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Exponential dichotomy and invariant manifolds of semi-linear differential equations on the line

Trinh Viet Duoc and Nguyen Ngoc Huy

Abstract. In this paper we investigate the homogeneous linear differential equation $v'(t) = A(t)v(t)$ and the semi-linear differential equation

$$v'(t) = A(t)v(t) + g(t, v(t))$$

in Banach space X, in which $A : \mathbb{R} \to \mathcal{L}(X)$ is a strongly continuous function, $g : \mathbb{R} \times X \to X$ is continuous and satisfies φ-Lipschitz condition. The first we characterize the exponential dichotomy of the associated evolution family with the homogeneous linear differential equation by space pair $(\mathcal{E}, \mathcal{E}_\infty)$, this is a Perron type result. Applying the achieved results, we establish the robustness of exponential dichotomy. The next we show the existence of stable and unstable manifolds for the semi-linear differential equation and prove that each a fiber of these manifolds is differentiable submanifold of class C^1.

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1. Introduction

The exponential dichotomy for the homogeneous linear differential equation $v'(t) = A(t)v(t)$ was extensively studied by mathematicians, for instance, Perron [15], Massera and Schäffer [13], Daleckii and Krein [5], Coppel [4], Chicone and Latushkin [3]. To characterize the exponential dichotomy for the homogeneous linear differential equation, Perron's method has played an underlying role up to now. Some

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efforts improved Perron's result by following two directions: one is to extend the notion of exponential dichotomy [7, 1], and the other is to change admissible space pair (input-output spaces) [16, 12, 19, 9, 17, 18].

Huy [9] had characterized the exponential dichotomy of evolution equations on a half-line by using the notion of admissible Banach function space. Through this notion, his group got some extended results for the existence of stable and unstable manifolds of evolution equations [10, 11].

In [1], the authors investigated the exponential dichotomy for the homogeneous linear differential equation $v'(t) = A(t)v(t)$, in which $A : \mathbb{R} \to \mathcal{L}(X)$ is a strongly continuous function. The notion of exponential dichotomy in [1] was with respect to the family of norms $\|\cdot\|_t$ on X for $t \in \mathbb{R}$. It was characterized by space pair (Y, Y), where $Y = C_b(\mathbb{R}, X)$ is equipped with the norm $\|v\|_\infty = \sup_{t\in\mathbb{R}} \|v(t)\|_t$, for $v \in Y$. So, the paper [1] has inspired us to investigate the exponential dichotomy for the homogeneous linear differential equation $v'(t) = A(t)v(t)$ in the present paper. Different from [1], in this paper we consider Banach space X with a fixed norm but our space pair is wider.

It is the aim of this paper to investigate the homogeneous linear differential equation $v'(t) = A(t)v(t)$ and the semi-linear differential equation $v'(t) = A(t)v(t) + g(t, v(t))$ in Banach space X, in which $A : \mathbb{R} \to \mathcal{L}(X)$ is a strongly continuous function, $g : \mathbb{R} \times X \to X$ is continuous and satisfies φ-Lipschitz condition. In Section 2 we use Perron's method to characterize the exponential dichotomy of the associated evolution family with the homogeneous linear differential equation by space pair $(\mathcal{E}, \mathcal{E}_\infty)$, the achieved result is a significant improvement compared to previous results for the homogeneous linear differential equation. As an application of this characterization, we get the robustness of exponential dichotomy.

The stable manifold theorem is one of the most important results in the local qualitative theory of autonomous nonlinear differential equations, see [2, 8, 14]. It was extended for the semi-linear differential equation $v'(t) = A(t)v(t) + g(t, v(t))$ in Banach space X, where g satisfies constant Lipschitz condition, i.e, there exists $q > 0$ such that $\|g(t, x) - g(t, y)\| \le q\|x - y\|$ for all $t \in \mathbb{R}$ and $x, y \in X$, see [5]. In Section 3 we show the existence of stable and unstable manifolds for the semi-linear differential equation $v'(t) = A(t)v(t) + g(t, v(t))$, in which g satisfies φ-Lipschitz condition, i.e, $\|g(t, x) - g(t, y)\| \le \varphi(t)\|x - y\|$ for all $t \in \mathbb{R}$ and $x, y \in X$. Different from the constant Lipschitz case, the semi-linear differential equation surely exists solution on positive semi-axis if initial value lies in a fiber of stable manifold and on negative semi-axis if initial value lies in a fiber of unstable manifold. The same as autonomous nonlinear differential equations, each a fiber of these manifolds is differentiable submanifold of class C^1 if the map $g(t, \cdot)$ is continuously differentiable (in the sense Fréchet derivative) on X for each fixed $t \in \mathbb{R}$.

The remainder in this section, we recall some notions on Banach function spaces on the line in the paper [6]. Denote by $\mathcal{B}$ the Borel algebra and by λ the Lebesgue measure on $\mathbb{R}$. The space $L_{1,loc}(\mathbb{R})$ of real-valued locally integrable functions on $\mathbb{R}$ becomes a Fréchet space for the seminorms $p_n(f) := \int_{J_n} |f(t)|dt$, where $J_n = [n, n+1]$ for each $n \in \mathbb{Z}$ (see [13, Chapt. 2, §20]).

Definition 1.1. A vector space E of real-valued Borel-measurable functions on $\mathbb{R}$ is called *a Banach function space* (over $(\mathbb{R}, \mathcal{B}, \lambda)$) if

1) E is Banach lattice with respect to a norm $\|\cdot\|_E$, i.e., $(E, \|\cdot\|_E)$ is a Banach space, and if $\varphi \in E$ and ψ is a real-valued Borel-measurable function such that $|\psi(\cdot)| \le |\varphi(\cdot)|$, λ-a.e., then $\psi \in E$ and $\|\psi\|_E \le \|\varphi\|_E$,
2) the characteristic functions χ_A belong to E for all $A \in \mathcal{B}$ of finite measure, and $\sup_{t\in\mathbb{R}} \|\chi_{[t,t+1]}\|_E < \infty$ and $\inf_{t\in\mathbb{R}} \|\chi_{[t,t+1]}\|_E > 0$,
3) $E \hookrightarrow L_{1,loc}(\mathbb{R})$, i.e., for each seminorm p_n of $L_{1,loc}(\mathbb{R})$ there exists a number $\beta_{p_n} > 0$ such that $p_n(f) \le \beta_{p_n} \|f\|_E$ for all $f \in E$.

