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# A strong convergence algorithm for approximating a common solution of variational inequality and fixed point problems in real Hilbert space

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**Abstract.** In this paper, we propose an iterative algorithm for approximating a common solution of a variational inequality and fixed point problem. The algorithm combines the subgradient extragradient technique, inertial method and a modified viscosity approach. Using this algorithm, we state and prove a strong convergence algorithm for obtaining a common solution of a pseudomonotone variational inequality problem and fixed point of an $\eta$-demimetric mapping in a real Hilbert space. We give an application of this result to some theoretical optimization problems. Furthermore, we report some numerical examples to show the efficiency of our method by comparing with previous methods in the literature. Our result extend, improve and unify many other results in this direction in the literature.

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**Keywords:** Variational inequality, pseudomonotone operator, strong convergence, fixed point, extragradient algorithm, linesearch rule, Hilbert space.

## 1. Introduction

In this paper, we consider the Variational Inequality Problem (VIP) which consists of finding a point $x^* \in K$ such that

$$\langle Fx^*, x - x^* \rangle \geq 0, \ \forall \ x \in K, \tag{1.1}$$

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where $K$ is a nonempty, closed and convex subset of a real Hilbert space $H$, $F : H \to H$ is a nonlinear single-valued mapping, $\langle \cdot, \cdot \rangle$ respectively $\| \cdot \|$ are inner products and norm defined on $H$. We denote by $VIP(K, F)$, the set of solutions of the VIP (1.1). A wide range of problems in science and engineering, optimization theory, equilibrium theory and differentiation equation leads to the study of the variational inequality problems. For this reason, there have been several researches into the study of iterative algorithms for approximating the solutions of VIP and related optimization problems, (see [1, 2, 4, 6, 30, 45, 48, 51, 50, 52]).

One of the simplest and earliest known method for solving VIP is the gradient projection method as a result of a fixed point formulation which involves the metric projection. The method is given as

$$x_{k+1} = P_K(x_k - \lambda F x_k), \; x_1 \in K, \; k \geq 1,$$

where $P_K$ is the metric projection of $H$ onto $K$ and $\lambda \in (0, 1/L)$ with $L$ the Lipschitz constant of the cost operator $F$. For the convergence of this method, it is required that the operator $F$ is strongly monotone (see [22, 23, 21, 31]).

Another method for solving the VIP is the so-called Extragradient Method (EGM) initially proposed by Korpelevich for solving the saddle points problem (see also, Antipin [7]). For solving the VIP, the EGM is given as follows: $x_1 \in K$

$$\begin{cases} y_k = P_K(x_k - \lambda F x_k), \\ x_{k+1} = P_K(x_k - \lambda F y_k). \; k \geq 1 \end{cases} \tag{1.2}$$

The EGM (1.2) requires executing projection onto feasible set $K$ twice per iteration. Considerable efforts have been made to modify and improve this method, one of which is to reduce the projection from two to one onto feasible sets. In particular, one of such modifications is the Subgradient Extragradient Method (SEGM) by Censor et. al (see [14, 15]). In this method, the second projection of the extragradient method was replaced by a projection onto a half-space whose formula can be easily executed. The SEGM is given as follows: $x_1 \in K$ :

$$\begin{cases} y_k = P_K(x_k - \lambda F x_k), \\ T_k = \{x \in H : \langle x_k - \lambda F x_k - y_k, x - y_k \rangle \leq 0\}, \\ x_{k+1} = P_{T_k}(x_k - \lambda F y_k), \; k \geq 1 \end{cases} \tag{1.3}$$

Another drawback of the EGM is the dependence of the constant $\lambda$ on the Lipschitz constant of the associated cost operator. For this reason, many authors have proposed several methods which avoid the prior knowledge or use of the Lipschitz constant. One of such is the use of well defined linesearch rule (see [11]) and the references therein. One other popular method for avoiding the use of Lipschitz constant is to construct an adaptable step size (see, [51, 52]) for more.

On the other hand, the Fixed Point Problem (FPP) consists of finding a point $x^* \in K$ such that

$$x^* = Sx^*, \tag{1.4}$$

where $K$ is a nonempty, closed and convex subset of a real Hilbert space $H$ and $S : K \to K$ is a nonlinear mapping. We denote by $Fix(S)$, the fixed point set

of a mapping $S$. The FPP finds application in proving the existence of solution of many nonlinear problems arising in many real life problems. From the existence of solution of differential equation to integral equations and evolutionary equations. The approximation of fixed points of several nonlinear operators in Hilbert, Banach and Hadamard spaces have been considered in the literature (see [18, 20, 26, 36, 53]).
In this paper, we consider the problem of finding a common solution of the VIP (1.1) and FPP (1.4). That is, finding a point $x^* \in K$ such that

$$x^* \in VIP(K, F) \cap Fix(S). \tag{1.5}$$

The problem (1.5) has many real life applications which include signal recovery problems, beam-forming problems, power-control problems, bandwith allocation problems and optimal control problems (see [25, 43] and the references therein).
For obtaining a solution of (1.5) in the case where $F : H \to H$ is inverse strongly monotone and $S : K \to K$ is nonexpansive, Takahashi and Toyoda [49] introduced an algorithm whose sequence $\{x_k\}$ is generated by the following recursive formula:

$$\begin{cases} y_k = P_K(x_k - \lambda F x_k), \\ x_{k+1} = (1 - \alpha_k)x_k + \alpha_k S y_k, \end{cases} \tag{1.6}$$

where $P_K$ is the metric projection of $H$ onto $K$ and $\{\alpha_k\}$ is a sequence in $(0, 1)$ satisfying some conditions.
Kraikaew and Saejung [34], for solving problem (1.5) combined the SEGM and Halpern method to propose an algorithm they called the Halpern Subgradient Extragradient Method (HSEGM). The HSEGM is given as

$$\begin{cases} x_1 \in H, \\ y_k = P_K(x_k - \lambda F x_k), \\ T_k = \{x \in H : \langle x_k - \lambda F x_k - y_k, x - y_k \rangle \leq 0\}, \\ z_k = \alpha_k x_1 + (1 - \alpha_k) P_{T_k}(x_k - \lambda F y_k), \\ x_{k+1} = \beta_k x_k + (1 - \beta_k) S z_k, \end{cases} \tag{1.7}$$

where $\lambda \in (0, 1/L)$, $\alpha_k \subset (0, 1)$ satisfying $\lim_{k\to\infty} \alpha_k = 0$, $\sum_{k=1}^{\infty} \alpha_k = \infty$, $\{\beta_k\} \subset [a, b] \subset (0, 1)$ and $S : H \to H$ is a quasi-nonexpansive mapping.
Recently, Thong and Hieu [50] introduced two viscosity-extragradient algorithms for approximating (1.5), where $S : H \to H$ is a $\eta$-demicontractive mapping and $F : H \to H$ is a $L$-Lipschitz monotone operator. The strong convergence of both algorithms were established under some mild conditions. One of these algorithms is presented as follows:

**Algorithm 1.1.** [50, Algorithm 3.1], *Viscosity-type Subgradient Extragradient Method (VSEM)*

**Initialization:** *Choose* $\lambda_0 > 0$, $\mu \in (0, 1)$, *and let* $x_0 \in K$ *be an arbitrary starting point.*

**Iterative steps:** *Calculate* $x_{k+1}$ *as follows:*

Step 1: *Compute*

$$y_k = P_K(x_k - \lambda_k F x_k).$$

Step 2: *Compute*

$$z_k = P_{T_k}(x_k - \lambda_k F y_k),$$

*where*

$$T_k = \{w \in H : \langle x_k - \lambda_k F x_k - y_k, w - y_k\rangle \leq 0\}.$$

Step 3: *Compute*

$$\begin{cases} v_k = (1-\beta_k)z_k + \beta_k S z_k, \\ x_{n+1} = \alpha_k f(x_k) + (1-\alpha_k)v_k \end{cases}$$

*and*

$$\lambda_{k+1} = \begin{cases} \min\{\frac{\mu\|w_k - y_k\|}{\|Fw_k - Fy_k\|},\ \lambda_k\} & \text{if } Fw_k - Fy_k \neq 0, \\ \lambda_k, & \text{otherwise.} \end{cases}$$

**Stopping criterion** *Set* $k := k+1$ *and return to* **Step 1**.

To speed up the convergence of iterative algorithm, the inertial technique has been widely employed (see [3, 8, 16, 38, 39, 47]). Inertial algorithms for variational inequality and other optimization problems have received due consideration by authors, see, e,g [51]. Very recently, Thong et al. [51] proposed the following inertial subgradient method:

**Algorithm 1.2.** *Inertial subgradient algorithm for VIP*

**Initialization:** *Choose* $\lambda_1 > 0$, $\mu \in (0,1)$, $\theta > 0$ *and let* $x_0, x_1 \in K$ *be an arbitrary starting point.*

**Iterative steps:** *Calculate* $x_{k+1}$ *as follows:*

Step 1: *Given* $x_k, x_{k-1}$, $k \geq 1$. *Set*

$$w_k = x_k + \theta_k(x_k - x_{k-1}),$$

*where*

$$\theta_k = \begin{cases} \min\left\{\dfrac{1}{k^2\|x_k - x_{k-1}\|^2}, \theta\right\} & \text{if } x_k \neq x_{k-1}, \\ \theta & \text{otherwise.} \end{cases}$$

Step 2: *Calculate*

$$y_k = P_K(w_k - \lambda_k F w_k).$$

*If* $y_k = w_k$ *or* $Fy_k = 0$ *then stop* ($y_k$ *is the solution of the VIP* (1.1) *). Otherwise go to* **Step 3**.

