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Weakly Picard mappings: Retraction-displacement condition, quasicontraction notion and weakly Picard admissible perturbation

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Abstract. Let (X,d) be a metric space, $f : X \to X$ be a mapping and $G(\cdot, f(\cdot))$ be an admissible perturbation of f. In this paper we study the following problems: In which conditions imposed on f and G we have the following:

(DDE) data dependence estimate for the mapping f perturbation;
(UH) Ulam-Hyers stability for the equation, $x = f(x)$;
(WP) well-posedness of the fixed point problem for f;
(OP) Ostrowski property of the mapping f.
Some research directions are suggested.

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Keywords: Metric space, fixed point equation, Picard mapping, weakly Picard mapping, admissible perturbation, retraction-displacement condition, data dependence estimate, Ulam-Hyers stability, well-posedness, Ostrowski property, quasicontraction.

1. Introduction

Let X be a nonempty set and $f : X \to X$ be a mapping. To define a perturbation of f we consider a mapping $G : X \times X \to X$ with the following properties:

(A_1) $G(x,x) = x$, $\forall\ x \in X$;
(A_2) $x, y \in X$, $G(x,y) = x$ implies $y = x$.

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Now, we consider the operator, $f_G : X \to X$ defined by,

$$f_G(x) := G(x, f(x)).$$

It is clear that, $F_f = F_{f_G}$, i.e., the fixed point equations,

$$x = f(x) \text{ and } x = f_G(x)$$

are equivalent.

By definition, the mapping f_G is an admissible perturbation of the mapping f corresponding to the mapping G.

Let us consider an example. For other examples see [53].

Example 1.1. Let $\mathbb{B}$ be a Banach space, $f : \mathbb{B} \to \mathbb{B}$ be a mapping and $G : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ be defined by,

$$G(x, y) := (1 - \lambda)x + \lambda y$$

for some $\lambda \in \mathbb{R}^*$. Then f_G is an admissible perturbation of f. We denote it by, f_λ.

Remark 1.2. If $X \subset \mathbb{B}$ is a nonempty convex subset of $\mathbb{B}$, $f : X \to X$ is a mapping and $G(x, y) := (1 - \lambda)x + \lambda y$ for some $\lambda \in]0, 1[$, then f_λ is an admissible perturbation of f, i.e., Krasnoselskii perturbation of f. For more considerations of this perturbation see [52], [3], [12], [20], [21].

Let (X, d) be a metric space, $f : X \to X$ be a mapping and $G(\cdot, f(\cdot))$ be an admissible perturbation of f. In this paper we shall study the following problems:

In which conditions imposed on f and G we have the following (all or one!) :

(DDE) data dependence estimate for the general perturbation of f;

(UH) Ulam-Hyers stability for the equation, $x = f(x)$;

(WP) well-posedness of the fixed point problem for f;

(OP) Ostrowski property of the mapping f.

Some research direction are suggested.

Throughout this paper the notations and terminology given in [8], [38], [56] and [57] are used.

Instead of long preliminaries we give the following references:

- Picard and weakly Picard mappings: [48], [56], [57], [61], [64];
- Ulam-Hyers stability: [55], [56], [57], [64];
- Well-posedness of fixed point problem: [56], [57], [9], [10], [35], [50], [33];
- Ostrowski property of a mapping (limit shadowing property): [35], [17], [22], [46], [56], [57], [61], [64], [13], [34], [32].

2. Retractions on the fixed point set and retraction-displacement conditions

Let (X, d) be a metric space and $f : X \to X$ be a mapping with $F_f \neq \varnothing$. Let $r : X \to F_f$ be a set retraction, i.e., $r|_{F_f} = 1_{F_f}$. Then,

$$X = \bigcup_{x \in F_f} r^{-1}(x)$$

is a partition of X. If $x^* \in F_f$ then we denote, $X_{x^*} := r^{-1}(x^*)$. By definition, the partition $X = \bigcup_{x^* \in F_f} X_{x^*}$ is a fixed point partition of X corresponding to the retraction r (see [59]).

Remark 2.1. In general, X_{x^*} is not an invariant subset for f.

Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function with $\psi(0) = 0$ and continuous at 0. By definition, the condition,

$$d(x, r(x)) \leq \psi(d(x, f(x))),\ \forall\ x \in X,$$

is a retraction-displacement condition on f corresponding to the retraction r.