The following lemma is very useful in the later sections.

Lemma 1.2. *Let E be a Banach function space. Let φ and ψ be real-valued, measurable functions on $\mathbb{R}$ such that they coincide with each other outside a compact interval and they are essentially bounded on this compact interval. Then $\varphi \in E$ if only if $\psi \in E$.*

Definition 1.3. Let now E be a Banach function space and X a Banach space. The set

$$\mathcal{E} := \mathcal{E}(\mathbb{R}, X) := \{f : \mathbb{R} \to X : f \text{ is strongly measurable and } \|f(\cdot)\| \in E\}$$

is endowed the norm

$$\|f\|_{\mathcal{E}} := \|\|f(\cdot)\|\|_E.$$

Then, $\mathcal{E}$ is a Banach space and is called *Banach space corresponding to the Banach function space E.*

Definition 1.4. The Banach function space E is called *admissible* if

1. there is a constant $M \ge 1$ such that for every compact interval $[a, b] \subset \mathbb{R}$ we have
$$\int_a^b |\varphi(t)| dt \le \frac{M(b-a)}{\|\chi_{[a,b]}\|_E} \|\varphi\|_E \text{ for all } \varphi \in E, \tag{1.1}$$
2. for $\varphi \in E$ the function $\Lambda_1\varphi$ defined by $\Lambda_1\varphi(t) := \int_t^{t+1} \varphi(\tau)d\tau$ belongs to E.
3. E is T_τ^+-invariant and T_τ^--invariant, where T_τ^+ and T_τ^- are defined by
$$T_\tau^+\varphi(t) := \varphi(t - \tau) \text{ for } t \in \mathbb{R},$$
$$T_\tau^-\varphi(t) := \varphi(t + \tau) \text{ for } t \in \mathbb{R},$$
and there exists constants N_1, N_2 such that $\|T_\tau^+\| \le N_1$, $\|T_\tau^-\| \le N_2$ for all $\tau \in \mathbb{R}_+$.

Remark 1.5. It can be easily seen that if E is an admissible Banach function space then $E \hookrightarrow \mathbf{M}(\mathbb{R})$, where

$$\mathbf{M}(\mathbb{R}) = \left\{ f \in L_{1,\mathrm{loc}}(\mathbb{R}) : \sup_{t\in\mathbb{R}} \int_t^{t+1} |f(\tau)| d\tau < \infty \right\}.$$

We now collect some properties of admissible Banach function space in the following proposition, see [6, Proposition 2.3] for complete proof.

Proposition 1.6. *Let E be an admissible Banach function space. The following assertions hold.*

(a) Let $\varphi \in L_{1,\,loc}(\mathbb{R})$ such that $\varphi \geq 0$ and $\Lambda_1\varphi \in E$, where $\Lambda_1\varphi$ is defined as in Definition 1.4(ii). For $\sigma > 0$ we define functions $\Lambda_\sigma\varphi$ and $\bar{\Lambda}_\sigma\varphi$ by

$$\Lambda_\sigma\varphi(t) = \int_{-\infty}^{t} e^{-\sigma(t-s)}\varphi(s)ds,$$
$$\bar{\Lambda}_\sigma\varphi(t) = \int_{t}^{\infty} e^{-\sigma(s-t)}\varphi(s)ds.$$

Then, $\Lambda_\sigma\varphi$ and $\bar{\Lambda}_\sigma\varphi$ belong to E, and

$$\|\Lambda_\sigma\varphi\|_E \leq \frac{N_1}{1-e^{-\sigma}}\|\Lambda_1\varphi\|_E, \quad \|\bar{\Lambda}_\sigma\varphi\|_E \leq \frac{N_2}{1-e^{-\sigma}}\|\Lambda_1\varphi\|_E.$$

In particular, if $\sup_{t\in\mathbb{R}}\int_t^{t+1}|\varphi(\tau)|d\tau < \infty$ (this will be satisfied if $\varphi \in E$ (see Remark 1.5)) then $\Lambda_\sigma\varphi$ and $\bar{\Lambda}_\sigma\varphi$ are bounded. Moreover, denoted by $\|\cdot\|_\infty$ for sup-norm, we have

$$\|\Lambda_\sigma\varphi\|_\infty \leq \frac{N_1}{1-e^{-\sigma}}\|\Lambda_1\varphi\|_\infty \text{ and } \|\bar{\Lambda}_\sigma\varphi\|_\infty \leq \frac{N_2}{1-e^{-\sigma}}\|\Lambda_1\varphi\|_\infty.$$

(b) E contains exponentially decaying functions $\psi(t) = e^{-\alpha|t|}$ for $t \in \mathbb{R}$ and $\alpha > 0$.

(c) E does not contain exponentially growing functions $f(t) = e^{bt}$ for $t \in \mathbb{R}$ and $b \neq 0$.

The associate space of Banach function space is defined as follows.

Definition 1.7. Let E be an admissible Banach function space and denote by $S(E)$ the unit sphere in E. Recall that $L_1(\mathbb{R}) = \{g : \mathbb{R} \to \mathbb{R} \mid g$ is Borel measurable and $\int_{-\infty}^{\infty}|g(t)|dt < \infty\}$. The set E' of all real-valued Borel-measureable functions ψ on $\mathbb{R}$ such that

$$\varphi\psi \in L_1(\mathbb{R}), \quad \int_{-\infty}^{\infty}|\varphi(t)\psi(t)|dt \leq k \quad \text{for all } \varphi \in S(E),$$

where k depends only on ψ. Then, E' is a normed space with the norm given by

$$\|\psi\|_{E'} := \sup\left\{\int_{-\infty}^{\infty}|\varphi(t)\psi(t)|dt : \varphi \in S(E)\right\} \qquad \text{for } \psi \in E'.$$

We call E' being *the associate space* of E.

Let E be an admissible Banach function space and E' be its associate space. Then, the following "Hölder-type inequality" holds:

$$\int_{-\infty}^{\infty}|\varphi(t)\psi(t)|dt \leq \|\varphi\|_E\|\psi\|_{E'} \qquad \text{for all } \varphi \in E,\ \psi \in E'. \tag{1.2}$$

Definition 1.8. Let E be an admissible Banach function space and E' be its associate space. A positive function $\varphi \in E'$ is called *exponentially E-invariant* if for any fixed $\nu > 0$, the function h_ν defined by

$$h_\nu(t) := \|e^{-\nu|t-\cdot|}\varphi(\cdot)\|_{E'} \qquad \text{for } t \in \mathbb{R}$$

belongs to E.