Step 3: *Compute*

$$z_k = P_{T_k}(w_k - \lambda_k F y_k),$$

*where*

$$T_k = \{w \in H : \langle w_k - \lambda_k F w_k - y_k, w - y_k\rangle \leq 0\}.$$

Step 4 *Compute*

$$x_{k+1} = \alpha_k f(z_k) + (1-\alpha_k)z_k.$$

*Update*

$$\lambda_{k+1} = \begin{cases} \min\left\{\frac{\mu\|w_k - y_k\|}{\|Fw_k - Fy_k\|},\ \lambda_k\right\} & \text{if } Fw_k - Fy_k \neq 0, \\ \lambda_k, & \text{otherwise.} \end{cases}$$

*Set $k := k+1$ and return to* **Step1**.

In this paper, motivated by the works of Attouch and Alvarez [8], Censor et al. [14] and [51], we proposed an inertial self adaptive subgradient extragradient algorithm for approximating a solution of VIP and FPP in real Hilbert space. Combining this method with a modified viscosity approach, we proved a strong convergence theorem for approximating the solution of a pseudomonotone VIP and FPP for $\eta$-demimetric mapping. The following highlight some of the advantages of our method and work over previous ones in the literature.

(i) Unlike the work of Gang et al. [11] where the linesearch rule (a linesearch means that at each outer iteration, an inner loop is executed until some finite stopping criterion is reached which can be time consuming) was employed, we used a carefully chosen self adaptive step size.
(ii) Also, by using self adaptive step size, our work does not depend on the prior knowledge of the Lipschitz constant in practice which makes the execution of the algorithm easy for computation.
(iii) Our algorithm is used for approximating a common solution of a VIP for pseudomonotone operator and a fixed point of an $\eta$-demimetric mapping thus including the work of [51] as a special consideration.
(iv) We employed an inertial technique to speed up the convergence rate of the sequence generated by our method. Our numerical experiments confirm that our method perform better than some existing methods in literature.

The paper is organized as follows: In Section 2, we present some preliminary results and definitions that are useful in establishing our main result. We present the main result in Section 3, by first introducing our algorithm and then establishing the strong convergence of the sequence generated by this algorithm. In Section 4, we give two theoretical applications of our main result. We reported some numerical experiments in Section 5 to demonstrate the performance of our method as well as comparing it with some related methods in the literature. Finally, in Section 6, we gave a conclusion of the paper.

## 2. Preliminaries

Throughout this paper, we denote the set of positive integers and the set of real numbers by $\mathbb{N}$ and $\mathbb{R}$ respectively. Let $H$ be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm given by $\|\cdot\|$ respectively. For a sequence $\{x_k\} \subset H$, we denote the weak and strong convergence of $\{x_k\}$ to a point $x \in H$ by $x_k \rightharpoonup x$ and $x_k \to x$ respectively.

Let $K$ be a nonempty, closed and convex subset of a real Hilbert space $H$. A mapping $S : K \to K$ is said to be:

(i) $L$-Lipschitz with a constant $L > 0$, if
$$\|Sx - Sy\| \leq L\|x - y\|, \ \forall\ x, y \in H;$$
(ii) a contraction respectively nonexpansive if $L \in (0, 1)$ respectively $L = 1$;
(iii) firmly nonexpansive, if
$$\langle Sx - Sy, x - y\rangle \geq \|Sx - Sy\|^2, \ \forall\ x, y \in H;$$
(iv) quasi-nonexpansive, if $Fix(S) \neq \emptyset$ and
$$\|Sx - Sx^*\| \leq \|x - x^*\|,$$
for any $x \in H$ and $x^* \in Fix(S)$;
(v) $k$-strictly pseudocontractive in the sense of Browder and Petryshyn [9], if there exists $k \in [0, 1)$, such that
$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|x - y - (Sx - Sy)\|^2, \ \forall\ x, y \in H;$$
(vi) [41]. $\eta$-demimetric with $\eta \in (-\infty, 1)$ , if $Fix(S) \neq \emptyset$ and
$$\langle x - x^*, x - Sx\rangle \geq \frac{1}{2}(1 - \eta)\|x - Sx\|^2, \text{for any } x \in K \text{ and } x^* \in Fix(S).$$
Equivalently, $S$ is $\eta$-demimetric, if there exists $\eta \in (-\infty, 1)$ such that
$$||Sx - x^*||^2 \leq ||x - x^*||^2 + \eta||x - Tx||^2, \quad \forall x \in H \quad \text{and} \quad x^* \in Fix(S).$$

**Remark 2.1.** [41]. The class $\eta$-demimetric mappings covers the class of strictly pseudo-contractive mappings with nonempty fixed points and many other important nonlinear mappings.

For each $x, y \in H$ and $t \in (0, 1)$, it is known that
$$||x + y||^2 \leq ||x||^2 + 2\langle y, x + y\rangle$$
and
$$||tx + (1 - t)y||^2 = t||x||^2 + (1 - t)||y||^2 - t(1 - t)||x - y||^2, \quad \text{(see, [28, 37])}.$$
Let $K$ be a nonempty, closed and convex subset of a real Hilbert space $H$. For every point $x \in H$, there exists a unique nearest point $P_K x \in K$, such that
$$||x - P_K x|| \leq ||x - y||, \quad \forall\ y \in K.$$
$P_K$ is called the metric projection (also nearest point mapping) of $H$ onto $K$, see [17, 29].

**Lemma 2.2.** [40]. *Let $K$ be a nonempty, closed and convex subset of a real Hilbert space $H$. Given $x \in H$ and $z \in K$. Then*
$$z = P_K x \iff \langle x - z, z - y\rangle \geq 0, \ \forall\ y \in K.$$

**Lemma 2.3.** [32, 40]. *Let $K$ be be a nonempty, closed and convex subset of a real Hilbert space $H$. Given $x \in H$, then*

(a) $\|P_K x - P_K y\| \leq \langle P_K x - P_K y, x - y\rangle, \ \forall\ y \in K;$

(b) $\|x-y\| - \|x - P_K x\| \geq \|P_K x - y\|$;
(c) $\|(I-P_K)x - (I-P_K)y\|^2 \leq \langle (I-P_K)x - (I-P_K)y, x-y\rangle, \ \forall\, y \in K$.

**Lemma 2.4.** [32, Lemma 2.1]. *Consider $VIP(K,F)$ (1.1) with $K$ being a nonempty, closed and convex subset of a real Hilbert space $H$ and $F : K \to H$ being a pseudomonotone and continuous operator. Then $x^* \in VIP(K,F)$ if and only if*

$$\langle Fx, x - x^*\rangle \geq 0, \ \forall\, x \in K.$$

**Lemma 2.5.** [9]. *Let $H$ be a real Hilbert space and $S : H \to H$ be a $\eta$-demimetric mapping with $(-\infty, 1)$ such that $F(S) \neq \emptyset$. $S_\eta x := (1-\eta)x + \eta Sx$. Then, $S_\eta$ is a quasi-nonexpansive mapping and $F(S_\eta) = F(S)$.*

**Lemma 2.6.** [49]. *Let $\{\alpha_k\}$ be a sequence of nonnegative real numbers satisfying*

$$\alpha_{k+1} \leq (1-\gamma_k)\alpha_k + \delta_k,$$

*where $\{\gamma_k\}$ is a sequence in (0,1) and $\delta_k$ is a sequence such that*

(i) $\sum\limits_{k=1}^{\infty} \gamma_k = \infty$ *and* $\lim\limits_{k\to\infty} \gamma_k = 0$;
(ii) $\sum\limits_{k=1}^{\infty} |\delta_k| < \infty$ *and* $\lim\limits_{k\to\infty} \dfrac{\delta_k}{\gamma_k} \leq 0$.

*Then $\alpha_k \to 0$ as $k \to \infty$.*

**Lemma 2.7.** [42, 46] *Let $\{\Upsilon_k\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\Upsilon_{k_j}\}$ of $\{\Upsilon_k\}$ such that $\Upsilon_{k_j} < \Upsilon_{k_j+1}$ for all $j \geq 0$. Also consider the sequence of integers $\{\tau(k)\}_{k\geq k_0}$ defined by*

$$\tau(k) = \max\{n \leq k : \Upsilon_k < \Upsilon_{k+1}\}.$$

*Then, $\{\tau(k)\}_{k\geq k_0}$ is a nondecreasing sequence verifying $\lim\limits_{k\to\infty} \tau(k) = \infty$ and, for all $k \geq k_0$,*

$$\max\{\Upsilon_{\tau(k)}, \Upsilon_k\} \leq \Upsilon_{\tau(k)+1}.$$

## 3. Main result

In this section, we present our main result of this paper.
For the convergence of our method, we assume the following conditions:

**Assumption 3.1.**

(C1) *The feasible set $K$ is nonempty, closed and convex on $H$.*
(C2) *The mapping $F : H \to H$ is pseudomonotone, L-Lipschitz continuous on $H$ and sequentially weakly continuous on $K$.*
(C3) *The solution set $\Gamma = VIP(K,F) \cap Fix(S)$ is nonempty, where $S : H \to H$ is an $\eta$-demimetric mapping.*

In addition to this, we assume that $\{\tau_k\}$ as used in Algorithm 3.2 is a positive sequence such that $\lim\limits_{k\to\infty} \dfrac{\tau_k}{\alpha_k} = 0$ (that is $\tau_k = \circ(\alpha_k)$), where $\{\alpha_k\} \subset (0,1)$ such that

(C4) $\lim\limits_{k\to\infty} \alpha_k = 0$ and $\sum\limits_{k=1}^{\infty} \alpha_k = \infty$,

(C5) $\alpha_k + \beta_k + \gamma_k = 1$.

