1$, which is a contradiction to the way $v^\star$ is defined. Therefore $v^\star$ must satisfy (3.5) or equivalently $v^\star \in \operatorname{argmax}_{x\in\mathcal{P}}$. Since by assumption, the argmin–set is a singleton, we are led to $v^\star = x^\star$, which concludes our proof. □

4. Conclusion and future work

In this paper we have considered two known NP hard problems namely maximizing the distance to a reference point over (i) an intersection of balls and (ii) an intersection of half-spaces. We have provided some particular cases of the above mentioned problems where algorithms of polynomial complexity exist. In both cases, our restrictions are in the form of some relation between the given fixed reference point and the set over which the maximum is searched for.

Consider the first problem (i): for a given finite intersection of equal radii balls, one can choose the reference point anywhere in the $\mathbb{R}^{n\times 1}$ to form a problem. Our algorithm provides a P time solution to all these choices except for a finite measure set "near" the search space, that is, this paper does not offer guarantees for the reference points whom distance to the search space is less than the radius of the intersecting balls. It is not known if "conquering" this last region is even possible, but obviously reducing it might be the subject of future work. As a first improvement one can try to provide an P time algorithm which allows the given reference point to be anywhere outside of the convex hull of the centers of the intersecting balls.

The approach to the first problem is based on a novel feasibility criteria for the intersection of convex sets which we apply to a non-convex optimization problem. The restrictions we obtain, are sufficient to actually transform the non-convex problem in a convex one.

The approach to the second problem, maximizing the distance to a point over a polytope, is somehow inspired from the first problem, by observing that if the exterior point is far enough, then in some situations the optimal point is actually obtained by solving a linear program.

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Marius Costandin
Babeş-Bolyai University,
Faculty of Mathematics and Computer Sciences,
Cluj-Napoca, Romania
e-mail: costandinmarius@gmail.com

Bogdan Gavrea
The work of this author was done
when the author was with the
Department of Mathematics,
Technical University of Cluj-Napoca
e-mail: bogdan.gavrea@math.utcluj.ro