2. Exponential dichotomy

Let $X = (X, \|\cdot\|)$ be a Banach space and $\mathcal{L}(X)$ be the set of all bounded linear operators on X. Assume that $A : \mathbb{R} \to \mathcal{L}(X)$ is strongly continuous function (that means the mapping $t \mapsto A(t)x$ is continuous on $\mathbb{R}$ for each $x \in X$). Then, the linear differential equation

$$v' = A(t)v, \quad t \in \mathbb{R} \tag{2.1}$$

generates an evolution family $(T(t,\tau))_{t,\tau\in\mathbb{R}}$ on the Banach space X. This evolution family is strongly continuous, exponentially unbounded, differentiable and invertible (see [5] to more detailed informations), also called the associated evolution family with Eq. (2.1). In this section we characterize the exponential dichotomy of the associated evolution family with Eq. (2.1) and show that the exponential dichotomy is invariant under small perturbations. Firstly, we recall the concept of the exponential dichotomy of the evolution family $(T(t,\tau))_{t,\tau\in\mathbb{R}}$ on the line.

Definition 2.1. The associated evolution family $(T(t,\tau))_{t,\tau\in\mathbb{R}}$ is said to have an exponential dichotomy on the line if there exist bounded linear projections $P(t)$, $t \in \mathbb{R}$ on X and positive constants N, η, ν such that

(a)

$$T(t,\tau)P(\tau) = P(t)T(t,\tau), \quad t,\tau \in \mathbb{R}; \tag{2.2}$$

(b) for all $x \in X$ and $t \geq \tau$,

$$\begin{aligned} \|T(t,\tau)P(\tau)x\| &\leq Ne^{-\eta(t-\tau)}\|x\|, \\ \|T(\tau,t)Q(t)x\| &\leq Ne^{-\eta(t-\tau)}\|x\|; \end{aligned} \tag{2.3}$$

(c) for all $x \in X$ and $t \leq \tau$,

$$\begin{aligned} \|T(t,\tau)P(\tau)x\| &\leq Ne^{-\nu(t-\tau)}\|x\|, \\ \|T(\tau,t)Q(t)x\| &\leq Ne^{-\nu(t-\tau)}\|x\|; \end{aligned} \tag{2.4}$$

in which $Q(t) = I - P(t)$, $t \in \mathbb{R}$.

Note that Definition 2.1 is derived from the concept of the exponential dichotomy in [1] when the family of norms is a fixed norm for all $t \in \mathbb{R}$. This definition is also equivalent to the concept of the exponential dichotomy of a strongly continuous, exponentially bounded, and invertible evolution family.

To characterize the exponential dichotomy of the associated evolution family $(T(t,\tau))_{t,\tau\in\mathbb{R}}$, we define Banach space $\mathcal{E}_\infty$ as follows

$$\mathcal{E}_\infty = \mathcal{E} \cap C_b(\mathbb{R}, X) \ \text{ with the norm } \ \|f\|_{\mathcal{E}_\infty} = \max\{\|f\|_{\mathcal{E}}, \|f\|_\infty\}.$$

The next part we will characterize the exponential dichotomy of the associated evolution family with Eq. (2.1) by space pair $(\mathcal{E}, \mathcal{E}_\infty)$. From the properties of the admissible Banach function space, we see that the output solution has better information than the input function. So the output space is smaller than the input space in our results. We now give necessary condition for the exponential dichotomy in the following theorem.

Theorem 2.2. *Assume that the associated evolution family* $(T(t,\tau))_{t,\tau\in\mathbb{R}}$ *has exponential dichotomy on the line. Then,*

1) *for each* $y \in \mathcal{E}$ *there exists a unique* $v \in \mathcal{E}_\infty$ *that is absolutely continuous on each* $[a,b] \subset \mathbb{R}$ *and satisfies*
$$v'(t) - A(t)v(t) = y(t) \quad \textit{for a.e.} \quad t \in \mathbb{R}; \tag{2.5}$$
2) *there exist* $K, \alpha > 0$ *such that*
$$\|T(t,\tau)\| \le Ke^{\alpha|t-\tau|} \quad \textit{for} \quad t,\tau \in \mathbb{R}. \tag{2.6}$$

Remark 2.3. The absolute continuity of v on each $[a,b] \subset \mathbb{R}$ guarantees that v is differentiable almost everywhere and furthermore the Newton-Leibniz formula for Bochner integral holds for v.

Proof. Take $y \in \mathcal{E}$, for $t \in \mathbb{R}$ we define
$$v(t) = \int_{-\infty}^{t} T(t,\tau)P(\tau)y(\tau)d\tau - \int_{t}^{\infty} T(t,\tau)Q(\tau)y(\tau)d\tau. \tag{2.7}$$
It follows from (2.3) and Proposition 1.6 that
$$\begin{aligned}\|v(t)\| &\le \int_{-\infty}^{t} \|T(t,\tau)P(\tau)y(\tau)\|d\tau + \int_{t}^{\infty} \|T(t,\tau)Q(\tau)y(\tau)\|d\tau \\ &\le N\int_{-\infty}^{t} e^{-\eta(t-\tau)}\|y(\tau)\|d\tau + N\int_{t}^{\infty} e^{-\eta(\tau-t)}\|y(\tau)\|d\tau \\ &= N\Lambda_\eta\varphi(t) + N\bar{\Lambda}_\eta\varphi(t),\end{aligned}$$
where $\varphi(t) = \|y(t)\|$. So $v(t)$ is well defined, continuous and bounded. On the other hand, by Banach lattice property of E we also obtain
$$\|v\|_{\mathcal{E}} \le \frac{NN_1}{1-e^{-\eta}}\|\Lambda_1\varphi\|_E + \frac{NN_2}{1-e^{-\eta}}\|\Lambda_1\varphi\|_E.$$
Therefore, $v \in \mathcal{E}_\infty$ and $\|v\|_{\mathcal{E}_\infty} \le N(N_1+N_2)(1-e^{-\eta})^{-1}\|\Lambda_1\varphi\|_{\mathcal{E}_\infty}$.
Moreover, given $t_0 \in \mathbb{R}$, by directly computing we have
$$\begin{aligned}v(t) &= T(t,t_0)v(t_0) + \int_{t_0}^{t} T(t,\tau)y(\tau)d\tau \\ &= T(t,t_0)\Big[v(t_0) + \int_{t_0}^{t} T(t_0,\tau)y(\tau)d\tau\Big],\end{aligned} \tag{2.8}$$
for $t \in \mathbb{R}$. Since $T(t,\tau)$ is the evolution family of Eq. (2.1) and property of Bochner integral, it follows from (2.8) that the function $v : \mathbb{R} \to X$ is differentiable almost everywhere and that identity (2.5) holds for a.e. $t \in \mathbb{R}$. Because $T(t_0,\tau)y(\tau)$ is locally Bochner-integrable function so
$$v(t_0) + \int_{t_0}^{t} T(t_0,\tau)y(\tau)d\tau, \quad t \in \mathbb{R}$$
is absolutely continuous function on each $[a,b] \subset \mathbb{R}$. On the other hand, $T(t,t_0)$ and $T(t_0,t)$ are continuously differentiable on $\mathbb{R}$ follow uniform topology in $\mathcal{L}(X)$. Therefore, $T(t,t_0)f(t)$ and $T(t_0,t)f(t)$ are absolutely continuous functions on each