**Algorithm 3.2.** *Iterative Algorithm*

**Initialization:** *Let $f : K \to K$ be a $\kappa$-contractive mapping. Choose $\lambda_1 > 0$, $\eta_k \subset (0,1)$, $\mu \in (0,1)$, $\theta > 0$ and let $x_0, x_1 \in K$ be an arbitrary starting point.*

**Iterative steps:** *Given $x_k, x_{k-1}$, choose $\theta_k$ such that $0 \le \theta_k \le \bar{\theta}_k$, where*

$$\bar{\theta}_k = \begin{cases} \min\left\{\theta, \dfrac{\tau_k}{\|x_k - x_{k-1}\|}\right\} & \text{if } x_k \neq x_{k-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{3.1}$$

*Calculate $x_{k+1}$ and $\lambda_k$ for each $k \ge 1$ as follows:*

Step 1: *Compute*

$$w_k = x_k + \theta_k(x_k - x_{k-1}). \tag{3.2}$$

Step : *Calculate*

$$y_k = P_K(w_k - \lambda_k F w_k). \tag{3.3}$$

Step 2: *Compute*

$$z_k = P_{T_k}(w_k - \lambda_k F y_k), \tag{3.4}$$

*where*

$$T_k = \{w \in H : \langle w_k - \lambda_k F w_k - y_k, w - y_k\rangle \le 0\}.$$

Step 3: *We obtain $x_{k+1}$ by*

$$x_{k+1} = \alpha_k f(x_k) + \beta_k x_k + \gamma_k S_{\eta_k} z_k \tag{3.5}$$

*and*

$$\lambda_{k+1} = \begin{cases} \min\left\{\frac{\mu\|w_k - y_k\|}{\|Fw_k - Fy_k\|},\ \lambda_k\right\} & \text{if } Fw_k - Fy_k \neq 0, \\ \lambda_k, & \text{otherwise.} \end{cases}$$

**Stopping criterion** *If $x_{k+1} = w_k = y_k = Sz_k$ for some $k \ge 1$ then stop. Otherwise set $k := k+1$ and return to* **Iterative step**.

The following result was stated and proved in [52]. It is easy to adapt for our situation. We state the lemma without proof.

**Lemma 3.3.** [52]. *The sequence $\{\lambda_k\}$ defined in Algorithm 3.2 is a nonincreasing sequence and*

$$\lim_{k\to\infty} \lambda_k = \lambda \ge \min\left\{\lambda_1, \frac{\mu}{L}\right\}.$$

The following is required for establishing the solution of the VIP (1.1).

**Lemma 3.4.** *Assume that Assumption 3.1 hold and $\{w_k\}$ is a sequence generated by Algorithm 3.2. If there exists a subsequence $\{w_{k_j}\}$ of $\{w_k\}$ convergent weakly to a point $\bar{x} \in H$ and $\lim\limits_{j\to\infty} \|w_{k_j} - y_{k_j}\| = 0$, then $\bar{x} \in VIP(K,F)$.*

*Proof.* First we show that $\liminf_{j\to\infty}\langle Fy_{k_j}, z-y_{k_j}\rangle \geq 0$. Indeed, we have by the definition of $\{y_k\}$ and Lemma 2.2, that

$$\langle w_{k_j} - \lambda_{k_j} F w_{k_j} - y_{k_j}, z - y_{k_j}\rangle \leq 0,\ \forall\ z \in K,$$

which implies

$$\frac{1}{\lambda_{k_j}}\langle w_{k_j} - y_{k_j}, z - y_{k_j}\rangle \leq \langle Fw_{k_j}, z - y_{k_j}\rangle\ \forall\ z \in K.$$

Consequence of this, we get that

$$\frac{1}{\lambda_{k_j}}\langle w_{k_j} - y_{k_j}\rangle + \langle Fw_{k_j}, y_{k_j} - w_{k_j}\rangle \leq \langle Fw_{k_j}, z - w_{k_j}\rangle,\ \forall\ z \in K. \tag{3.6}$$

Since $\{w_{k_j}\}$ is convergent, it is bounded. Then, since $F$ is Lipschitz continuous, $\{Fw_{k_j}\}$ is bounded. We obtain also that $\{y_{k_j}\}$ is bounded since $\|w_{k_j} - y_{k_j}\| \to 0$ as $j \to \infty$ and $\lambda_{k_j} \geq \min\left\{\lambda_1, \frac{\mu}{L}\right\}$. Passing limit over (3.6) as $j \to \infty$, we obtain

$$\liminf_{j\to\infty}\langle Fw_{k_j}, z - w_{k_j}\rangle \geq 0.$$

Observe that

$$\langle Fw_{k_j}, z - y_{k_j}\rangle = \langle Fy_{k_j} - Fw_{k_j}, z - y_{k_j}\rangle + \langle Fy_{k_j}, z - w_{k_j}\rangle + \langle Fy_{k_j}, w_{k_j} - y_{k_j}\rangle. \tag{3.7}$$

We obtain from $\lim_{j\to\infty}\|w_{k_j} - y_{k_j}\| = 0$ and the Lipschitz continuity of $F$, that $\lim_{j\to\infty}\|Fw_{k_j} - Fy_{k_j}\| = 0$. Thus, we get from (3.7), that

$$\liminf_{j\to\infty}\langle Fy_{k_j}, z - y_{k_j}\rangle \geq 0.$$

Next we show that $\bar{x} \in VIP(K,F)$. We choose a subsequence $\{\epsilon_j\}$ of positive numbers decreasing such that $\epsilon_j \to 0$ as $j \to \infty$. For each $j$, let $N_j$ be the smallest nonnegative integer such that

$$\langle Fy_{k_i}, z - y_{k_i}\rangle + \epsilon_j \geq 0,\ \forall\ i \geq N_j. \tag{3.8}$$

Since $\{\epsilon_j\}$ is decreasing, it is obvious that $N_j$ is increasing. Further, for each $j \in \mathbb{N}$, $\{y_{N_j}\} \subset K$. Suppose $Fy_{N_j} \neq 0$ so that $y_{N_j}$ is not a solution of the $VIP(K,F)$, set

$$\nu_{N_j} = \frac{Fy_{N_j}}{\|Fy_{N_j}\|^2},$$

so that $\langle Fy_{N_j}, \nu_{N_j}\rangle = 1$ for each $j$. We see from this and (3.8), that

$$\langle Fy_{N_j}, z + \epsilon_j \nu_{N_j} - y_{N_j}\rangle \geq 0.$$

Since $F$ is pseudomonotone on $H$, we have

$$F(z + \epsilon_j \nu_{N_j}), z + \epsilon_j \nu_{N_j} - y_{N_j}\rangle \geq 0$$

and thus

$$\langle Fz, z - y_{N_j}\rangle \geq \langle Fz - F(z + \epsilon_j \nu_{N_j}), z + \epsilon_j \nu_{N_j} - y_{N_j}\rangle - \epsilon_j\langle Fz, \nu_{N_j}\rangle. \tag{3.9}$$

Now, we show that $\epsilon_j \nu_{N_j} \to 0$ as $j \to \infty$. To see this, from the hypothesis we get that $y_{N_j} \rightharpoonup \bar{x}$ as $j \to \infty$. By $\{y_k\} \subset K$, we have that $\bar{x} \in K$. Since $F$ is sequentially weakly continuous on $K$, we have $Fy_{N_j} \rightharpoonup F\bar{x}$. Suppose that $F\bar{x} \neq 0$ so that $\bar{x} \in VIP(K,F)$. Since $\|\cdot\|$ is sequentially weakly continuous, we have

$$0 < \|F\bar{x}\| \leq \liminf_{j\to\infty} \|Fy_{N_j}\|.$$

From $\{y_{N_j}\} \subset \{y_{k_j}\}$ and $\epsilon_j \to 0$ as $j \to \infty$, we have

$$0 \leq \lim_{j\to\infty} \|\epsilon_j \nu_{N_j}\| = \lim_{j\to\infty} \left(\frac{\epsilon_j}{\|Fy_{k_j}\|}\right) \leq \frac{0}{\|F\bar{x}\|} = 0,$$

which shows that $\epsilon_j \nu_{N_j} \to 0$. Now letting $j \to \infty$, we obtain by the continuity of $F$ that the right hand side of (3.9) tends to zero, $\{w_{N_j}\}$, $\{\nu_{N_j}\}$ are bounded and $\lim_{j\to\infty} \epsilon_j \nu_{N_j} = 0$. Therefore,

$$\liminf_{j\to\infty} \langle Fz, z - y_{N_j}\rangle \geq 0.$$

Hence for all $z \in K$, we have

$$\langle Fz, z - \bar{x}\rangle = \lim_{j\to\infty} \langle Fz, z - y_{N_j}\rangle = \liminf_{j\to\infty} \langle Fz, z - y_{N_j}\rangle \geq 0.$$

By Lemma 2.4 we have $\bar{x} \in VIP(K,F)$. The proof is thus complete.

