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On a coupled system of viscoelastic wave equation of infinite memory with acoustic boundary conditions

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Abstract. This work deals with a coupled system of viscoelastic wave equation of infinite memory with mixed Dirichlet-Neumann boundary conditions. The coupling is via by the acoustic boundary conditions on a portion of the boundary. The semigroup theory is used to show the well posedness and regularity of the initial and boundary value problem. Moreover, we investigate exponential stability of the system taking into account Gearhart-Prüss' theorem.

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1. Introduction

In this paper, we consider the following viscoelastic wave equation coupled with mixed boundary conditions

$$\begin{cases} u_{tt} - \operatorname{div}(\mathrm{A}\nabla u) + \int_0^{+\infty} g(s)\operatorname{div}(\mathrm{A}\nabla u(t-s))ds = 0 & \text{in} \quad \Omega \times \mathbb{R}_+ \\ u = 0 & \text{on} \quad \Gamma_0 \times \mathbb{R}_+ \\ \frac{\partial u}{\partial \nu_{\mathrm{A}}} - \int_0^{+\infty} g(t-s)\frac{\partial u}{\partial \nu_{\mathrm{A}}}(s)ds = z_t & \text{on} \quad \Gamma_1 \times \mathbb{R}_+ \\ hz_{tt} + fz_t + mz + u_t = 0 & \text{on} \quad \Gamma_1 \times \mathbb{R}_+ \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{for} \quad x \in \Omega \\ z(x,0) = z_0(x),\ z_t(x,0) = z_1(x) & \text{for} \quad x \in \Gamma_1, \end{cases} \tag{1.1}$$

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where Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$ with a smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, such that Γ_0 and Γ_1 are closed and disjoint and $\nu = (\nu_1, \cdots, \nu_N)$ represents the unit outward normal to Γ. The term $\int_0^{+\infty} g(s)\mathrm{div}(\mathrm{A}\nabla u(t-s))\mathrm{d}s$ is the infinite memory (past history) responsible for the viscoelastic damping, where g is called the relaxation function. The functions $h, f, m : \Gamma_1 \to \mathbb{R}^+$ are essentially bounded. There exist three positive constants f_0, m_0, and h_0 such that $f(x) \geq f_0$, $m(x) \geq m_0$ and $h(x) \geq h_0$ for a.e. $x \in \Gamma_1$. The initial conditions $u_0, u_1 : \Omega \to \mathbb{C}$, $z_0, z_1 : \Gamma_1 \to \mathbb{C}$ are given functions. The operator $\mathrm{A} = (a_{ij}(x))_{i,j};\ i,j = 1, \ldots, N$; and $\frac{\partial u}{\partial \nu_\mathrm{A}} = \sum\limits_{i,j=1}^{N} a_{ij}(x) \frac{\partial u}{\partial x_j} \nu_i$.

The above model would be to describe the motion of fluid particles from rest in the domain Ω into part of the surface at a given point $x \in \Gamma_1$, which can be expressed by the pressure at that point. The relationship between the velocity potential $u_t = u_t(x,t)$ at a point on the surface and the normal displacement $z = z(x,t)$ is proportional to the pressure. It is called the acoustic impedance. This impedance may be complex in the case of the velocity potential was not in phase with the pressure. The coupling of our model (1.1) is via by the impenetrability boundary condition $(1.1)_3$ and the acoustic boundary condition $(1.1)_4$.

The partial differential equation (PDE) system of viscoelastic wave equation with acoustic boundary conditions was first introduced by Morse and Ingard [15] and developed by Beale [5]. In [5], the problem was formulated as an initial value problem in a Hilbert space and semigroup methods were used to solve it. The loss of decay has obtained by [5] provided that the term z_{tt} was included. Recently, the result concerning existence and asymptotic behavior of smooth, as well as weak solution of wave equation with acoustic boundary conditions have been established by many authors, see [10, 13]. Boukhatem and Benabderrahmane [8] studied the global existence and exponential decay of solution of finite memory of the system (1.1) in the absence of the second derivative z_{tt}. This absence brings us some difficulties in the study because of the abnormality of the system. It can not apply directly the semigroups or Faedo-Galerkin's theories. They added in the arguments the term εz_{tt} when $\varepsilon \to 0$ to overcome the difficulty. Mentionned the work of Peralta [16] who bringing an analysis of wave equation involving mixed Dirichlet-Neumann boundary conditions, delay and acoustic conditions where both are localized on a portion of the boundary

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in} \quad \Omega \times \mathbb{R}_+ \\ u = 0 & \text{on} \quad \Gamma_0 \times \mathbb{R}_+ \\ \frac{\partial u}{\partial \nu} - z_t = -a u_t(.,. - \tau) - k u_t & \text{on} \quad \Gamma_1 \times \mathbb{R}_+ \\ h z_{tt} + f z_t + m z + u_t = 0 & \text{on} \quad \Gamma_1 \times \mathbb{R}_+ \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{for} \quad x \in \Omega \\ u(x,t) = \varphi(x,t) & \text{for} \quad (x,t) \in \Omega \times (-\tau, 0) \\ z(x,0) = z_0(x),\ z_t(x,0) = z_1(x) & \text{for} \quad x \in \Gamma_1, \end{cases} \tag{1.2}$$