$[a,b] \subset \mathbb{R}$ if so is f. This means that v is absolutely continuous on each $[a,b] \subset \mathbb{R}$. We now show that v is the unique function in $\mathcal{E}_\infty$ satisfying (2.5) for a.e. $t \in \mathbb{R}$.

Indeed, let $v_1 \in \mathcal{E}_\infty$ be absolutely continuous function on each $[a,b] \subset \mathbb{R}$ and satisfy (2.5) for a.e. $t \in \mathbb{R}$. So that

$$v_1'(t) - A(t)v_1(t) = y(t) \quad \text{for a.e.} \quad t \in \mathbb{R}.$$

Put $z(t) = T(t_0,t)v_1(t)$. Then, z is absolutely continuous on each $[a,b] \subset \mathbb{R}$, differentiable almost everywhere and

$$z'(t) = T(t_0,t)y(t) \quad \text{for a.e.} \quad t \in \mathbb{R}.$$

Thus,

$$z(t) - z(t_0) = \int_{t_0}^{t} z'(\tau)d\tau = \int_{t_0}^{t} T(t_0,\tau)y(\tau)d\tau.$$

This implies that

$$v_1(t) = T(t,t_0)z(t) = T(t,t_0)v_1(t_0) + \int_{t_0}^{t} T(t,\tau)y(\tau)d\tau.$$

Put $w(t) = v(t) - v_1(t)$, we have $w \in \mathcal{E}_\infty$ and $w(t) = T(t,t_0)w(t_0)$ for $t, t_0 \in \mathbb{R}$. For $\tau \geq 0$, using (2.2) and (2.3) we obtain

$$\begin{aligned}
\|P(t)w(t)\| &= \|T(t,t-\tau)P(t-\tau)w(t-\tau)\| \leq Ne^{-\eta\tau}\|w\|_{\mathcal{E}_\infty},\\
\|Q(t)w(t)\| &= \|T(t,t+\tau)Q(t+\tau)w(t+\tau)\| \leq Ne^{-\eta\tau}\|w\|_{\mathcal{E}_\infty}.
\end{aligned}$$

Sending $\tau \to \infty$ yields that $P(t)w(t) = Q(t)w(t) = 0$ for $t \in \mathbb{R}$. Therefore, $w(t) = 0$ for $t \in \mathbb{R}$. So, v is unique.
In order to prove (2.6), we use (2.3) and (2.4). For $t \geq \tau$,

$$\begin{aligned}
\|T(t,\tau)x\| &\leq \|T(t,\tau)P(\tau)x\| + \|T(t,\tau)Q(\tau)x\|\\
&\leq Ne^{-\eta(t-\tau)}\|x\| + Ne^{\nu(t-\tau)}\|x\| \leq 2Ne^{\nu(t-\tau)}\|x\|;
\end{aligned}$$

and for $t \leq \tau$,

$$\begin{aligned}
\|T(t,\tau)x\| &\leq \|T(t,\tau)P(\tau)x\| + \|T(t,\tau)Q(\tau)x\|\\
&\leq Ne^{\nu(\tau-t)}\|x\| + Ne^{-\eta(\tau-t)}\|x\| \leq 2Ne^{\nu(\tau-t)}\|x\|.
\end{aligned}$$

Thus, (2.6) holds with $K = 2N$ and $\alpha = \nu$. $\square$

The next we show that (2.5) and (2.6) are also sufficient condition for the exponential dichotomy of associated evolution family $(T(t,\tau))_{t,\tau\in\mathbb{R}}$.

Theorem 2.4. *Assume that the assertions 1) and 2) in Theorem 2.2 are true. Then, associated evolution family $(T(t,\tau))_{t,\tau\in\mathbb{R}}$ has exponential dichotomy on the line.*

Proof. The proof scheme is the same as [1, Theorem 2.3]. For the sake of completeness, we still present the complete proof.

Linear operator $H : \mathcal{D}(H) \subset \mathcal{E}_\infty \to \mathcal{E}$ is defined as follows:

$$\begin{aligned}(Hv)(t) &= v'(t) - A(t)v(t), \quad t \in \mathbb{R},\\ \mathcal{D}(H) &= \{v \in \mathcal{E}_\infty \text{ is absolutely continuous function}\\ &\qquad \text{on each } [a,b] \subset \mathbb{R} \text{ such that } Hv \in \mathcal{E}\}.\end{aligned} \tag{2.9}$$

Then, $(H, \mathcal{D}(H))$ is closed operator. Indeed, let $\{v_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathcal{D}(H)$ such that $v_k \to v$ in $\mathcal{E}_\infty$ and $y_k := Hv_k \to y$ in $\mathcal{E}$. For each fixed $\tau \in \mathbb{R}$ and $t \geq \tau$, we have

$$\begin{aligned}v(t) - v(\tau) &= \lim_{k\to\infty}(v_k(t) - v_k(\tau)) = \lim_{k\to\infty}\int_\tau^t v_k'(s)ds\\ &= \lim_{k\to\infty}\int_\tau^t (y_k(s) + A(s)v_k(s))ds.\end{aligned}$$