**Lemma 3.5.** *Let $\{z_k\}$ be given as in Algorithm 3.2 and $x^* \in \Gamma$, then there holds the inequality*

$$\|z_k - x^*\|^2 \leq \|w_k - x^*\|^2 - \left(1 - \frac{\lambda_k}{\lambda_{k+1}}\right)[\|w_k - y_k\|^2 + \|y_k - z_k\|^2]. \tag{3.10}$$

*Proof.* Using Lemma 2.3 and (3.4), we have

$$\begin{aligned}
\|z_k - x^*\|^2 &= \|P_{T_k}(w_k - \lambda_k F y_k) - x^*\|^2 \\
&\leq \|w_k - \lambda_k F y_k - x^*\|^2 - \|w_k - \lambda_k F y_k - z_k\|^2 \\
&= \|w_k - x^*\|^2 - 2\lambda_k \langle w_k - x^*, Fy_k\rangle - \|w_k - z_k\|^2 + 2\lambda_k \langle w_k - z_k, Fy_k\rangle \\
&= \|w_k - x^*\|^2 - \|w_k - z_k\|^2 - 2\lambda_k \langle z_k - x^*, Fy_k\rangle \\
&= \|w_k - x^*\|^2 - \|w_k - z_k\|^2 - 2\lambda_k \langle z_k - y_k, Fy_k\rangle - 2\lambda_k \langle y_k - x^*, Fy_k\rangle \\
&= \|w_k - x^*\|^2 - \|w_k - y_k + y_k - z_k\|^2 \\
&- 2\lambda_k \langle z_k - y_k, Fy_k\rangle - 2\lambda_k \langle y_k - x^*, Fy_k\rangle \\
&= \|w_k - x^*\|^2 - \|w_k - y_k\|^2 - \|y_k - z_k\|^2 + \langle z_k - y_k, w_k - y_k\rangle \\
&- 2\lambda_k \langle z_k - y_k, Fy_k\rangle - 2\lambda_k \langle y_k - x^*, Fy_k\rangle \\
&= \|w_k - x^*\|^2 - \|w_k - y_k\|^2 - \|y_k - z_k\|^2 \\
&- 2\lambda_k \langle y_k - z_k, w_k - \lambda_k F y_k - y - k\rangle - 2\lambda_k \langle y_k - x^*, Fy_k\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \|w_k - x^*\|^2 - \|w_k - y_k\|^2 - \|y_k - z_k\|^2 \\
&- 2\langle y_k - z_k, w_k - \lambda_k F w_k - y_k\rangle + 2\lambda_k \langle z_k - y_k, F w_k - F y_k\rangle \\
&- 2\lambda_k \langle y_k - x^*, F y_k\rangle \\
&= \|w_k - x^*\|^2 - \|w_k - y_k\|^2 - \|y_k - z_k\|^2 \\
&- 2\langle y_k - z_k, w_k - \lambda_k F w_k - y_k\rangle + 2\lambda_k \|z_k - y_k\|\|F w_k - F y_k\| \\
&- 2\lambda_k \langle y_k - x^*, F y_k\rangle \\
&= \|w_k - x^*\|^2 - \|w_k - y_k\|^2 - \|y_k - z_k\|^2 \\
&- 2\langle y_k - z_k, w_k - \lambda_k F w_k - y_k\rangle + 2\frac{\lambda_k}{\lambda_{k+1}} \|z_k - y_k\|\|F w_k - F y_k\| \\
&- 2\lambda_k \langle y_k - x^*, F y_k\rangle \\
&\leq \|w_k - x^*\|^2 - \|w_k - y_k\|^2 - \|y_k - z_k\|^2 \\
&- 2\langle y_k - z_k, w_k - \lambda_k F w_k - y_k\rangle - 2\lambda_k \langle y_k - x^*, F y_k\rangle \\
&+ \frac{\lambda_k}{\lambda_{k+1}} (\|z_k - y_k\|^2 + \|F w_k - F y_k\|^2) \\
&= \|w_k - x^*\|^2 - \left(1 - \frac{\lambda_k}{\lambda_{k+1}}\right) [\|w_k - y_k\|^2 + \|y_k - z_k\|^2] \\
&- 2\langle y_k - z_k, w_k - \lambda_k F w_k - y_k\rangle - 2\lambda_k \langle y_k - x^*, F y_k\rangle. \qquad (3.11)
\end{aligned}
$$

Since $x^* \in \Gamma$, $y_k \in K$ and the fact that $F$ is pseudomonotone we have that

$$\langle y_k - x^*, F x^*\rangle \geq 0$$

which implies

$$\langle y_k - x^*, F y_k\rangle \geq 0.$$

Also from $z_k \in T_k$, we get that

$$\langle y_k - z_k, w_k - \lambda_k F w_k - y_k\rangle \geq 0.$$

Therefore, we obtain from (3.11) that

$$\|z_k - x^*\|^2 \leq \|w_k - x^*\|^2 - \left(1 - \frac{\lambda_k}{\lambda_{k+1}}\right) [\|w_k - y_k\|^2 + \|y_k - z_k\|^2], \qquad (3.12)$$

as required.

**Lemma 3.6.** *The sequence $\{x_k\}$ generated by Algorithm 3.2 is bounded.*

*Proof.* From $x^* \in \Gamma$ and (3.2), we have

$$
\begin{aligned}
\|w_k - x^*\| &\|x_k + \theta_k (x_k - x_{k-1}) - x^*\| \\
&\leq \|x_k - x^*\| + \theta_k \|x_k - x_{k-1}\| \\
&= \|x_k - x^*\| + \alpha_k \cdot \frac{\theta_k}{\alpha_k} \|x_k - x_{k-1}\|.
\end{aligned}
$$

Since $\dfrac{\theta_k}{\alpha_k}\|x_k - x_{k-1}\| \to 0$, there exists $M_1 > 0$ such that

$$\frac{\theta_k}{\alpha_k}\|x_k - x_{k-1}\| \leq M_1,\ \ k \geq 1,$$

hence

$$\|w_k - x^*\| \leq \|x_k - x^*\| + \alpha_k M_1.$$

It is easy to see from Lemma 3.5, that

$$\|z_k - x^*\| \leq \|w_k - x^*\| \leq \|x_k - x^*\| + \alpha_k M_1.$$

Furthermore, from (3.5), we have

$$\begin{aligned}
\|x_{k+1} - x^*\| &= \|\alpha_k f(x_k) + \beta_k x_k + \gamma_k S_{\eta_k} z_k - x^*\| \\
&\leq \alpha_k\|f(x_k) - x^*\| + \beta_k\|x_k - x^*\| + \gamma_k\|S_{\eta_k} z_k - x^*\| \\
&\leq \alpha_k\|f(x_k) - f(x^*)\| + \alpha_k\|f(x^*) - x^*\| + \beta_k\|x_k - x^*\| + \gamma_k\|z_k - x^*\| \\
&\leq \alpha_k\kappa\|x_k - x^*\| + \beta_k\|x_k - x^*\| \\
&+ \alpha_k\|f(x^*) - x^*\| + \gamma_k(\|x_k - x^*\| + \alpha_k M_1) \\
&= \alpha_k\kappa\|x_k - x^*\| + \beta_k\|x_k - x^*\| \\
&+ \alpha_k\|f(x^*) - x^*\| + \gamma_k\|x_k - x^*\| + \gamma_k\alpha_k M_1 \\
&= \alpha_k\kappa\|x_k - x^*\| + (1-\alpha_k)\|x_k - x^*\| + \alpha_k\|f(x^*) - x^*\| + \gamma_k\alpha_k M_1 \\
&= [1 - \alpha_k(1-\kappa)]\|x_k - x^*\| + \alpha_k\|f(x^*) - x^*\| + \gamma_k\alpha_k M_1 \\
&\leq \max\left\{\|x_k - x^*\|, \frac{\|f(x^*) - x^*\| + \theta_k\alpha_k M_1}{1-\kappa}\right\} \\
&\leq \vdots \\
&\leq \max\left\{\|x_0 - x^*\|, \frac{\|f(x^*) - x^*\| + \theta_k\alpha_k M_1}{1-\kappa}\right\},\ \forall\ k \geq 1. \qquad (3.13)
\end{aligned}$$

Therefore the sequence $\{x_k\}$ is bounded. Consequently, the sequences $\{z_k\}$, $\{y_k\}$ and $\{Sz_k\}$ are bounded.

**Lemma 3.7.** *Let $\{x_k\}$ be the sequence generated by Algorithm 3.2. Then, for $x^* \in \Gamma$, it holds that*

$$\begin{aligned}
\|x_{k+1} - x^*\|^2 \leq &\left(1 - \frac{2\alpha_k(1-\kappa)}{(1-\alpha_k\kappa)}\right)\|x_k - x^*\|^2 \\
&+ \frac{2\alpha_k(1-\kappa)}{(1-\alpha_k\kappa)}\left(\frac{\alpha_k}{1-\kappa}\|x_k - x^*\|^2 + \frac{1}{1-\kappa}\langle f(x^*) - x^*, x_{k+1} - x^*\rangle\right. \\
&\left. + \frac{\theta_k\gamma_k}{\alpha_k(1-\kappa)}\|x_k - x^*\|\|x_k - x_{k-1}\| + \frac{\theta_k^2}{2\alpha_k(1-\kappa)}\|x_k - x_{k-1}\|^2\right). \qquad (3.14)
\end{aligned}$$

*Proof.* From (3.5) and $x^* \in \Gamma$, we have

$$\begin{aligned}
\|x_{k+1} - x^*\|^2 &\le \|\alpha_k f(x_k) + \beta_k x_k + \gamma_k S_{\eta_k} z_k - x^*\| \\
&\le \|\beta_k(x_k - x^*) + \gamma_k(S_{\eta_k} z_k - x^*)\|^2 + 2\alpha_k\langle f(x_k) - x^*, x_{k+1} - x^*\rangle \\
&\le \|\beta_k(x_k - x^*) + \gamma_k(z_k - x^*)\|^2 + 2\alpha_k\langle f(x_k) - x^*, x_{k+1} - x^*\rangle \\
&\le \|\beta_k(x_k - x^*) + \gamma_k(w_k - x^*)\|^2 + 2\alpha_k\langle f(x_k) - x^*, x_{k+1} - x^*\rangle \\
&\le [\beta_k\|x_k - x^*\| + \gamma_k(\|x_k - x^*\| + \theta_k\|x_k - x_{k-1}\|)]^2 \\
&+ 2\alpha_k\langle f(x_k) - f(x^*), x_{k+1} - x^*\rangle \\
&+ 2\alpha_k\langle f(x^*) - x^*, x_{k+1} - x^*\rangle \\
&\le [\beta_k\|x_k - x^*\| + \gamma_k\|x_k - x^*\| + \gamma_k\theta_k\|x_k - x_{k-1}\|]^2 \\
&+ 2\alpha_k\kappa\|x_k - x^*\| + \|x_{k+1} - x^*\| \\
&+ 2\alpha_k\langle f(x^*) - x^*, x_{k+1} - x^*\rangle \\
&\le [(1-\alpha_k)\|x_k - x^*\| + \theta_k\gamma_k\|x_k - x_{k-1}\|]^2 \\
&+ \alpha_k\kappa(\|x_k - x^*\|^2 + \|x_{k+1} - x^*\|) \\
&+ 2\alpha_k\langle f(x^*) - x^*, x_{k+1} - x^*\rangle \\
&\le (1-\alpha_k)^2\|x_k - x^*\|^2 + 2\theta_k\gamma_k(1-\alpha_k)\|x_k - x^*\|\|x_k - x_{k-1}\| \\
&+ \gamma_k\theta_k\|x_k - x_{k-1}\|^2 \\
&+ \alpha_k\kappa(\|x_k - x^*\|^2 + \|x_{k+1} - x^*\|) + 2\alpha_k\langle f(x^*) - x^*, x_{k+1} - x^*\rangle,
\end{aligned}$$