Here, $\tau > 0$ is a constant delay parameter and $a, k \geq 0$. He proved the existence and uniqueness of solutions of (1.2) using semigroup theory for bounded linear operators. Moreover, if the delay factor is less than the damping factor $(a < k)$, the exponential stability result is shown using the energy multiplier method. In the case of equality $(a = k)$, he showed that the energy decays to zero asymptotically using variational methods. In addition, the stability results have been considered in [16], where the term $-f_0 z_t$ was included in the right hand of side of the oscillator equation $(1.2)_4$. If $f > f_0$ then we show that the energy of the solution decays to zero exponentially. In the case $f = f_0$, the solutions have an asymptotically decaying energy. Moreover, Gao et al. [11] presented a new method to obtain uniform decay rates for (1.2) with nonlinear acoustic boundary conditions in the absene of delay. The system contains an internal localized damping term $w(x)u_t$ in $(1.2)_1$ and damping and potential in the boundary displacement equation are nonlinear, where the terms $f(z_t)$ and $m(z)$ are replaced by fz_t and mz in $(1.2)_4$, respectively.

The primary discussion touched upon by several authors is to use the integral term of relaxation function g instead the frictional damping term u_t. The question that have been focused their attention as an important works is the viscoelastic damping of memory effect should be strong enough to procreate the decay of the system.

One of important motivations to studying exponential stability of the associated semigroup comes from the spectral analysis. This purpose recalls the related results given by Gearhart-Prüss' theorem (see [14, 17]). It is shown all eigenvalues approach a line that parallel to the imaginary axis. Moreover, the resolvent operator is bounded for all eigenvalues of the generator associated. The proof is the combination of the contradiction argument with a PDE technique. Let us mention some papers on weakly dissipative coupled systems. In [12], the exponential decay is established for each of the wave equations that have been damped on the boundary. Prüss [18] gave the optimal results to characterize polynomial as well as exponential decay rates for viscoelastic materials. Apalara et al. [4] studied the exponential stability of laminated beams when the frictional damping acts on the effective rotation angle. For weak damping acting only one equation, the following coupled wave equation

$$\begin{cases} u_{tt} - \Delta u + \int_0^\infty g(s)\Delta u(s)\mathrm{d}s + \alpha v = 0 & \text{in} \quad \Omega \times \mathbb{R}_+ \\ v_{tt} - \Delta v + \alpha u = 0 & \text{in} \quad \Omega \times \mathbb{R}_+ \\ u = v = 0 & \text{on} \quad \Gamma \times \mathbb{R}_+ \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{for} \quad x \in \Omega \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x) & \text{for} \quad x \in \Omega \end{cases} \tag{1.3}$$

has been considered by Almeida and Santos [3] (see also [9]). In [3, 9], they proved the lack of exponential decay to system (1.3). The authors obtained the optimal polynomial decay by using the recent results due to Borichev and Tomilov [7]. The method used in this contexts introduced by Alabau [1] and developed by Alabau-Cannarsa-Komornik [2]. For memory damping acting on the acoustic boundary, Benomar and Benaissa [6] established polynomial energy decay rates for system (1.2) without delay in one dimensional space.

Our main result is devoted to study the well posedness and exponential decay of the system (1.1), in which we analyze the spectral distribution in the complex plane. The semigroup theory is used to show, in Sect. 3, the global existence of energy-associated solution which its real part decreases with time. Motivated by the mentioned works above concerning Gearhart-Prüss' theorem, the exponential stability of the corresponding semigroup is concluded in Sect. 4.

2. Preliminary

In this section, we give some notations and we present some assumptions needed for our work. Let $\mathrm{H}(\mathrm{div},\Omega)=\{u\in \mathrm{H}^1(\Omega);\mathrm{div}(\mathrm{A}\nabla u)\in \mathrm{L}^2(\Omega)\}$ be the Hilbert space equipped with the norm

$$\|u\|_{\mathrm{H}(\mathrm{div},\Omega)}=\left(\|u\|^2_{\mathrm{H}^1(\Omega)}+\|\mathrm{div}(\mathrm{A}\nabla u)\|^2_2\right)^{1/2},$$

where $\mathrm{H}^1(\Omega)$ is the Sobolev space of first order, $\|.\|_2$ is an L^2-norm and $(.,.)$, $\langle .,.\rangle_{\Gamma_1}$ are the scalar product in $\mathrm{L}^2(\Omega)$, $\mathrm{L}^2(\Gamma_1)$, respectively.