On the other hand, by (1.1)

$$\left\|\int_\tau^t y_k(s)ds - \int_\tau^t y(s)ds\right\| \leq \int_\tau^t \|y_k(s) - y(s)\|ds \leq \frac{M(t-\tau)}{\|\chi_{[\tau,t]}\|_E}\|y_k - y\|_{\mathcal{E}}.$$

Therefore,

$$\lim_{k\to\infty}\int_\tau^t y_k(s)ds = \int_\tau^t y(s)ds.$$

Similarly,

$$\begin{aligned}\left\|\int_\tau^t A(s)v_k(s)ds - \int_\tau^t A(s)v(s)ds\right\| &\leq M_1\int_\tau^t \|v_k(s) - v(s)\|ds\\ &\leq M_1(t-\tau)\|v_k - v\|_{\mathcal{E}_\infty}\end{aligned}$$

with $M_1 = \sup\{\|A(s)\| : s \in [\tau, t]\}$. Thus,

$$\lim_{k\to\infty}\int_\tau^t A(s)v_k(s)ds = \int_\tau^t A(s)v(s)ds.$$

So,

$$v(t) - v(\tau) = \int_\tau^t (A(s)v(s) + y(s))ds.$$

This implies that $v(t)$ is absolutely continuous on each $[a,b] \subset \mathbb{R}$, differentiable almost everywhere and $v'(t) = A(t)v(t) + y(t)$ for a.e. $t \in \mathbb{R}$. So, $Hv = y$ and $v \in \mathcal{D}(H)$. Therefore, $(H, \mathcal{D}(H))$ is closed operator.

By the assumption, $H : \mathcal{D}(H) \to \mathcal{E}$ is bijective. So the operator H has an inverse operator $G : \mathcal{E} \to \mathcal{D}(H)$. Because G is closed operator and $\mathcal{D}(G) = \mathcal{E}$ is Banach space so G is bounded.

We now construct stable and unstable subspaces, for $\tau \in \mathbb{R}$

$$F_\tau^s = \{x \in X : \chi_{[\tau,\infty)}(\cdot)T(\cdot,\tau)x \in \mathcal{E} \quad \text{and} \quad \sup_{t\geq\tau}\|T(t,\tau)x\| < \infty\}, \tag{2.10}$$

$$F_\tau^u = \{x \in X : \chi_{(-\infty,\tau]}(\cdot)T(\cdot,\tau)x \in \mathcal{E} \quad \text{and} \quad \sup_{t\leq\tau}\|T(t,\tau)x\| < \infty\}. \tag{2.11}$$

Then, F^s_τ and F^u_τ are subspaces. The next we show that the associated evolution family $(T(t,\tau))_{t,\tau\in\mathbb{R}}$ has exponential dichotomy corresponding to F^s_τ and F^u_τ subspaces. To track easily we will split the proof process into lemmas below.

Lemma 2.5. *$X = F^s_\tau \oplus F^u_\tau$ for each $\tau \in \mathbb{R}$.*

Proof. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth function supported on $[\tau, \infty)$ such that $0 \le \phi \le 1$, $\phi = 1$ on $[\tau + 1, \infty)$ and $\sup_{t\in\mathbb{R}} |\phi'(t)| < \infty$. Given $x \in X$, put $g(t) = \phi'(t)T(t,\tau)x$. By (2.6) we get

$$\begin{aligned}\|g(t)\| &= \|\chi_{[\tau,\tau+1]}(t)\phi'(t)T(t,\tau)x\| \\ &\le \chi_{[\tau,\tau+1]}(t)\sup_{t\in\mathbb{R}}|\phi'(t)|Ke^{\alpha}\|x\| \qquad \text{for all } \ t \in \mathbb{R}.\end{aligned}$$

By Banach lattice property then $\|g(\cdot)\| \in E$. Thus, $g \in \mathcal{E}_\infty$. Because H is bijective so there exists unique $v \in \mathcal{D}(H) \subset \mathcal{E}_\infty$ such that $Hv = g$ for all $t \in \mathbb{R}$. Denoted $w(t) = (1 - \phi(t))T(t,\tau)x + v(t)$ for $t \in \mathbb{R}$, we check easily $Hw = 0$. Therefore, w is a solution of Eq. (2.1). For $t \ge \tau$, we get

$$\|w(t)\| \le \chi_{[\tau,\tau+1]}(t)Ke^{\alpha}\|x\| + \|v(t)\|.$$

This implies $\chi_{[\tau,\infty)}(\cdot)\|w(\cdot)\| \in E$. Thus, $w(\tau) \in F^s_\tau$.
On the other hand, $w(t) - T(t,\tau)x$ is also a solution of Eq. (2.1). For $t \le \tau$, we have $w(t) - T(t,\tau)x = v(t)$ so thus $\chi_{(-\infty,\tau]}(\cdot)(w(\cdot) - T(\cdot,\tau)x) \in \mathcal{E}$ and

$$\sup_{t\le\tau}\|w(t) - T(t,\tau)x\| < \infty.$$

Therefore, $w(\tau) - x \in F^u_\tau$. Hence, $x \in F^s_\tau + F^u_\tau$ for all $x \in X$.

If $x \in F^s_\tau \cap F^u_\tau$ then $u(\cdot) := T(\cdot,\tau)x \in \mathcal{E}_\infty$. Furthermore, u is absolutely continuous function on each compact interval in $\mathbb{R}$. Therefore, $u \in \mathcal{D}(H)$. Since H is invertible and $Hu = 0$ so $u = 0$ for a.e. $t \in \mathbb{R}$. Because u is continuous function so $u = 0$ for all $t \in \mathbb{R}$. Thus, $x = 0$. So, $F^s_\tau \cap F^u_\tau = \{0\}$. □

The decomposition in Lemma 2.5 determines a complementary projection pair $P(\tau) : X \to F^s_\tau$ and $Q(\tau) : X \to F^u_\tau$ for each $\tau \in \mathbb{R}$. These projections are uniformly bounded.