this implies that

$$\begin{aligned}
\|x_{k+1} - x^*\|^2 &\le \frac{(1-\alpha_k)^2 + \alpha_k\kappa}{1-\alpha_k\kappa}\|x_k - x^*\|^2 \\
&+ \frac{2\alpha_k}{1-\alpha_k\kappa}\langle f(x^*) - x^*, x_{k+1} - x^*\rangle + \frac{\theta_k}{1-\alpha_k\kappa}\|x_k - x_{k-1}\|^2 \\
&+ \frac{2\gamma_k\theta_k}{1-\alpha_k\kappa}\|x_k - x^*\|\|x_k - x_{k-1}\| \\
&= \frac{1 - 2\alpha_k + \alpha_k\kappa}{1-\alpha_k\kappa}\|x_k - x^*\|^2 \\
&+ \frac{\alpha_k^2}{1-\alpha_k\kappa}\|x_k - x^*\|^2 + \frac{2\alpha_k}{1-\alpha_k\kappa}\langle f(x^*) - x^*, x_{k+1} - x^*\rangle \\
&+ \frac{\theta_k}{1-\alpha_k\kappa}\|x_k - x_{k-1}\|^2 + \frac{2\gamma_k\theta_k}{1-\alpha_k\kappa}\|x_k - x^*\|\|x_k - x_{k-1}\| \\
&= \left(1 - \frac{2\alpha_k(1-\kappa)}{1-\alpha_k\kappa}\right)\|x_k - x^*\|^2 \\
&+ \frac{2\alpha_k^2(1-\kappa)}{2(1-\alpha_k\kappa)(1-\kappa)}\|x_k - x^*\|^2 + \frac{2\theta_k^2(1-\kappa)}{2(1-\alpha_k\kappa)(1-\kappa)}\|x_k - x_{k-1}\|^2 \\
&+ \frac{2\alpha_k(1-\kappa)}{(1-\alpha_k\kappa)(1-\kappa)}\langle f(x^*) - x^*, x_{k+1} - x^*\rangle
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2\theta_k\gamma_k(1-\kappa)}{(1-\alpha_k)(1-\kappa)}\|x_k-x^*\|\|x_k-x_{k-1}\| \\
&=\left(1-\frac{2\alpha_k(1-\kappa)}{1-\alpha_k\kappa}\right)\|x_k-x^*\|^2 \\
&+\frac{2\alpha_k(1-\kappa)}{(1-\alpha_k\kappa)}\left(\frac{\alpha_k}{2(1-\kappa)}\|x_k-x^*\|^2+\frac{\theta_k^2}{\alpha_k(1-\kappa)}\|x_k-x_{k-1}\|^2\right. \\
&\left.+\frac{\theta_k\gamma_k}{\alpha_k(1-\kappa)}\|x_k-x^*\|\|x_k-x_{k-1}\|+\frac{1}{(1-\kappa)}\langle f(x^*)-x^*,x_{k+1}-x^*\rangle\right). \qquad (3.15)
\end{aligned}
$$

**Theorem 3.8.** *Assume that condition $C1-C5$ hold. Then the sequence $\{x_k\}$ generated by Algorithm 3.2 converges to a common solution $x^* \in \Gamma$, which is also a unique solution of the variational inequality*

$$\langle f(x^*)-x^*, x^*-\bar{x}\rangle \geq 0, \ \forall\ \bar{x}\in\Gamma.$$

*Proof.* Let $x^*\in\Gamma$, the proof of this theorem is divided into two cases.
**Case I:** Suppose there exists $k_0\in\mathbb{N}$ such that $\|\{x_k-x^*\|\}$ is monotonically non-increasing. Then, by Lemma 3.6, it follows that $\|\{x_k-x^*\|\}$ is a convergent sequence and thus

$$\|x_{k-1}-x^*\|^2-\|x_k-x^*\|^2\to 0 \text{ as } k\to\infty.$$

Consider

$$
\begin{aligned}
\|w_k-x^*\|^2 &= \|x_k-x^*+\theta_k(x_k-x_{k-1})\|^2 \\
&= \|x_k-x^*\|^2+2\theta_k\langle x_k-x^*,x_k-x_{k-1}\rangle+\theta_k^2\|x_k-x_{k-1}\|^2 \\
&\leq \|x_k-x^*\|+\theta_k\|x_k-x_{k-1}\|(2\|x_k-x^*\|+\theta_k\|x_k-x_{k-1}\|). \qquad (3.16)
\end{aligned}
$$

From (3.5) and Lemma 3.10, we have

$$
\begin{aligned}
\|x_{k+1}-x^*\|^2 &\leq \alpha_k\|f(x_k)-x^*\|^2+\beta_k\|x_k-x^*\|^2+\gamma_k\|S_{\eta_k}z_k-x^*\|^2 \\
&\leq \alpha_k\|f(x_k)-x^*\|^2+\beta_k\|x_k-x^*\|^2+\gamma_k\|z_k-x^*\|^2 \\
&\leq \alpha_k\|f(x_k)-x^*\|^2+\beta_k\|x_k-x^*\|^2 \\
&+\gamma_k(\|w_k-x^*\|^2-\left(1-\frac{\lambda_k}{\lambda_{k+1}}\right)[\|w_k-y_k\|^2+\|y_k-z_k\|^2]) \\
&\leq \alpha_k\|f(x_k)-x^*\|^2+\beta_k\|x_k-x^*\|^2+\gamma_k\|x_k-x^*\|^2 \\
&+\frac{\gamma_k\theta_k\alpha_k}{\alpha_k}\|x_k-x_{k-1}\|(2\|x_k-x^*\|+\theta_k\|x_k-x_{k-1}\|) \\
&-\gamma_k\left(1-\frac{\lambda_k}{\lambda_{k+1}}\right)[\|w_k-y_k\|^2+\|y_k-z_k\|^2] \\
&= \alpha_k\|f(x_k)-x^*\|^2+\|x_k-x^*\|^2 \\
&+\frac{\gamma_k\theta_k\alpha_k}{\alpha_k}\|x_k-x_{k-1}\|(2\|x_k-x^*\|+\theta_k\|x_k-x_{k-1}\|) \\
&-\gamma_k\left(1-\frac{\lambda_k}{\lambda_{k+1}}\right)[\|w_k-y_k\|^2+\|y_k-z_k\|^2]), \qquad (3.17)
\end{aligned}
$$

which implies

$$\begin{aligned}\gamma_k\left(1-\frac{\lambda_k}{\lambda_{k+1}}\right)[\|w_k-y_k\|^2+\|y_k-z_k\|^2] \le{}& \alpha_k\|f(x_k)-x^*\|^2+\|x_k-x^*\|^2\\ &-\|x_{k+1}-x_k\|^2\\ &+\frac{\gamma_k\theta_k\alpha_k}{\alpha_k}\|x_k-x_{k-1}\|(2\|x_k-x^*\|\\ &+\theta_k\|x_k-x_{k-1}\|)\to 0 \text{ as } k\to\infty. \quad (3.18)\end{aligned}$$

Therefore, we obtain from the definition of $\lambda_k$, that

$$\lim_{k\to\infty}\|w_k-y_k\|=0=\lim_{k\to\infty}\|y_k-z_k\|. \quad (3.19)$$

Note also that

$$\|w_k-x_k\|=\theta_k\|x_k-x_{k-1}\|=\alpha_k\cdot\frac{\theta_k}{\alpha_k}\|x_k-x_{k-1}\|\to 0 \text{ as } k\to\infty. \quad (3.20)$$