Example 2.2. (see [57]; see also [42], [37], [36]). Let (X, d) be a complete metric space and $f : X \to X$ be a graphic l-contraction. In addition we suppose that,

$$d(f(f^n(x)), f(f^\infty(x))) \to 0 \text{ as } n \to \infty,$$

for all $x \in X$. Then f is weakly Picard mapping.

The mapping $f^\infty : X \to F_f$ is a set-retraction and

$$d(x, f^\infty(x)) \leq \frac{1}{1-l} d(x, f(x)),\ \forall\ x \in X.$$

In this case, $f(X_{x^*}) \subset X_{x^*}$, $\forall\ x^* \in F_f$, i.e., $X = \bigcup_{x^* \in F_f} X_{x^*}$ is an invariant fixed point partition of X corresponding to the retraction f^∞.

Example 2.3. (Browder [11] and Bruck [14], pp. 6, 33). Let H be a Hilbert space, $X \subset H$ be a convex, closed and bounded subset of H and $f : X \to X$ be a nonexpansive mapping. Let $r_1(x) = \lim_{n\to\infty} x_n(x)$, where x_n is the unique solution of,

$$x_n(x) = \frac{1}{n}x + \big(1 - \frac{1}{n}\big) f(x_n(x)),\ n \in \mathbb{N}^*,\ x \in X,$$

and

$$r_2(x) = w - lim \frac{1}{n}(1_X + f + \ldots + f^{n-1})(x),\ n \in \mathbb{N}^*,\ x \in X.$$

Then the mappings, $r_1, r_2 : X \to F_f$ are nonexpansive retractions. In general, $r_1 \neq r_2$.

In this case we have two distinct fixed point partitions of X corresponding to r_1 and to r_2.

Remark 2.4. The notion *fixed point partition of the space with respect to a retraction* is a relevant one. For example, in terms of this notion we can give the following definitions.

Let (X, d) be a metric space, $f : X \to X$ be a mapping with $F_f \neq \varnothing$, $r : X \to F_f$ be a set retraction and $X = \bigcup_{x^* \in F_f} X_{x^*}$ be the fixed point partition of X, corresponding to the retraction r.

Definition 2.5. The fixed point problem for the mapping f is well-posed with respect to the partition $X = \bigcup_{x^* \in F_f} X_{x^*}$ if the following implication holds:

$$x^* \in F_f,\ x_n \in X_{x^*},\ n \in \mathbb{N},\ d(x_n, f(x_n)) \to 0 \text{ as } n \to \infty \\ \Rightarrow\ x_n \to x^* \text{ as } n \to \infty.$$

Definition 2.6. The mapping f has the Ostrowski property with respect to the partition, $X = \bigcup_{x^* \in F_f} X_{x^*}$, if the following implication holds:

$$x^* \in F_f,\ x_n \in X_{x^*},\ n \in \mathbb{N},\ d(x_{n+1}, f(x_n)) \to 0 \text{ as } n \to \infty \\ \Rightarrow\ x_n \to x^* \text{ as } n \to \infty.$$

3. Results for (DDE), (UH) and (WP) problems

3.1. (DDE) problem

Let (X, d) be a metric space, $f : X \to X$ be a mapping and f_G be an admissible perturbation. Let $g : X \to X$ be a mapping such that,

$$d(f(x), g(x)) \le \eta,\ \forall\ x \in X,\ \text{for some } \eta \in \mathbb{R}_+^*.$$

We suppose that, $F_f = \{x^*\}$ and $F_g \neq \varnothing$.

The problem is to find in which conditions imposed on f and G, there exists an increasing, $\theta : \mathbb{R}_+ \to \mathbb{R}_+$, with $\theta(0) = 0$ and continuous in 0 such that,

$$d(y^*, x^*) \le \theta(\eta),\ \forall\ y^* \in F_g.$$

We have the following result.

Theorem 3.1. *We suppose that:*

(1) f_G *is a* ψ*-Picard mapping* ($F_{f_G} = \{x^*\}$)*;*
(2) $d(x, f_G(x)) \le cd(x, f(x))$, $\forall\ x \in X$ *with some* $c \in \mathbb{R}_+^*$*;*
(3) $d(g(x), f(x)) \le \eta$, $\forall\ x \in X$ *with some* $\eta \in \mathbb{R}_+^*$.