Lemma 2.6. *There exists $M > 0$ such that*

$$\|P(\tau)x\| \le M\|x\| \tag{2.12}$$

for $x \in X$ and $\tau \in \mathbb{R}$.

Proof. Using the same notation as in the proof of Lemma 2.5, we get

$$\begin{aligned}\|P(\tau)x\| = \|w(\tau)\| &\le \|x\| + \|v(\tau)\| \le \|x\| + \|v\|_{\mathcal{E}_\infty} \\ &= \|x\| + \|Gg\|_{\mathcal{E}_\infty} \le \|x\| + \|G\|\|g\|_{\mathcal{E}}.\end{aligned}$$

Moreover, we have

$$\|g\|_{\mathcal{E}} \le \|\chi_{[\tau,\tau+1]}\|_E LKe^{\alpha}\|x\| \le \sup_{\tau\in\mathbb{R}}\|\chi_{[\tau,\tau+1]}\|_E LKe^{\alpha}\|x\|,$$

where $L = \sup_{t\in\mathbb{R}} |\phi'(t)| < \infty$. Therefore,

$$\|P(\tau)x\| \leq (1 + \|G\|LKe^{\alpha} \sup_{\tau\in\mathbb{R}} \|\chi_{[\tau,\tau+1]}\|_E)\|x\|. \qquad \square$$

We prove that property (2.2) holds in the following lemma.

Lemma 2.7.

$$T(t,\tau)P(\tau) = P(t)T(t,\tau) \quad \textit{for} \quad t,\, \tau \in \mathbb{R}.$$

Proof. Using the same notation as in the proof of Lemma 2.5. We prove this lemma in several steps.
Step 1. We show that $T(t,\tau)w(\tau) \in F_t^s$. Indeed,

$$\begin{aligned}\chi_{[t,\infty)}(\xi)T(\xi,t)T(t,\tau)w(\tau) &= \chi_{[t,\infty)}(\xi)T(\xi,\tau)w(\tau)\\ &= \begin{cases} 0 & \text{if} \quad \xi < t,\\ w(t) & \text{if} \quad \xi \geq t.\end{cases}\end{aligned}$$

Thus, $\chi_{[t,\infty)}(\cdot)T(\cdot,t)T(t,\tau)w(\tau) \in \mathcal{E}_\infty$. This implies $T(t,\tau)w(\tau) \in F_t^s$.
Step 2. We prove that $T(t,\tau)v(\tau) \in F_t^u$. Indeed, by $Hv = g$ we have

$$v(t) = T(t,\tau)v(\tau) + \int_\tau^t T(t,\xi)g(\xi)d\xi \quad \text{for} \quad t,\, \tau \in \mathbb{R}.$$

Therefore,

$$\begin{aligned}T(t,\tau)v(\tau) &= v(t) - \int_\tau^t T(t,\xi)g(\xi)d\xi\\ &= v(t) - \int_\tau^t T(t,\xi)\phi'(\xi)T(\xi,\tau)xd\xi\\ &= v(t) - \int_\tau^t \phi'(\xi)T(t,\tau)xd\xi\\ &= \begin{cases} v(t) & \text{if} \quad t \leq \tau,\\ v(t) - \phi(t)T(t,\tau)x & \text{if} \quad t \geq \tau.\end{cases}\end{aligned}$$

Hence,

$$\begin{aligned}\chi_{(-\infty,t]}(\xi)T(\xi,t)T(t,\tau)v(\tau) &= \begin{cases} 0 & \text{if} \quad \xi > t,\\ T(\xi,\tau)v(\tau) & \text{if} \quad \xi \leq t,\end{cases}\\ &= \begin{cases} 0 & \text{if} \quad \xi > t,\\ v(\xi) - \phi(\xi)T(\xi,\tau)x & \text{if} \quad \tau \leq \xi \leq t,\\ v(\xi) & \text{if} \quad \xi < \tau.\end{cases}\end{aligned}$$

Putting

$$f(\xi) = \begin{cases} 0 & \text{if} \quad \xi \geq \tau,\\ v(\xi) & \text{if} \quad \xi < \tau.\end{cases}$$

Then, $f \in \mathcal{E}_\infty$. By Lemma 1.2, we have $\chi_{(-\infty,t]}(\cdot)T(\cdot,t)T(t,\tau)v(\tau) \in \mathcal{E}_\infty$. Therefore, $T(t,\tau)v(\tau) \in F_t^u$.

Step 3. We also have

$$\begin{aligned} T(t,\tau)P(\tau)x &= T(t,\tau)w(\tau) \\ &= T(t,\tau)x + T(t,\tau)v(\tau). \end{aligned}$$

Let projection $P(t)$ act on both sides of the above equality, we obtain

$$T(t,\tau)P(\tau)x = P(t)T(t,\tau)x \quad \text{for all} \quad x \in X. \qquad \square$$

Lemma 2.8. *There exists constants $N, \eta > 0$ such that*

$$\|T(t,\tau)x\| \le Ne^{-\eta(t-\tau)}\|x\| \tag{2.13}$$

for $x \in P(\tau)X$ and $t \ge \tau$.

Proof. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth function which has support on $[\tau, +\infty)$ such that $0 \le \psi \le 1$, $\psi = 1$ on $[\tau+1, +\infty)$ and $\sup_{t\in\mathbb{R}} |\psi'(t)| \le 2$. Given $x \in F_\tau^s$, let u be a solution of Eq. (2.1) with $u(\tau) = x$, i.e, $u(t) = T(t,\tau)x$ for $t \in \mathbb{R}$. We have

$$\begin{aligned} \|\psi(t)u(t)\| = \|\psi(t)T(t,\tau)x\| &= \|\chi_{[\tau,+\infty)}(t)\psi(t)T(t,\tau)x\| \\ &\le \|\chi_{[\tau,+\infty)}(t)T(t,\tau)x\|. \end{aligned}$$

From $x \in F_\tau^s$ and using (2.10) we get $\chi_{[\tau,+\infty)}(\cdot)T(\cdot,\tau)x \in \mathcal{E}_\infty$. Therefore, by Banach lattice property then we have $\psi(\cdot)u(\cdot) \in \mathcal{E}_\infty$. Moreover, we have $H(\psi u) = \psi' u$ and

$$\|(\psi' u)(\xi)\| \le 2\chi_{[\tau,\tau+1]}(\xi)\|u(\xi)\| \le 2\chi_{[\tau,\tau+1]}(\xi)Ke^{\alpha}\|x\|, \quad \xi \in \mathbb{R}.$$