Denoting $\gamma_0:\mathrm{H}^1(\Omega)\to \mathrm{L}^2(\Gamma)$ and $\gamma_1:\mathrm{H}(\mathrm{div},\Omega)\to \mathrm{L}^2(\Gamma)$ defined by $\gamma_0(u)=u_{|\Gamma}$ and $\gamma_1(u)=\left(\frac{\partial u}{\partial\nu_{\mathrm{A}}}\right)_\Gamma$ for all u in $\mathrm{H}(\mathrm{div},\Omega)$. Some times to simplify the notations we write u and $\frac{\partial u}{\partial\nu_{\mathrm{A}}}$ instead $\gamma_0(u)$ and $\gamma_1(u)$, respectively.

We denote by

$$\mathrm{H}^1_{\Gamma_0}(\Omega)=\{u\in \mathrm{H}^1(\Omega)\mid \ \gamma_0(u)=0 \text{ on } \Gamma_0\}$$

the closure subspace of $\mathrm{H}^1(\Omega)$ equipped with the norm equivalent to the usual norm in $\mathrm{H}^1_0(\Omega)$. The Poincaré inequality holds in $\mathrm{H}^1_{\Gamma_0}(\Omega)$.

In this study, we will need the following assumptions:
$(\mathbf{A}_1)$ The operator $\mathrm{A}=(a_{ij}(x))_{i,j},\ i,j=1,\dots,N$; where the coefficient a_{ij} in $\mathcal{C}^1(\overline{\Omega})$ is symmetric and there exists a constant $a_0>0$ such that

$$\sum_{i,j=1}^N a_{ij}(x)\zeta_i\zeta_j\geq a_0|\zeta|^2,\qquad \forall x\in\overline{\Omega},\ \ \forall\zeta\in\mathbb{C}^N. \tag{2.1}$$

$(\mathbf{A}_2)$ The kernel function $g:\mathbb{R}_+\to\mathbb{R}_+$ is a bounded $\mathcal{C}^1$ function satisfying

$$\lim_{t\to\infty} g(t)=0,\quad g(0)>0,\quad 1-\int_0^\infty g(s)\mathrm{d}s=\ell>0, \tag{2.2}$$

and there exists a constant $\alpha>0$ such that

$$g'(t)\leq-\alpha g(t),\quad \forall t\geq 0. \tag{2.3}$$

Furthermore, we define

$$a(u(t),v(t))=(Au(t),v(t))=\sum_{i,j=1}^N\int_\Omega a_{ij}(x)\frac{\partial u(t)}{\partial x_i}\overline{\frac{\partial v(t)}{\partial x_j}}\mathrm{d}x.$$

By using the hypothesis $(\mathbf{A}_1)$, we verify that the sesquilinear form $a(.,.):\mathrm{H}^1_{\Gamma_0}(\Omega)\times \mathrm{H}^1_{\Gamma_0}(\Omega)\to\mathbb{C}$ is continuous, and by (2.1), we deduce that a is coercive.

3. The well posedness

In this section, we will show the well posedness of the system (1.1).

Let us introduce a new variable η as follows

$$\eta(x,s,t)=u(x,t)-u(x,t-s),\quad x\in\Omega,\ t,s\in\mathbb{R}_+.$$

Then, the system (1.1) becomes

$$\left\{\begin{array}{lll} u_{tt}-\ell\mathrm{div}(\mathrm{A}\nabla u)-\int_0^{+\infty}g(s)\mathrm{div}(\mathrm{A}\nabla\eta(s))\mathrm{d}s=0 & \text{in} & \Omega\times\mathbb{R}_+ \\ \eta_t+\eta_s-u_t=0 & \text{in} & \Omega\times\mathbb{R}_+\times\mathbb{R}_+ \\ hz_{tt}+fz_t+mz+u_t=0 & \text{on} & \Gamma_1\times\mathbb{R}_+ \\ u=0 & \text{on} & \Gamma_0\times\mathbb{R}_+ \\ \ell\frac{\partial u}{\partial\nu_{\mathrm{A}}}+\int_0^{+\infty}g(s)\frac{\partial\eta}{\partial\nu_{\mathrm{A}}}(s)\mathrm{d}s=z_t & \text{on} & \Gamma_1\times\mathbb{R}_+ \\ u(0)=u_0,\ u_t(0)=u_1 & \text{in} & \Omega \\ \eta(s,0)=u_0-u(-s)=\eta_0(s) & \text{for} & s\in\mathbb{R}_+ \\ z(0)=z_0,\ z_t(0)=z_1 & \text{in} & \Gamma_1 \end{array}\right. \tag{3.1}$$

In order to give a reformulation as first-order evolution system, we denote by

$$U=(u,v,\eta,z,\delta)^T \quad \text{with} \quad v=u_t \quad \text{and} \quad \delta=z_t.$$