It is easy to see from above that

$$\lim_{k\to\infty}\|y_k-x_k\|=0=\lim_{k\to\infty}\|z_k-x_k\|. \quad (3.21)$$

Next we show that $\|Sz_k-z_k\|\to 0$ as $k\to\infty$. From the definition of $S_{\eta_k}$ and $x^*\in\Gamma$, we have

$$\begin{aligned}\|S_{\eta_k}z_k-x^*\|^2&\|(1-\eta_k)(z_k-x^*)+\eta_k(Sz_k-x^*)\|^2\\ &(1-\eta_k)\|z_k-x^*\|^2+\eta_k\|Sz_k-x^*\|^2-\eta_k(1-\eta_k)\|Sz_k-z_k\|^2\\ &\le(1-\eta_k)\|z_k-x^*\|^2+\eta_k\|z_k-x^*\|^2-\eta_k(1-\eta_k)\|Sz_k-z_k\|^2\\ &=\|z_k-x^*\|^2-\eta_k(1-\eta_k)\|Sz_k-z_k\|^2,\end{aligned}$$

which implies from Lemma 3.10, that

$$\|S_{\eta_k}z_k-x^*\|^2\le\|w_k-x^*\|^2-\eta_k(1-\eta_k)\|Sz_k-z_k\|^2.$$

Using this in (3.5), we get

$$\begin{aligned}\|x_{k+1}-x^*\|^2&\le\alpha_k\|f(x_k)-x^*\|^2+\beta_k\|x_k-x^*\|^2+\gamma_k\|S_{\eta_k}z_k-x^*\|^2\\ &\le\alpha_k\|f(x_k)-x^*\|^2+\beta_k\|x_k-x^*\|^2\\ &+\gamma_k(\|w_k-x^*\|^2-\eta_k(1-\eta_k)\|Sz_k-z_k\|^2)\\ &\le\alpha_k\|f(x_k)-x^*\|^2+\beta_k\|x_k-x^*\|^2\\ &+\gamma_k\|w_k-x^*\|^2-\eta_k(1-\eta_k)\gamma_k\|Sz_k-z_k\|^2\\ &\le\alpha_k\|f(x_k)-x^*\|^2+\beta_k\|x_k-x^*\|^2+\gamma_k\|x_k-x^*\|^2\\ &+\gamma_k\theta_k\|x_k-x_{k-1}\|(2\|x_k-x^*\|\|x_k-x_{k-1}\|)\\ &-\eta_k(1-\eta_k)\gamma_k\|Sz_k-z_k\|^2\\ &\le\alpha_k\|f(x_k)-x^*\|^2+\|x_k-x^*\|^2\\ &+\gamma_k\theta_k\|x_k-x_{k-1}\|(2\|x_k-x^*\|\|x_k-x_{k-1}\|)\\ &-\eta_k(1-\eta_k)\gamma_k\|Sz_k-z_k\|^2. \quad (3.22)\end{aligned}$$

We obtain from this that

$$\eta_k(1-\eta_k)\gamma_k\|Sz_k-z_k\|^2 \leq \alpha_k\|f(x_k)-x^*\|^2+\|x_k-x^*\|^2-\|x_{k+1}-x^*\|^2 + \frac{\gamma_k\theta_k}{\alpha_k}\|x_k-x_{k-1}\|(2\|x_k-x^*\|\|x_k-x_{k-1}\|) \to 0 \text{ as } k\to\infty,$$

hence

$$\lim_{k\to\infty}\|Sz_k-z_k\|=0. \tag{3.23}$$

It is not difficult to obtain from this, that

$$\lim_{k\to\infty}\|S_{\eta_k}z_k-z_k\|=0. \tag{3.24}$$

Observe that

$$\|x_{k+1}-x^*\|^2 \leq \alpha_k\|f(x_k)-z_k\|^2+\beta_k\|x_k-z_k\|^2+\gamma_k\|S_{\eta_k}z_k-z_k\|^2, \tag{3.25}$$

thus, we have from (3.21), (3.24) and condition (i), that

$$\|x_{k+1}-z_k\|\to 0 \text{ as } k\to\infty.$$

Using this and (3.21), we obtain

$$\lim_{k\to\infty}\|x_{k+1}-x_k\|=\lim_{k\to\infty}(\|x_{k+1}-z_k\|+\|z_k-x_k\|)=0. \tag{3.26}$$

By the conclusion of Lemma 3.6, there exists a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ such that $\{x_{k_j}\}$ converge weakly to $\bar{x}\in H$ satisfying

$$\limsup_{k\to\infty}\langle f(x^*)-x^*,x_k-x^*\rangle=\lim_{j\to\infty}\langle f(x^*)-x^*,x_{k_j}-x^*\rangle. \tag{3.27}$$

By (3.19) and Lemma 3.5, we obtain $\bar{x}\in VIP(F,K)$. Also from (3.23), (3.24) and Lemma 2.5, we have $\bar{x}\in F(S_{\eta_k})=F(S)$. Hence $x^*\in\Gamma$. It is clear that $P_\Gamma f$ is a contraction. Using Banach's principle of contraction, $P_\Gamma f$ has a unique fixed point, say $x^*\in H$. That is $x^*=P_\Gamma f(x^*)$. It follows from Lemma 2.2, that

$$\langle f(x^*)-x^*,\bar{x}-x^*\rangle\leq 0. \tag{3.28}$$

Thus, we have that

$$\begin{aligned}\limsup_{k\to\infty}\langle f(x^*)-x^*,x_k-x^*\rangle &= \lim_{j\to\infty}\langle f(x^*)-x^*,x_{k_j}-x^*\rangle\\ &\langle f(x^*)-x^*,\bar{x}-x^*\rangle\leq 0.\end{aligned} \tag{3.29}$$

Hence by (3.26) and (3.29), we have

$$\begin{aligned}\limsup_{k\to\infty}\langle f(x^*)-x^*,x_{k+1}-x^*\rangle &\leq \limsup_{k\to\infty}\langle f(x^*)-x^*,x_{k+1}-x_k\rangle\\ &\quad+\limsup_{k\to\infty}\langle f(x^*)-x^*,x_k-x^*\rangle \leq 0.\end{aligned} \tag{3.30}$$

By applying Lemma 2.6, Lemma 3.7, and (3.30), we have $x_k\to 0$ as $k\to\infty$.
**Case II:** There exists a subsequence $\{\|x_{k_j}-x^*\|\}$ of $\{\|x_k-x^*\|\}$ such that

$$\|x_{k_j}-x^*\|^2\leq\|x_{k_j+1}-x^*\|^2$$

for all $j \in \mathbb{N}$. By Lemma 2.7, there exists a nondecreasing sequence $\{m_n\}$ of $\mathbb{N}$ such that $\lim\limits_{n\to\infty} m_n = \infty$ and there hold

$$\|x_{m_n} - x^*\|^2 \leq \|x_{m_n+1} - x^*\|^2 \quad \text{and} \quad \|x_k - x^*\|^2 \leq \|x_{m_n+1} - x^*\|^2, \ \forall\, n \in \mathbb{N}. \tag{3.31}$$

By (3.17) and Lemma 3.7, we have

$$\begin{aligned}
\|x_{m_n} - x^*\|^2 \leq \|x_{m_n+1} - x^*\|^2 &\leq \alpha_{m_n}\|f(x_{m_n}) - x^*\|^2 \\
&+ \beta_{m_n}\|x_{m_n} - x^*\|^2 + \gamma_{m_n}\Bigg(\|w_{m_n} - x^*\|^2 \\
&- \left(1 - \frac{\lambda_{m_n}}{\lambda_{m_n}+1}\right)[\|w_{m_n} - y_{m_n}\|^2 + \|y_{m_n} - z_{m_n}\|^2]\Bigg) \\
&\leq \alpha_{m_n}\|f(x_{m_n}) - x^*\|^2 + \beta_{m_n}\|x_{m_n} - x^*\|^2 + \gamma_{m_n}\|x_{m_n} - x^*\|^2 \\
&+ \gamma_{m_n}\theta_{m_n}\|x_{m_n} - x_{m_n-1}\|(2\|x_{m_n} - x^*\| + \theta_{m_n}\|x_{m_n} - x_{m_n-1}\|) \\
&- \gamma_{m_n}\left(1 - \frac{\lambda_{m_n}}{\lambda_{m_n}+1}\right)[\|w_{m_n} - y_{m_n}\|^2 + \|y_{m_n} - z_{m_n}\|^2] \\
&= \alpha_{m_n}\|f(x_{m_n}) - x^*\|^2 + (1 - \alpha_{m_n})\|x_{m_n} - x^*\| \\
&- \gamma_{m_n}\left(1 - \frac{\lambda_{m_n}}{\lambda_{m_n}+1}\right)[\|w_{m_n} - y_{m_n}\|^2 + \|y_{m_n} - z_{m_n}\|^2] \\
&+ \gamma_{m_n}\theta_{m_n}\|x_{m_n} - x_{m_n-1}\|(2\|x_{m_n} - x^*\| + \theta_{m_n}\|x_{m_n} - x_{m_n-1}\|).
\end{aligned}$$

Since $\alpha_{m_n} \to 0$ as $n \to \infty$, it follows from above that

$$\lim_{n\to\infty} \gamma_{m_n}\left(1 - \frac{\lambda_{m_n}}{\lambda_{m_n}+1}\right)[\|w_{m_n} - y_{m_n}\|^2 + \|y_{m_n} - z_{m_n}\|^2] = 0,$$

hence

$$\lim_{n\to\infty} \|w_{m_n} - y_{m_n}\| = \|y_{m_n} - z_{m_n}\| = 0. \tag{3.32}$$

By using similar arguments as in **Case I**, the following are easy to establish:

$$\lim_{n\to\infty} \|S_{\eta_{m_n}} z_{m_n} - z_{m_n}\| = \|S z_{m_n} - z_{m_n}\| = 0, \tag{3.33}$$

$$\lim_{n\to\infty} \|w_{m_n} - x_{m_n}\| = \|x_{m_n+1} - x_{m_n}\| = 0. \tag{3.34}$$

and

$$\limsup_{n\to\infty} \langle f(x^*) - x^*, x_{m_n+1} - x^*\rangle \leq 0.$$

It follows from (3.14), that

$$\begin{aligned}\|x_{m_n+1}-x^*\|^2 \le \Big(1-\frac{2\alpha_{m_n}(1-\kappa)}{1-\alpha_{m_n}\kappa}\Big)\|x_{m_n}-x^*\|^2 \\ &+ \frac{2\alpha_{m_n}(1-\kappa)}{1-\alpha_{m_n}\kappa}\Bigg(\frac{\alpha_{m_n}}{1-\kappa}\|x_{m_n}-x^*\|^2 \\ &+ \frac{1}{1-\kappa}\langle f(x^*)-x^*, x_{m_n+1}-x^*\rangle + \frac{\theta_{m_n}^2}{2\alpha_{m_n}(1-\kappa)}\|x_{m_n}-x_{m_n-1}\|^2 \\ &+ \frac{\theta_{m_n}\gamma_{m_n}}{\alpha_{m_n}(1-\kappa)}\|x_{m_n}-x^*\|\|x_{m_n}-x_{m_n-1}\|\Bigg),\end{aligned}$$

which implies that

$$\begin{aligned}\|x_{m_n+1}-x^*\|^2 &\le \frac{\alpha_{m_n}}{1-\kappa}\|x_{m_n}-x^*\|^2 + \frac{1}{1-\kappa}\langle f(x^*)-x^*, x_{m_n+1}-x^*\rangle \\ &+ \frac{\theta_{m_n}^2}{2\alpha_{m_n}(1-\kappa)}\|x_{m_n}-x_{m_n-1}\|^2 \\ &+ \frac{\theta_{m_n}\gamma_{m_n}}{\alpha_{m_n}(1-\kappa)}\|x_{m_n}-x^*\|\|x_{m_n}-x_{m_n-1}\|.\end{aligned}$$