Thus,

$$\|\|\psi' u\|\|_E = \|\psi' u\|_{\mathcal{E}} \le 2Ke^{\alpha}\|\chi_{[\tau,\tau+1]}\|_E\|x\|.$$

- For $t \ge \tau + 1$,

$$\begin{aligned} \|u(t)\| = \|\psi(t)u(t)\| = \|G(\psi' u)(t)\| &\le \|G(\psi' u)\|_{\mathcal{E}_\infty} \\ &\le \|G\|\|\psi' u\|_{\mathcal{E}} \le 2\|G\|Ke^{\alpha}\|\chi_{[\tau,\tau+1]}\|_E\|x\|. \end{aligned}$$

- For $\tau \le t \le \tau + 1$,

$$\|u(t)\| = \|T(t,\tau)x\| \le Ke^{\alpha}\|x\|.$$

Therefore,

$$\|u(t)\| \le C\|x\| \quad \text{for} \quad t \ge \tau, \tag{2.14}$$

where $C = Ke^{\alpha}\max\{2\|G\|\sup_{\tau\in\mathbb{R}}\|\chi_{[\tau,\tau+1]}\|_E, 1\}$.

The next, we show that there exists $m \in \mathbb{N}$ such that

$$\|u(t)\| \le \frac{1}{2}\|x\| \quad \text{for} \quad t - \tau \ge m,\ \tau \in \mathbb{R}. \tag{2.15}$$

In order to prove (2.15), let

$$y(\xi) = \chi_{[\tau,t]}(\xi)u(\xi) \text{ and } v(\xi) = u(\xi)\int_{-\infty}^{\xi}\chi_{[\tau,t]}(s)ds.$$

It can be seen that $y \in \mathcal{E}$, $v \in \mathcal{D}(H) \subset \mathcal{E}_\infty$ and $Hv = y$. Therefore,

$$\|v\|_{\mathcal{E}_\infty} = \|Gy\|_{\mathcal{E}_\infty} \le \|G\|\|y\|_{\mathcal{E}}.$$

On the other hand,

$$\|y(\xi)\| \le \begin{cases} \chi_{[\tau,\tau+1]}(\xi)Ke^{\alpha}\|x\|, \ \xi \in [\tau,\tau+1], \\ G(\psi' u)(\xi), \ \xi \in [\tau+1,+\infty). \end{cases}$$

Thus,

$$\|y\|_{\mathcal{E}} \le \max\{Ke^{\alpha} \sup_{\tau\in\mathbb{R}} \|\chi_{[\tau,\tau+1]}\|_E, 2Ke^{\alpha}\|G\| \sup_{\tau\in\mathbb{R}} \|\chi_{[\tau,\tau+1]}\|_E\}\|x\| =: K_1\|x\|.$$

So,

$$\|v\|_{\mathcal{E}_\infty} \le \|G\|K_1\|x\|.$$

We have

$$(t-\tau)\|u(t)\| = \|v(t)\| \le \|v\|_{\mathcal{E}_\infty} \le \|G\|K_1\|x\|.$$

Therefore,

$$\|u(t)\| \le \frac{\|G\|K_1}{t-\tau}\|x\|.$$

Hence, if $t-\tau \ge 2K_1\|G\|$ then $\|u(t)\| \le \frac{1}{2}\|x\|$. Taking $m > 2K_1\|G\|$, we obtain (2.15). Finally, take $t \ge \tau$ and write $t-\tau = km + r$ with $k \in \mathbb{N}$ and $0 \le r < m$. By (2.12), (2.14), (2.15), and Lemma 2.7 we get

$$\begin{aligned} \|T(t,\tau)P(\tau)x\| &= \|T(\tau+km+r,\tau)P(\tau)x\| \le C\|T(\tau+km,\tau)P(\tau)x\| \\ &\le \frac{C}{2^k}\|P(\tau)x\| \le 2CMe^{-(t-\tau)\frac{\ln 2}{m}}\|x\| \quad \text{for} \quad x \in X. \end{aligned}$$

□

Lemma 2.9. *There exists constants $N, \eta > 0$ such that*

$$\|T(t,\tau)x\| \le Ne^{-\eta(\tau-t)}\|x\| \tag{2.16}$$

for $x \in \operatorname{Ker}P(\tau) = Q(\tau)X$ and $t \le \tau$.

Proof. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth function supported on $(-\infty,\tau]$ such that $0 \le \psi \le 1$, $\psi = 1$ on $(-\infty,\tau-1]$ and $\sup_{t\in\mathbb{R}} |\psi'(t)| \le 2$. Given $x \in F^u_\tau$, let u be a solution of Eq. (2.1) with $u(\tau) = x$. We have

$$\|\psi(t)u(t)\| = \|\chi_{(-\infty,\tau]}(t)\psi(t)T(t,\tau)x\| \le \|\chi_{(-\infty,\tau]}(t)T(t,\tau)x\|.$$

From $x \in F^u_\tau$ and using (2.11) we get $\chi_{(-\infty,\tau]}(\cdot)T(\cdot,\tau)x \in \mathcal{E}_\infty$. Therefore, $\psi(\cdot)u(\cdot) \in \mathcal{E}_\infty$. Furthermore, we can also easily verify that $H(\psi u) = \psi' u$. We have

$$|(\psi' u)(\xi)\| \le 2\chi_{[\tau-1,\tau]}(\xi)\|u(\xi)\| \le 2\chi_{[\tau-1,\tau]}(\xi)Ke^{\alpha}\|x\|, \quad \xi \in \mathbb{R}.$$

Thus,

$$\|\|\psi' u\|\|_E = \|\psi' u\|_{\mathcal{E}} \le 2Ke^{\alpha}\|\chi_{[\tau-1,\tau]}\|_E\|x\|.$$

- For $t \le \tau - 1$,

$$\begin{aligned} \|u(t)\| &= \|\psi(t)u(t)\| = \|G(\psi' u)(t)\| \le \|G(\psi' u)\|_{\mathcal{E}_\infty} \\ &\le \|G\|\|\psi' u\|_{\mathcal{E}} \le 2\|G\|Ke^{\alpha}\|\chi_{[\tau-1,\tau]}\|_E\|x\|. \end{aligned}$$