We consider the product Hilbert spaces

$$\mathcal{H}=\mathrm{H}^1_{\Gamma_0}(\Omega)\times\mathrm{L}^2(\Omega)\times\mathrm{L}^2_g(\mathbb{R}_+;\mathrm{H}^1_{\Gamma_0}(\Omega))\times\mathrm{L}^2(\Gamma_1)\times\mathrm{L}^2(\Gamma_1),$$

endowed with the following inner product

$$\begin{aligned}\left\langle U,\tilde{U}\right\rangle_{\mathcal{H}} &= \ell a(u(t),\tilde{u}(t))+\int_\Omega v(t)\overline{\tilde{v}(t)}\mathrm{d}x+\langle\eta(t),\tilde{\eta}(t)\rangle_{\mathrm{L}^2_g} \\ &\quad +\langle mz(t),\tilde{z}(t)\rangle_{\Gamma_1}+\left\langle h\delta(t),\tilde{\delta}(t)\right\rangle_{\Gamma_1},\end{aligned} \tag{3.2}$$

where $\mathrm{L}^2_g(\mathbb{R}_+;\mathrm{H}^1_{\Gamma_0}(\Omega))$ denotes the Hilbert space of $\mathrm{H}^1_{\Gamma_0}(\Omega)-$valued functions on $\mathbb{R}_+$, endowed with the inner product

$$\langle\eta(t),\tilde{\eta}(t)\rangle_{\mathrm{L}^2_g(\mathbb{R}_+;\mathrm{H}^1_{\Gamma_0}(\Omega))}=\int_0^{+\infty}g(s)a(\eta(s,t),\tilde{\eta}(s,t))\mathrm{d}s, \tag{3.3}$$

for every $U=(u,v,\eta,z,\delta)^T$ and $\tilde{U}=(\tilde{u},\tilde{v},\tilde{\eta},\tilde{z},\tilde{\delta})^T$ in $\mathcal{H}$.
Thus, the system (3.1) can be rewritten in the following

$$\left\{\begin{array}{l} U_t(t)=\mathcal{A}U(t), \quad \forall t\geq 0 \\ U(0)=U_0=(u_0,u_1,\eta_0,z_0,z_1)^T \end{array}\right. \tag{3.4}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}U(t)=\left(\begin{array}{c} v(t) \\ \ell\mathrm{div}(\mathrm{A}\nabla u(t))+\int_0^{+\infty}g(s)\mathrm{div}(\mathrm{A}\nabla\eta(t,s))\mathrm{d}s \\ v(t)-\eta_s(t,s) \\ \delta(t) \\ \frac{1}{h(x)}\left(-v(t)-m(x)z(t)-f(x)\delta(t)\right) \end{array}\right) \tag{3.5}$$

The domain of $\mathcal{A}$ is given by

$$D(\mathcal{A})=\left\{\begin{array}{c|l} U & \ell u+\int_0^{+\infty}g(s)\eta(s)\mathrm{d}s\in\mathrm{H}(\mathrm{div},\Omega);\ v\in\mathrm{H}^1_{\Gamma_0}(\Omega); \\ & \eta\in\mathrm{L}^2_g(\mathbb{R}_+;\mathrm{H}^1_{\Gamma_0}(\Omega));\ z,\delta\in\mathrm{L}^2(\Gamma_1); \\ & \ell\frac{\partial u}{\partial\nu_{\mathrm{A}}}+\int_0^{+\infty}g(s)\frac{\partial\eta}{\partial\nu_{\mathrm{A}}}(s)\mathrm{d}s=\delta \ \text{ on } \Gamma_1 \end{array}\right\} \tag{3.6}$$

Set the energy functional E of the system (3.1)

$$E(t) = \frac{1}{2} \langle U, U \rangle_{\mathcal{H}}. \tag{3.7}$$

Lemma 3.1. *The energy functional (3.7), along the solution of (3.1), is a nonincreasing function satisfying, for all* $t \geq 0$

$$E'(t) = \frac{1}{2} \int_0^{+\infty} g'(s) a(\eta(s), \eta(s)) \mathrm{d}s - \|f^{1/2}\delta(t)\|_{2,\Gamma_1}^2. \tag{3.8}$$

Proof. Taking the scalar product of $(3.1)_1$ with u_t and $(3.1)_3$ with z_t in $\mathrm{L}^2(\Omega)$ and $\mathrm{L}^2(\Gamma_1)$, respectively, then adding it to the inner product (3.3) of $(3.1)_2$ with η. Using Green's formula and the properties of η. Taking its real part, we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|u_t(t)\|_2^2 + \ell a(u(t), u(t)) + \|\eta(t)\|_{\mathrm{L}_g^2} + \|m^{1/2} z(t)\|_{2,\Gamma_1}^2 + \|h^{1/2}\delta(t)\|_{2,\Gamma_1}^2\right) \\
= \;& -\int_0^{+\infty} g(s) a(\eta_s(t,s), \eta(t,s) ds - \|f^{1/2}\delta(t)\|_{2,\Gamma_1}^2.
\end{aligned} \tag{3.9}$$

Using (2.2) and the properties of η, we have

$$\int_0^{+\infty} g(s) a(\eta(s), \eta_s(s)) ds = -\frac{1}{2} \int_0^{+\infty} g'(s) a(\eta(s), \eta(s)) ds. \tag{3.10}$$