Then we have that:

(i) $d(x, x^*) \le \psi(cd(x, f(x)))$, $\forall\ x \in X$*;*
(ii) $d(y^*, x^*) \le \psi(c\eta)$, $\forall\ y^* \in F_g$.

Proof. Since f_G is a Picard mapping and an admissible perturbation of f we have that, $F_f = \{x^*\}$ and from (1),

$$d(x, x^*) \le \psi(d(x, f_G(x))),\ \forall\ x \in X.$$

From (2) we have (i).

If we take $x = y^* \in F_g$, then from (i) and (3),

$$d(y^*, x^*) \le \psi(cd(y^*, f(y^*))) = \psi(cd(g(y^*), f(y^*))) \le \psi(c\eta).$$ □

Example 3.2. Let $X := \mathbb{B}$ be a Banach space and $G(x, y) := (1 - \lambda)x + \lambda y$, with $\lambda \in \mathbb{R}_+^*$. We suppose that f_λ is an l-contraction for some $\lambda \in \mathbb{R}_+^*$. Then f_λ is $\frac{1}{1-l}$ -Picard mapping and $d(x, f_\lambda(x)) = \|x - f_\lambda(x)\| \le |\lambda|\|x - f(x)\|$.

Let, $\|f(x)-g(x)\| \leq \eta$, $\forall\ x \in \mathbb{B}$. Then by Theorem 3.1 we have that:

$$\|y^* - x^*\| \leq \frac{|\lambda|}{1-l}\eta,\ \forall\ y^* \in F_g.$$

Remark 3.3. For the mappings f_λ which are contractions or which satisfy other metric conditions, see Berinde [4] and Berinde-Păcurar [7].

Remark 3.4. With similar proof as the one given for Theorem 3.1, we have the following result.

Theorem 3.5. *We suppose that:*

(1) f_G *is a* ψ*-weakly Picard mapping;*
(2) $d(x, f_G(x)) \leq cd(x, f(x))$, $\forall\ x \in X$ *with some* $c \in \mathbb{R}_+^*$*;*
(3) $d(g(x), f(x)) \leq \eta$, $\forall\ x \in X$ *with some* $\eta \in \mathbb{R}_+^*$*.*

Then we have that:

(*i*) $d(x, f_G^\infty(x)) \leq \psi(cd(x, f(x)))$, $\forall\ x \in X$*;*
(*ii*) *if* $x^* \in F_f$*, then* $d(y^*, x^*) \leq \psi(c\eta)$, $\forall\ y^* \in F_g \cap X_{x^*}$*, where* $X = \bigcup\limits_{x^* \in F_f} X_{x^*}$ *is a fixed point partition of* X *corresponding to the retraction* f_G^∞*.*

3.2. (UH) **problem**

Let (X,d) be a metric space, $f : X \to X$ be a mapping and $f_G : X \to X$ be an admissible perturbation of f. For $\varepsilon \in \mathbb{R}_+^*$ we consider the inequation

$$d(y, f(y)) \leq \varepsilon.$$

Let y^* be a solution of this inequation. We suppose that f_G is a ψ-weakly Picard mapping and

$$d(x, f_G(x)) \leq cd(x, f(x)),\ \forall\ x \in X,\ \text{with some } c \in \mathbb{R}_+^*.$$

There exists $x^* \in F_f$ such that $y^* \in X_{x^*}$. For a such x^* we have that

$$d(y^*, x^*) \leq \psi(c\varepsilon).$$

So, we have the following result.

Theorem 3.6. *In the above conditions the fixed point equation,* $x = f(x)$ *is Ulam-Hyers stable.*

3.3. (WP) **problem**

By standard proof (see [56], [57]) and the above considerations, we have the following result for this problem.

Theorem 3.7. *Let* (X,d) *be a metric space,* $f : X \to X$ *be a mapping and* f_G *be an admissible perturbation. We suppose that:*

(1) f_G *is* ψ*-weakly Picard mapping;*
(2) $d(x, f_G(x)) \leq cd(x, f(x))$, $\forall\ x \in X$*, for some* $c \in \mathbb{R}_+^*$*.*

Then the fixed point problem for f *is well-posed.*

4. Notion of quasicontraction and (OP) problem

4.1. Quasicontractions

In [8] the following definition is given:

Let (X,d) be a metric space and $f : X \to X$ be a mapping with $F_f \neq \varnothing$. By definition f is a quasicontraction if there exists $l \in]0,1[$ such that

$$d(f(x),x^*) \leq ld(x,x^*),\ \forall\, x \in X,\ \forall\, x^* \in F_f.$$

It is clear that if f is a quasicontraction then $F_f = \{x^*\}$.