By (3.31), we obtain

$$\begin{aligned}\|x_k-x^*\|^2 &\le \|x_{m_n+1}-x^*\|^2 \\ &\le \frac{\alpha_{m_n}}{1-\kappa}\|x_{m_n}-x^*\|^2 + \frac{1}{1-\kappa}\langle f(x^*)-x^*, x_{m_n+1}-x^*\rangle \\ &+ \frac{\theta_{m_n}^2}{2\alpha_{m_n}(1-\kappa)}\|x_{m_n}-x_{m_n-1}\|^2 \\ &+ \frac{\theta_{m_n}\gamma_{m_n}}{\alpha_{m_n}(1-\kappa)}\|x_{m_n}-x^*\|\|x_{m_n}-x_{m_n-1}\|.\end{aligned}$$

Thus, we get that $\limsup_{k\to\infty}\|x_n-x^*\|^2 = 0$, which means that $\lim_{n\to\infty}\|x_n\| = x^*$. The proof is therefore complete.

## 4. Application

In this section, we give some applications of our main result.

### 4.1. Constrained optimization problem

Let $K$ be a nonempty, closed and convex subset of a real Hilbert space $H$. Let $h : H \to \mathbb{R}$ be a differentiable function on $K$ with its gradient $\nabla h$. The Constrained Optimization Problem (COP) is given as: Find $x^* \in K$ such that

$$h(x^*) \le h(x), \ \forall\, x \in K. \tag{4.1}$$

We denote by $Sol(h)$ the solution set of (4.1). It is well known (see [44]), that a point $x^*$ is a minimizer of (4.1) if and only if $x^*$ is a solution of the VIP (1.1) with $F = \nabla h$.

Thus by applying this formulations and substituting $F = \nabla h$ in Algorithm 3.2, we have the following result for finding a common solution of a COP and a FPP.

**Theorem 4.1.** *Let $K$ be a nonempty, closed and convex subset of a real Hilbert space $H$. Let $h : H \to \mathbb{R}$ be a differentiable function on $K$ with its gradient $\nabla h$. Let $S : H \to H$ be an $\eta$-demimetric mapping. Assume $Sol(h) \cap Fix(S) \neq \emptyset$. Then, the sequence $\{x_k\}$ generated by Algorithm 3.2 with $F$ replaced by $\nabla h$ converges strongly to a point $x^* = P_{Sol(h)\cap Fix(S)} f(x^*)$.*

### 4.2. Split feasibility problem

Let $K$ and $Q$ be nonempty, closed and convex subsets of real Hilbert spaces $H_1$ and $H_2$ respectively and $A : H_1 \to H_2$ be a bounded linear operator. The Split Feasibility Problem (SFP) in the sense of Censor and Elfving [13] is to find

$$x \in K \text{ such that } Ax \in Q. \tag{4.2}$$

We denote by $\Omega$ the solution set of (4.2). Many authors have considered the solution of the SFP (4.2). We note that whenever the SFP (4.2) is consistent (i.e, has a solution), then $x^* \in \Omega$ solves the fixed point equation

$$x^* = P_K(x - \lambda A^*(I - P_Q)Ax), \ \forall \ x \in K,$$

where $P_K$ and $P_Q$ are orthogonal projection of $H_1$ and $H_2$ onto $K$ and $Q$ respectively $\lambda > 0$ and $A^*$ is the adjoint of $A$. One of the most popular method for solving the SFP was the algorithm proposed by Bryne [10]. He gave a recursive formula $\{x_k\}$ generated by $x_1$ and

$$x_{k+1} = P_K(x_k - \lambda A^*(I - P_Q)Ax_k), \ k \in \mathbb{N}, \tag{4.3}$$

where $\lambda \in [0, 2/\gamma]$ with $\gamma$ the spectral radius of the operator $A^*A$.
For the adaptation of our main result to the solution of the SFP, we need the following proposition (see Ceng et al. [12]).

**Proposition 4.2.** [12] *Given $x^* \in H_1$, the following are equivalent*

(i) $x^* \in \Omega$;
(ii) $x^*$ *solves* (4.3)*;*
(iii) $x^*$ *solves the system of variational inequality problem: find $x^* \in K$ such that*

$$\langle A^*(I - P_Q)Ax^*, x - x^*\rangle \geq 0, \ \forall \ x \in K,$$

*where $A^*$ is the adjoint of $A$.*

By these adaptations, we have the following theorem for approximating a solution of an SFP and a FPP.

**Theorem 4.3.** *Let $K$ and $Q$ be nonempty, closed and convex subsets of real Hilbert spaces $H_1$ and $H_2$ respectively and $A : H_1 \to H_2$ be a bounded linear operator. Let $S : H \to H$ be an $\eta$-demimetric mapping. Assume $\Omega \cap Fix(S) \neq \emptyset$. Then, the sequence $\{x_k\}$ generated by Algorithm 3.2 with $F := A^*(I - P_Q)A$ converges strongly to*

$$x^* = P_{\Omega\cap Fix(S)} f(x^*).$$

## 5. Numerical examples

We next provide some numerical experiments to illustrate the performance of our method as well as comparing it with some related methods in the literature.

**Example 5.1.** Let $H = \mathbb{R}^m$ with the standard topology. Consider a mapping $F : \mathbb{R}^m \to \mathbb{R}^m$ given in the form $F(x) = Mx + q$ (see [19], also [35]) where

$$M = BB^T + P + Q$$

$q$ is a vector in $\mathbb{R}^m$, $B$ is an $m \times m$ matrix, $P$ is an $m \times m$ skew-symmetric matrix, $Q$ is a positive definite diagonal matrix, hence the variational inequality is consistent with a unique solution. We define the feasible set $K$ by $K := \{x \in H : \|x\| \leq 1\}$. Let $S : H \to H$ be defined by $S(x) = \frac{-3x}{2}$ for all $x \in H$ and $f(x) = x$. In this example, we choose $\alpha_k = \frac{1}{k+3}$, $\beta_k = \gamma_k = 0.5(1 - \alpha_k)$, $\eta_k = 0.8 - \alpha_k$, $\theta = \frac{1}{3}$, $\lambda_0 = \mu = 0.95$ and $\tau_k = \frac{1}{k^{1.9}}$. For VSEGM and HSEGM, we choose $\beta_k = 0.8 - \alpha_k$ and $\lambda_k = 0.75/L$ where $L = ||F||$. We terminate the iterations at $Tol = ||x_k - P_C(x_k - Fx_k)||_2 \leq \epsilon$ with $\epsilon = 10^{-4}$.
We compare Algorithm 3.2, VSEGM [50] and HSEGM [34] for different values of $m$. The results are presented in Figure 1.

**Example 5.2.** The following example was taken from [24],

$$\min g(x) = \frac{x^T Px + a^T x + a_0}{b^T x + b_0}$$
$$\text{subject to } x \in X = \{x \in \mathbb{R}^5 : b^T x + b_0 > 0\},$$

where

$$P = \begin{pmatrix} 5 & -1 & 2 & 0 \\ -1 & 5 & -1 & 3 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 0 & 5 \end{pmatrix}, \ a = \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix} \ b = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \ a_0 = -2, \ b_0 = 4.$$

Since $P$ is symmetric and positive definite, $g$ is pseudoconvex on $X$. We minimize $g$ on $K = \{x \in \mathbb{R}^4 : 1 \leq x_i \leq 10\} \subset X$.
Following our consideration in Theorem 4.1, we have

$$F(x) = \nabla g(x) = \frac{(b^T x + b_0)(2Px + a) - b(x^T Px + a^T x + a_0)}{(b^T x + b_0)^2}. \tag{5.1}$$

We define the mapping $S : H \to H$ by $S(x) = \frac{-3x}{2}$ and the function $f$ by $f(x) = \frac{x}{2}$. Since the Lipschitz constant of $F$ given by (5.1) is unknown, we compare Algorithm 3.2 with the VSEGM [50]. The following choices of parameters are made: $\alpha_k = \frac{1}{k+3}$, $\beta_k = \gamma_k = 0.5(1 - \alpha_k)$, $\eta_k = 0.5$, $\theta = \frac{1}{3}$, $\lambda_0 = \mu = 0.5$ and $\tau_k = \frac{1}{k^{1.5}}$.
We terminate the iterations at $Tol = ||x_k - P_C(x_k - Fx_k)||_2 \leq \epsilon$ with $\epsilon = 10^{-4}$. The results are presented in Figure 2 for varying initial values $x_0$ and $x_1$.

**Case1:** $x_0 = (10, 10, 10, 10)'$ and $x_1 = (5, 5, 5, 5)'$;
**Case2**: $x_0 = (5, 5, 5, 5)'$ and $x_1 = (20, 20, 20, 20)'$;
**Case3**: $x_0 = (1, 1, 1, 1)'$ and $x_1 = (-4, -4, -4, -4)'$.