Combining (3.10) and (3.9), we get (3.8). $\square$

Our aim is ensured by the following theorem:

Theorem 3.2. *The operator $\mathcal{A}$ is the infinitesimal generator of $\mathcal{C}_0-$semigroup of contractions over the Hilbert space $\mathcal{H}$. Thus, for any initial data $U_0 \in \mathcal{H}$, the problem (3.4) has a unique weak solution $U \in \mathcal{C}(\mathbb{R}_+; \mathcal{H})$. Moreover, if $U_0 \in D(\mathcal{A})$, then the solution $U \in \mathcal{C}(\mathbb{R}_+; D(\mathcal{A})) \cap \mathcal{C}^1(\mathbb{R}_+; \mathcal{H})$.*

Proof. We will use the Hille-Yosida theorem. For this purpose, $\mathcal{A}$ is dissipative. Indeed, using (3.4) and (3.7), we have, for $U \in D(\mathcal{A})$

$$E'(t) = \Re \langle U_t(t), U(t) \rangle_{\mathcal{H}} = \Re \langle \mathcal{A}U(t), U(t) \rangle_{\mathcal{H}}.$$

Therefore, we deduce from Lemma 3.1 that

$$\Re \langle \mathcal{A}U, U \rangle_{\mathcal{H}} = \frac{1}{2} \int_0^{+\infty} g'(s) a(\eta(t,s), \eta(t,s)) \mathrm{d}s - \|f^{1/2}\delta(t)\|_{\Gamma_1} \leq 0. \tag{3.11}$$

Next, $\mathrm{I} - \mathcal{A}$ is surjective. Indeed, for each $F = (f_1, f_2, f_3, f_4, f_5)^T \in \mathcal{H}$, we show that there exists $U \in D(\mathcal{A})$ such that

$$(\mathrm{I} - \mathcal{A})U = F.$$

Then, the previous equation reads

$$\begin{aligned} u - v &= f_1 && (3.12)\\ v - \ell \mathrm{div}(\mathrm{A}\nabla u) - \int_0^{+\infty} g(s)\mathrm{div}(\mathrm{A}\nabla \eta(s))\mathrm{d}s &= f_2 && (3.13)\\ \eta + \eta_s - v &= f_3 && (3.14)\\ z - \delta &= f_4 && (3.15)\\ h(x)\delta + v + f(x)\delta + m(x)z &= h(x)f_5. && (3.16)\end{aligned}$$

Suppose (u, z) are found in $\mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathrm{L}^2(\Gamma_1)$. Thus, (3.12) and (3.15) yield

$$\begin{cases} v = u - f_1, \\ \delta = z - f_4, \end{cases} \tag{3.17}$$

Then,

$$v \in \mathrm{H}^1_{\Gamma_0}(\Omega), \quad \text{and} \quad \delta \in \mathrm{L}^2(\Gamma_1).$$

From (3.14), we can determine

$$\eta(s) = -ve^{-s} + v + \int_0^s f_3(\tau)e^{\tau - s}d\tau, \quad \forall s \in \mathbb{R}_+, \tag{3.18}$$

that is $\eta(0) = 0$. According (3.18) with $(3.17)_1$, we have

$$\eta(s) = -ue^{-s} + u + \eta_1(s), \quad \forall s \in \mathbb{R}_+, \tag{3.19}$$

with $\eta_1 \in \mathrm{L}^2_g(\mathbb{R}_+; \mathrm{H}^1_{\Gamma_0}(\Omega))$ defined by

$$\eta_1(s) = f_1 e^{-s} - f_1 + \int_0^s f_3(\tau)e^{\tau - s}d\tau.$$

Then, $(3.1)_5$ becomes

$$\ell_g \frac{\partial u}{\partial \nu_{\mathrm{A}}} + \int_0^{+\infty} g(s)\frac{\partial \eta_1}{\partial \nu_{\mathrm{A}}}(s)\mathrm{d}s = z - f_4 \quad \text{on } \Gamma_1 \times \mathbb{R}_+,$$

where

$$\ell_g = \left(\ell + \int_0^{+\infty} g(s)(1 - e^{-s})\mathrm{d}s\right) > 0.$$

Inserting (3.17) and (3.19) into (3.13)-(3.16) and adding the results, we get

$$u - \ell_g \mathrm{div}(\mathrm{A}\nabla u) = f_1 + f_2 + \int_0^{+\infty} g(s)\mathrm{div}(\mathrm{A}\nabla \eta_1(s))\mathrm{d}s, \tag{3.20}$$

$$(h(x) + m(x) + f(x))z + u = h(x)f_5 + f_1 + (h(x) + f(x))f_4 \tag{3.21}$$

Taking the inner product of (3.20) with $\tilde{u}$ in $\mathrm{L}^2(\Omega)$, then adding it to the complex conjugate of the inner product of (3.21) with $\tilde{z}$ in $\mathrm{L}^2(\Gamma_1)$ and using Green's formula, we obtain the sesquilinear from $\mathfrak{B} : (\mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathrm{L}^2(\Gamma_1)) \times (\mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathrm{L}^2(\Gamma_1)) \to \mathbb{C}$ defined by