If $F_f \neq \varnothing$ and $r : X \to F_f$ is a set-retraction then we have the following definition.

Definition 4.1. Let (X,d) be a metric space, $f : X \to X$ be a mapping with $F_f \neq \varnothing$ and $r : X \to F_f$ be a set retraction. Then f is a quasicontraction with respect to the retraction r if there exists $l \in]0,1[$ such that,

$$d(f(x),r(x)) \leq ld(x,r(x)),\ \forall\, x \in X.$$

For example, if f is a weakly Picard mapping then f is a quasicontraction if,

$$d(f(x),f^{\infty}(x)) \leq ld(x,f^{\infty}(x)),\ \forall\, x \in X.$$

For more considerations on quasicontractions, see: [3], [17], [46], [56], [57], [67], [14], [13].

4.2. (OP) problem

Let (X,d) be a metric space, $f : X \to X$ be a mapping with $F_f \neq \varnothing$ and $r : X \to F_f$ be a set retraction. Let $X = \bigcup\limits_{x^* \in F_f} X_{x^*}$ be the partition of X corresponding to the retraction r. Let $x^* \in F_f$ and $x_n \in X_{x^*}$, $n \in \mathbb{N}$ such that,

$$d(x_{n+1},f(x_n)) \to 0 \text{ as } n \to \infty.$$

Let us suppose that the mapping f is a quasi l-contraction with respect to the retraction r, i.e.,

$$d(f(x),x^*) \leq ld(x,x^*),\ \forall\, x \in X_{x^*},\ \forall\, x^* \in F_f.$$

From this condition we have that,

$$\begin{aligned} d(x_{n+1},x^*) &\leq d(x_{n+1},f(x_n)) + d(f(x_n),x^*) \\ &\leq d(x_{n+1},f(x_n)) + ld(x_n,x^*) \\ &\leq d(x_{n+1},f(x_n)) + ld(x_n,f(x_{n-1})) + l^2 d(x_{n-1},x^*) \\ &\vdots \\ &\leq d(x_{n+1},f(x_n)) + ld(x_n,f(x_{n-1})) + \ldots + l^n d(x_1,f(x_0)) \to 0, \end{aligned}$$

as $n \to \infty$, from a Cauchy-Toeplitz lemma [63].

So we have,

Theorem 4.2. *Let (X,d) be a metric space, $f : X \to X$ be a mapping with $F_f \neq \varnothing$ and $r : X \to F_f$ be a set retraction. We suppose that f is a quasicontraction with respect to the retraction r. Then the mapping f has the Ostrowski property.*

For example let f_G be an admissible perturbation of f. If f_G is a weakly Picard mapping and the mapping f is a quasicontraction with respect to f_G^∞, then the mapping f has the Ostrowski property with respect to f_G^∞.

5. Research directions

5.1. To give relevant examples of iterative fixed point algorithms which generate retractions on a fixed point set.

References: [3], [10], [12], [17], [28], [31], [35], [45], [53], [58], [66], [65], [11].

5.2. To give relevant examples of quasicontractions with respect to retractions defined by iterative algorithms.

For theoretical and applicative point of view, from the considerations of this article, the following problems arise:

To give similar results for:

5.3. Zero point equations

References: [16], [43], [19], [3], [35], [55].

5.4. Coincidence point equations

References: [15], [55], [60].

5.5. Equations with nonself mappings

References: [6], [9], [18], [35], [54], [55], [61].

5.6. Equations in $\mathbb{R}^m_+$-metric spaces

References: [35], [47], [61], [48], [56], [63], [27], [34].

5.7. Equations in $s(\mathbb{R}_+)$-metric spaces

References: [68], [56], [57], [61], [63], [27].

5.8. Equations in dislocated metric spaces

References: [31], [51], [24], [25], [29], [2], [1], [5].

5.9. Equations in a set with two metrics

References: [48], [61], [49], [24], [47].

5.10. Equations in a set with an order relation and a metric

References: [41] and the references therein.

5.11. Equations with multivalued mappings

References: [40], [44], [61], [55], [62], [14], [30].

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