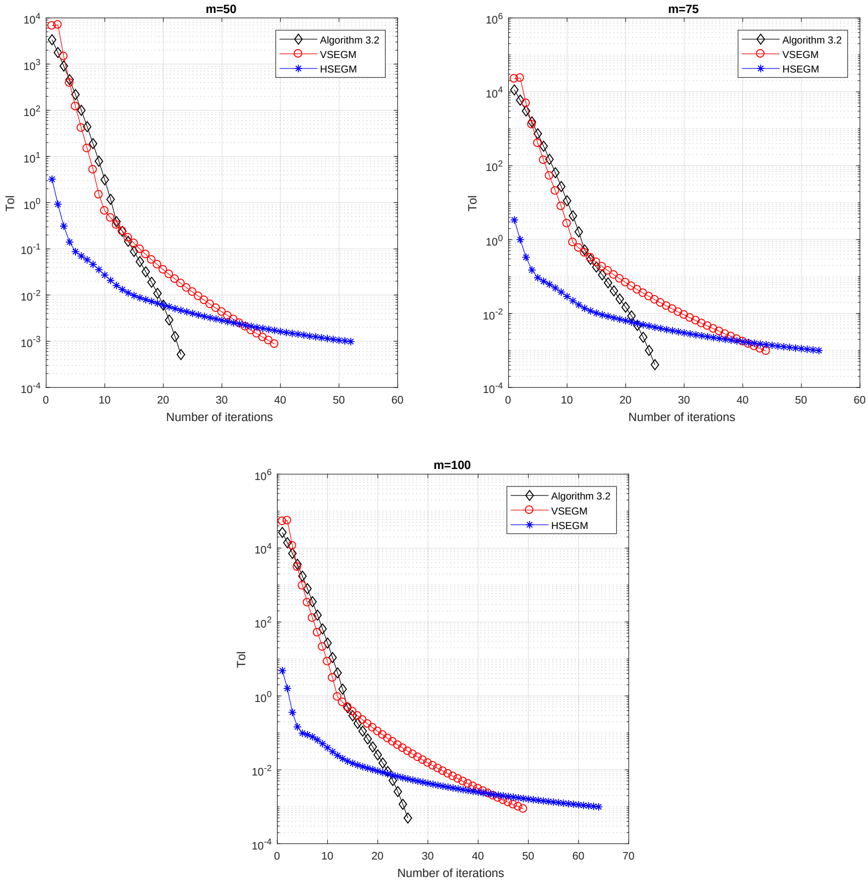


FIGURE 1. Performance of Algorithm 3.2 compared with VSEGM [50] and HSEGM [34].

**Example 5.3.** Let $H = L^2([0,1])$ with the inner product

$$\langle x, y\rangle = \int_0^1 x(t)y(t)dt, \ \forall \ x, y \in H$$

and the induced norm

$$\|x\| = \sqrt{\int_0^1 |x(t)|^2 dt}.$$

Let the mapping $F : H \to H$ be defined by $F(x) = \max\{0, x(t)\}, \ \forall x \in L^2([0,1]), \ t \in [0,1]$ for all $x \in H$ and the feasible $K := \{x \in H : \|x\| \le 1\}$. Define the mapping $T$

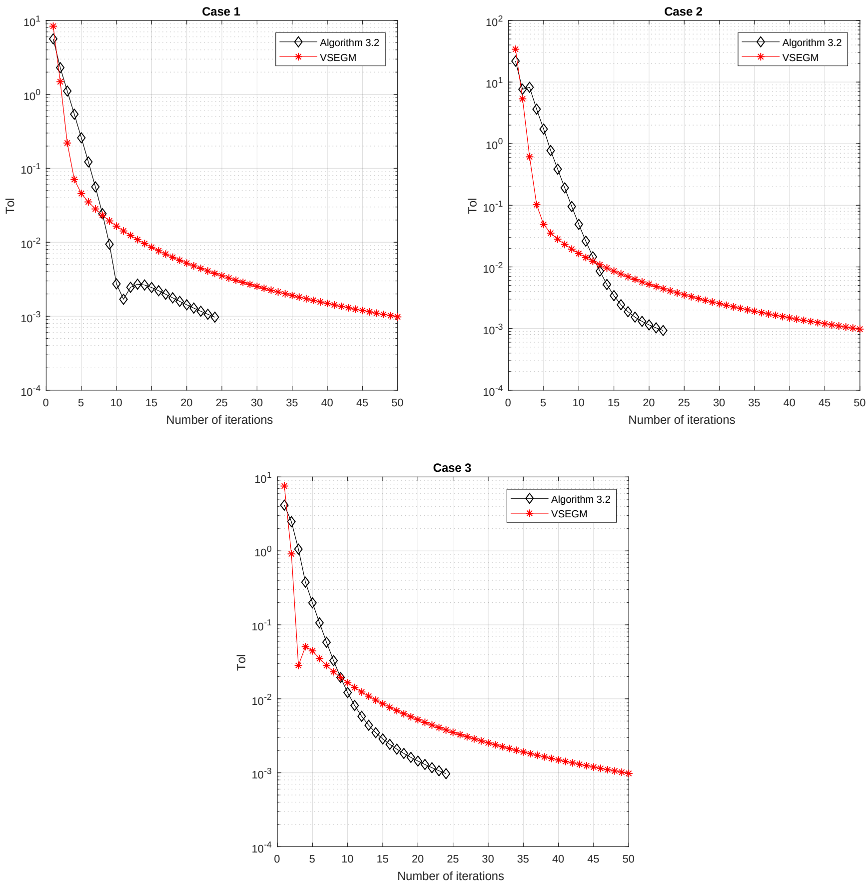


FIGURE 2. Performance of Algorithm 3.2 compared with VSEGM [50].

by

$$Tx(t) = \int_0^1 tx(t)dt, \ \forall x \in L^2([0,1]), \ t \in [0,1],$$

then $T$ is 0-demimetric. Also, let $f : H \to H$ be given by $f(x) = \frac{x}{2}$. For this example, we choose parameters for Algorithm 3.2, HSEGM [34] and VSEGM [50] as follows: $\alpha_k = \frac{1}{k+3}$, $\beta_k = \gamma_k = 0.5(1-\alpha_k)$, $\eta_k = \frac{1}{2k+1}$, $\theta = \frac{1}{3}$, $\lambda_0 = 0.25$, $\mu = 0.5$ and $\tau_k = \frac{1}{k^{1.9}}$. For the VSEGM and HSEGM, we choose $\beta_k = \frac{1}{2k+1}$. We make our comparisons with different initial values and present the result in Figure 3.

**Case i:** $x_0 = -5t$ and $x_1 = 2t$;
**Case ii**: $x_0 = 9t^3 + 11t$ and $x_1 = t^2$;
**Case iii**: $x_0 = \cos(2t) + 5$ and $x_1 = e^{-3t}$.

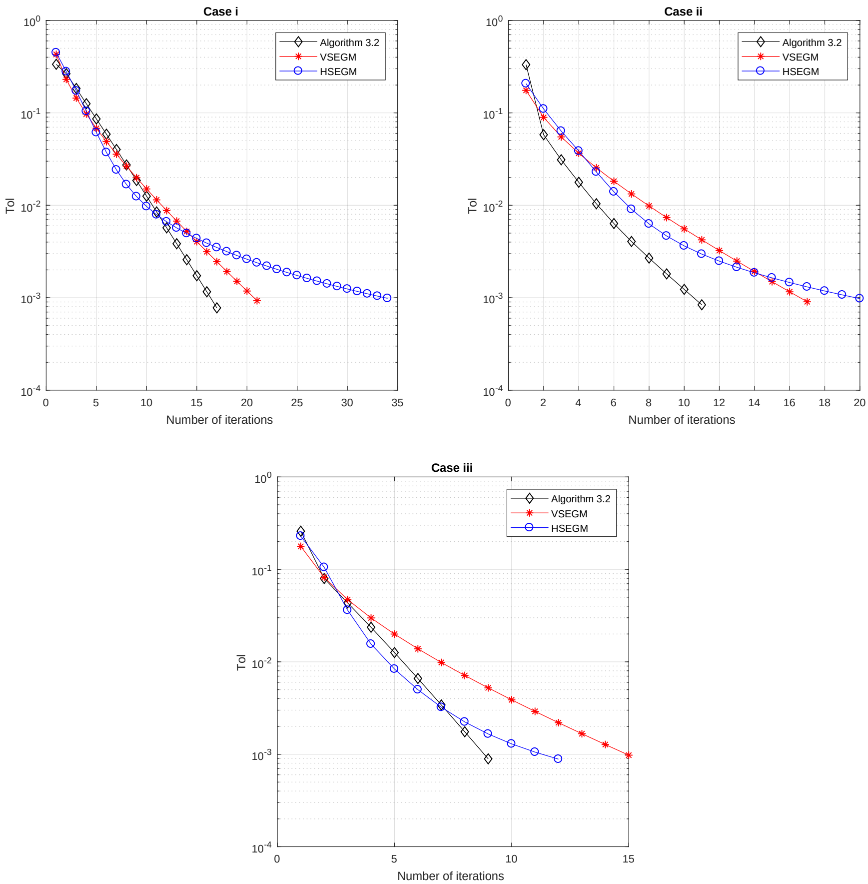


FIGURE 3. Performance of Algorithm 3.2 compared with VSEGM [50] and HSEGM [34].

## 6. Conclusion

In this paper, we considered the problem of finding a common element of the set of solution of VIP and FPP for $\eta$-demimetric mapping in real Hilbert space. We proposed a new iterative algorithm of inertial form and proved a strong convergence theorem under some mild conditions. Our proposed method uses a combination of subgradient extragradient method and a modified viscosity approach with self adaptable step size which avoids the knowledge of the Lipschitz constant of the cost operator in practice. Some applications to constrained optimization and split feasibility problems were considered. We finally gave some numerical experiments to illustrate the behaviour of our method and compare it with some related methods in the literature.

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