$$\begin{aligned}\mathfrak{B}((u, z)), (\tilde{u}, \tilde{z})) &= (u, \tilde{u}) + \ell_g a(u, \tilde{u}) - \langle z, \tilde{u}\rangle_{\Gamma_1} \\ &\quad + \langle u, \tilde{z}\rangle_{\Gamma_1} + \langle (h(x) + m(x) + f(x))z, \tilde{z}\rangle_{\Gamma_1}\end{aligned}$$

for every $(u,z),(\tilde{u},\tilde{z}) \in \mathrm{H}^1_{\Gamma_0}(\Omega)\times \mathrm{L}^2(\Gamma_1)$, and the antilinear from $\mathcal{G} : \mathrm{H}^1_{\Gamma_0}(\Omega)\times \mathrm{L}^2(\Gamma_1)\to \mathbb{C}$ defined by

$$\begin{aligned}\mathcal{G}(\tilde{u},\tilde{z}) &= (f_1+f_2,\tilde{u}) - \int_0^{+\infty} g(s)a(\eta_1(s),\tilde{u})\mathrm{d}s \\ &\quad + \langle h(x)f_5 + f_1 + (h(x)+f(x))f_4, \tilde{z}\rangle_{\Gamma_1},\end{aligned}$$

for every $(\tilde{u},\tilde{z}) \in \mathrm{H}^1_{\Gamma_0}(\Omega)\times \mathrm{L}^2(\Gamma_1)$.

It's easy to see that $\mathfrak{B}$ is a continuous sesquilinear form and coercive on $(\mathrm{H}^1_{\Gamma_0}(\Omega)\times \mathrm{L}^2(\Gamma_1))\times(\mathrm{H}^1_{\Gamma_0}(\Omega)\times \mathrm{L}^2(\Gamma_1))$ and $\mathcal{G}$ is a continuous antilinear form on $\mathrm{H}^1_{\Gamma_0}(\Omega)\times \mathrm{L}^2(\Gamma_1)$. Using complex Lax-Milgram's theorem, then there exists a unique solution $(u,z)\in \mathrm{H}^1_{\Gamma_0}(\Omega)\times \mathrm{L}^2(\Gamma_1)$, satisfying, for all $(\tilde{u},\tilde{z}) \in \mathrm{H}^1_{\Gamma_0}(\Omega)\times \mathrm{L}^2(\Gamma_1)$

$$\mathfrak{B}((u,z),(\tilde{u},\tilde{z})) = \mathcal{G}(\tilde{u},\tilde{z}). \tag{3.22}$$

Additionally, we proceed to get more regularity.
Taking $\tilde{z}=0$ in (3.22). Since $\mathcal{D}(\Omega)$ is dense in $\mathrm{H}^1_{\Gamma_0}(\Omega)$, we deduce that

$$\ell_g\left(\mathrm{div}(\mathrm{A}\nabla u),\tilde{u}\right) = \left(\int_0^{+\infty} g(s)\mathrm{div}(\mathrm{A}\nabla\eta_1(s))\mathrm{d}s,\tilde{u}\right) + \langle z,\tilde{u}\rangle_{\Gamma_1} - (u,\tilde{u}),$$

for every $\tilde{u}\in \mathrm{H}^1_{\Gamma_0}(\Omega)$. Hence, $u\in(\mathrm{H}(\mathrm{div},\Omega)\cap \mathrm{H}^1_{\Gamma_0}(\Omega))$.

Then $U \in D(\mathcal{A})$. Consequently, Lumper-Phillips' theorem guarantees the generator $\mathcal{A}$ of a $\mathcal{C}_0-$semigroup on $\mathcal{H}$. $\square$

4. Exponential stability

Here we will show the exponential stability of (3.4). The method that we will use in the following theorem is based on Gearhart-Prüss' theorem [14, 17] to complex value dissipative systems.

Theorem 4.1. *Let $T(t) := e^{\mathcal{A}t}$ be a $\mathcal{C}_0-$semigroup of contractions on Hilbert space $\mathcal{H}$. Then $T(t)$ is exponentially stable if and only if*

(i) *The resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$ contains the imaginary axis ($i\mathbb{R}\subset\rho(\mathcal{A})$),*
(ii) $\limsup_{|\lambda|\to\infty} \|(i\lambda\mathrm{I}-\mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$

Our starting point is to show that the semigroup associated to (3.4), generated by $\mathcal{A}$, is exponentially stable. The following Theorem gives our main result, that is to verify the conditions (i) and (ii) of Theorem 4.1.

Theorem 4.2. *Assume that $(\mathbf{A}_2)$ holds. Then, $e^{\mathcal{A}t}$ generated by $\mathcal{A}$ is exponentially stable, that is to say, there exist two constants $M\geq 1$ and $\epsilon>0$ such that*

$$\|e^{\mathcal{A}t}\| \leq Me^{-\epsilon t}.$$

Proof. We first show that the resolvent of the system (3.4) is located on the imaginary axes. Note that the resolvent equation $(\mathrm{i}\lambda \mathrm{I} - \mathcal{A})U = F \in \mathcal{H}$ is given by

$$\begin{aligned} \mathrm{i}\lambda u - v &= f_1 && (4.1)\\ \mathrm{i}\lambda v - \ell \mathrm{div}(\mathrm{A}\nabla u) - \int_0^{+\infty} g(s)\mathrm{div}(\mathrm{A}\nabla \eta(s))\mathrm{d}s &= f_2 && (4.2)\\ \mathrm{i}\lambda \eta + \eta_s - v &= f_3 && (4.3)\\ \mathrm{i}\lambda z - \delta &= f_4 && (4.4)\\ \mathrm{i}\lambda h(x)\delta + f(x)\delta + m(x)z + v &= h(x)f_5. && (4.5) \end{aligned}$$

It's means to show that $\mathrm{i}\mathbb{R} \cap \sigma(\mathcal{A}) = \emptyset$, where $\sigma(\mathcal{A})$ is the spectrum of $\mathcal{A}$.
Using contradiction arguments. Let us suppose that $\mathcal{A}$ has an imaginary eigenvalue. Then, we have

$$\mathcal{A}U = i\lambda U, \qquad \lambda \in \mathbb{R}. \tag{4.6}$$

Thus, $F \equiv 0$ in (4.1)-(4.5). From (3.11) and (4.6), we can get

$$0 = \Re \langle \mathcal{A}U, U\rangle_{\mathcal{H}} \le \frac{1}{2}\int_0^{+\infty} g'(s)a(\eta(t,s),\eta(t,s))\mathrm{d}s - \|f^{1/2}\delta(t)\|_{\Gamma_1} \le 0.$$

It follows that $\delta = 0$, and from the hypothesis of g that $\nabla\eta = 0$. Using the fact $u = 0$ in $\Gamma \times \mathbb{R}_+$ that $\eta = 0$. This implies by (4.1) and (3.18) that $u = v = 0$. From equation (4.5), we conclude that $z = 0$. Hence, $U \equiv 0$. We obtain a contradiction.

We now prove (ii) by a contradiction argument again. Suppose that (ii) is not true. Then there exist a sequence λ_n with $|\lambda_n| \to +\infty$ and a sequence of functions

$$U_n = (u_n, v_n, \eta_n, z_n, \delta_n)^T \in D(\mathcal{A}) \quad \text{with} \quad \|U_n\|_{\mathcal{H}} = 1, \tag{4.7}$$

such that, as $n \to +\infty$;

$$(\mathrm{i}\lambda_n \mathrm{I} - \mathcal{A})U_n \to 0 \quad \text{in} \quad \mathcal{H} \tag{4.8}$$

i.e,

$$\begin{aligned} \mathrm{i}\lambda_n u_n - v_n &\to 0 \quad \text{in} \quad \mathrm{H}^1_{\Gamma_0}(\Omega) && (4.9)\\ \mathrm{i}\lambda_n v_n - \ell\mathrm{div}(\mathrm{A}\nabla u_n) - \int_0^{+\infty} g(s)\mathrm{div}(\mathrm{A}\nabla\eta_n(s))\mathrm{d}s &\to 0 \quad \text{in} \quad \mathrm{L}^2(\Omega) && (4.10)\\ \mathrm{i}\lambda_n\eta_n + \partial_s\eta_n - v_n &\to 0 \quad \text{in} \quad \mathrm{L}^2_g && (4.11)\\ \mathrm{i}\lambda_n z_n - \delta_n &\to 0 \quad \text{in} \quad \mathrm{L}^2(\Gamma_1) && (4.12)\\ \mathrm{i}\lambda_n h(x)\delta_n + f(x)\delta_n + m(x)z_n + v_n &\to 0 \quad \text{in} \quad \mathrm{L}^2(\Gamma_1) && (4.13) \end{aligned}$$

Taking the inner product (3.2) of (4.8) with U_n and then taking its real part yields

$$-\Re\langle(\mathrm{i}\lambda_n\mathrm{I} - \mathcal{A})U_n, U_n\rangle_{\mathcal{H}} = -\frac{1}{2}\int_0^{+\infty} g'(s)a(\eta_n(s),\eta_n(s))\mathrm{d}s + \|f^{1/2}\delta_n\|^2_{2,\Gamma_1} \to 0. \tag{4.14}$$

Using (2.3), we find that

$$\begin{aligned} \eta_n &\to 0 \qquad \text{in } \mathrm{L}^2_g(\mathbb{R}_+; \mathrm{H}^1_{\Gamma_0}(\Omega)), && (4.15)\\ \delta_n &\to 0 \qquad \text{in } \mathrm{L}^2(\Gamma_1). && (4.16) \end{aligned}$$

On the other hand, taking the complex conjugate of the inner product of (4.9) with ℓu_n in $\mathrm{H}_0^1(\Omega)$, then adding it to the inner product of (4.10) with v_n in $\mathrm{L}^2(\Omega)$ and using Green's formula, we get

$$\mathrm{i}\left(-\ell a(u_n,u_n)+\|v_n\|_2^2\right)-\frac{1}{\lambda_n}\left\langle\delta_n,v_n\right\rangle_{\Gamma_1}+\frac{1}{\lambda_n}\int_0^{+\infty}g(s)a(\eta_n(s),v_n)\mathrm{d}s\to 0. \quad (4.17)$$

We can deduce from (4.9) that $\frac{1}{\lambda_n}\|\nabla v_n\|_2^2$ is uniformly bounded. By using (4.15) and (4.16), the last two terms in (4.17) converge to zero. Hence,

$$\ell a(u_n,u_n)-\|v_n\|_2^2\to 0. \quad (4.18)$$

Adding the complex conjugate of the inner product of (4.12) with $m(x)\delta_n$ to the inner product of (4.13) with z_n in $\mathrm{L}^2(\Gamma_1)$, we have

$$\mathrm{i}(\|m^{1/2}z_n\|_{2,\Gamma_1}^2+\|h^{1/2}\delta_n\|_{2,\Gamma_1}^2)+\frac{1}{\lambda_n}\|f^{1/2}\delta_n\|_{2,\Gamma_1}^2+\frac{1}{\lambda_n}\left\langle v_n,\delta_n\right\rangle_{\Gamma_1}\to 0.$$

By using (4.16) and the fact that $\frac{1}{\lambda_n}\|\nabla v_n\|_2^2$ and $\|f^{1/2}\delta_n\|_{2,\Gamma_1}^2$ are uniformly bounded, we obtain

$$\|m^{1/2}z_n\|_{2,\Gamma_1}^2\to 0. \quad (4.19)$$

Combining (4.7) with (4.15), (4.16) and (4.19). Then, using (4.18), we find that

$$a(u_n,u_n)\to\frac{1}{2}, \quad (4.20)$$

and

$$\|v_n\|_2^2\to\frac{1}{2}. \quad (4.21)$$

It's easy to see that $\frac{1}{\lambda_n}v_n\in\mathrm{L}_g^2(\mathbb{R}_+;\mathrm{H}_{\Gamma_0}^1(\Omega))$. Then, taking the inner product (3.3) of (4.11) with $\frac{1}{\lambda_n}v_n$, we have

$$\frac{1}{\lambda_n}\left\langle\eta_n(s),v_n\right\rangle_{\mathrm{L}_g^2}+\frac{1}{\lambda_n^2}\left\langle\partial_s\eta_n(s),v_n\right\rangle_{\mathrm{L}_g^2}-\frac{1}{\lambda_n^2}\left\langle v_n,v_n\right\rangle_{\mathrm{L}_g^2}\to 0. \quad (4.22)$$

Using again the fact that $\frac{v_n}{\lambda_n}$ is bounded in $\mathrm{H}_{\Gamma_0}^1(\Omega)$ and by using (4.15), we get that the first term of (4.22) converges to zero. This yields

$$\frac{(1-\ell)}{\lambda_n^2}a(v_n,v_n)-\underbrace{\frac{1}{\lambda_n^2}\int_0^{+\infty}g(s)a(\partial_s\eta_n(s),v_n)\mathrm{d}s}_{I_1}\to 0. \quad (4.23)$$

The second term (I_1) in (4.23) converges to zero. Indeed, from (2.3) and by using again that $\dfrac{v_n}{\lambda_n}$ is bounded in $\mathrm{H}^1_{\Gamma_0}(\Omega)$, we have

$$\begin{aligned} |I_1| &= \frac{1}{|\lambda_n|}\left|\int_0^{+\infty} g'(s)a(\eta_n(s),\frac{v_n}{\lambda_n})\mathrm{d}s\right| \\ &\leq \frac{\alpha a_1}{|\lambda_n|}\left\|\frac{\nabla v_n}{\lambda_n}\right\|_2\left(\frac{(1-\ell)}{a_0}\int_0^{+\infty} g(s)a(\eta_n(s),\eta_n(s))\mathrm{d}s\right)^{1/2} \to 0, \end{aligned}$$

where $a_1 = \max\limits_{j=\overline{1,n}}\left(\sum\limits_{i=1}^{n}\|a_{ij}\|^2_\infty\right)$. This with (4.23), leads to

$$\frac{v_n}{\lambda_n} \to 0 \quad \text{in } \mathrm{H}^1_{\Gamma_0}(\Omega). \tag{4.24}$$

Taking the inner product of (4.9) with ℓu_n in $\mathrm{H}^1_0(\Omega)$. Since u_n is bounded in $\mathrm{H}^1_{\Gamma_0}(\Omega)$. By using (4.24), we obtain

$$a(u_n,u_n) \to 0.$$

This contradicts (4.20). Therefore, the proof is completed. $\square$

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