- For $\tau - 1 \le t \le \tau$,

$$\|u(t)\| = \|T(t,\tau)x\| \le Ke^{\alpha}\|x\|.$$

Therefore,

$$\|u(t)\| \le C\|x\| \quad \text{for} \quad t \le \tau, \tag{2.17}$$

where $C = Ke^{\alpha} \max\{2\|G\| \sup_{\tau\in\mathbb{R}} \|\chi_{[\tau-1,\tau]}\|_E, 1\}$.
The next, we show that there exists $m \in \mathbb{N}$ such that

$$\|u(t)\| \le \frac{1}{2}\|x\| \quad \text{for} \quad \tau - t \ge m,\, \tau \in \mathbb{R}. \tag{2.18}$$

In order to prove (2.18), let

$$y(\xi) = -\chi_{[t,\tau]}(\xi)u(\xi) \text{ and } v(\xi) = u(\xi)\int_{\xi}^{\infty} \chi_{[t,\tau]}(s)ds.$$

It can be seen that $y \in \mathcal{E}$, $v \in \mathcal{D}(H) \subset \mathcal{E}_\infty$ and $Hv = y$. Therefore,

$$\|v\|_{\mathcal{E}_\infty} = \|Gy\|_{\mathcal{E}_\infty} \le \|G\|\|y\|_{\mathcal{E}}.$$

On the other hand,

$$\|y(\xi)\| \le \begin{cases} \chi_{[\tau-1,\tau]}(\xi)Ke^{\alpha}\|x\|,\ \xi \in [\tau-1,\tau], \\ G(\psi' u)(\xi),\ \xi \in (-\infty, \tau-1]. \end{cases}$$

Thus,

$$\|y\|_{\mathcal{E}} \le \max\{Ke^{\alpha} \sup_{\tau\in\mathbb{R}} \|\chi_{[\tau-1,\tau]}\|_E, 2Ke^{\alpha}\|G\| \sup_{\tau\in\mathbb{R}} \|\chi_{[\tau-1,\tau]}\|_E\}\|x\| =: K_2\|x\|.$$

So,

$$\|v\|_{\mathcal{E}_\infty} \le \|G\|K_2\|x\|.$$

We have

$$(\tau - t)\|u(t)\| = \|v(t)\| \le \|v\|_{\mathcal{E}_\infty} \le \|G\|K_2\|x\|.$$

Therefore,

$$\|u(t)\| \le \frac{\|G\|K_2}{\tau - t}\|x\|.$$

Hence, if $\tau - t \ge 2K_2\|G\|$ then $\|u(t)\| \le \frac{1}{2}\|x\|$. Taking $m > 2K_2\|G\|$, we obtain (2.18). In order to complete the proof, take $t \le \tau$ and write $\tau - t = km + r$ with $k \in \mathbb{N}$ and $0 \le r < m$. By (2.12), (2.17), (2.18), and Lemma 2.7 we get

$$\begin{aligned}\|T(t,\tau)Q(\tau)x\| &= \|T(\tau - km - r, \tau)Q(\tau)x\| \le C\|T(\tau - km, \tau)Q(\tau)x\| \\ &\le \frac{C}{2^k}\|Q(\tau)x\| \le 2C(1+M)e^{-(\tau-t)\frac{\ln 2}{m}}\|x\| \quad \text{for} \quad x \in X.\end{aligned}$$

□

So, we get (2.3) from (2.13) and (2.16). For $t \le \tau$, using (2.6) and (2.12) we obtain (2.4) as follows.

$$\begin{aligned}\|T(t,\tau)P(\tau)x\| &\le Ke^{\alpha|t-\tau|}\|P(\tau)x\| \le Ke^{\alpha|t-\tau|}M\|x\| = KMe^{-\alpha(t-\tau)}\|x\|, \\ \|T(\tau,t)Q(t)x\| &\le Ke^{\alpha|t-\tau|}\|Q(t)x\| \le Ke^{\alpha|t-\tau|}(1+M)\|x\| \\ &= K(1+M)e^{-\alpha(t-\tau)}\|x\|.\end{aligned}$$

Thus, the associated evolution family $(T(t,\tau))_{t,\tau\in\mathbb{R}}$ has exponential dichotomy on the line. □

In the remainder of this section we establish the robustness of the notion of exponential dichotomy. It is an application of Theorem 2.2 and Theorem 2.4.

Theorem 2.10. *Let $A, B : \mathbb{R} \to \mathcal{L}(X)$ be strongly continuous functions such that*

1. *the evolution family $(T(t,\tau))_{t,\tau\in\mathbb{R}}$ of Eq.* (2.1) *has exponential dichotomy on the line;*
2. *there exists $\varphi \in E$ such that*

$$\|B(t) - A(t)\| \leq \varphi(t) \quad \textit{for a.e.} \quad t \in \mathbb{R}. \tag{2.19}$$

Then, the evolution family $(U(t,\tau))_{t,\tau\in\mathbb{R}}$ of the equation $v' = B(t)v$ has exponential dichotomy on the line if $\|\varphi\|_E$ is sufficiently small.

Proof. Let H be the linear operator defined by (2.9) on the domain $\mathcal{D}(H)$. We define a linear operator $L : \mathcal{D}(L) \subset \mathcal{E}_\infty \to \mathcal{E}$ by

$$(Lv)(t) = v'(t) - B(t)v(t), \quad t \in \mathbb{R},$$

where $\mathcal{D}(L) = \{v \in \mathcal{E}_\infty$ is absolutely continuous function on each $[a,b] \subset \mathbb{R}$ such that $Lv \in \mathcal{E}\}$.

For $v \in \mathcal{E}_\infty$, denoted $(Pv)(t) := (B(t) - A(t))v(t)$. By (2.19) we get

$$\|(Pv)(t)\| \leq \varphi(t)\|v(t)\| \leq \varphi(t)\|v\|_{\mathcal{E}_\infty} \quad \text{for a.e} \quad t \in \mathbb{R}.$$

Therefore, $Pv \in \mathcal{E}$ and $\|Pv\|_{\mathcal{E}} \leq \|\varphi\|_E \|v\|_{\mathcal{E}_\infty}$. So, the mapping $P : \mathcal{E}_\infty \to \mathcal{E}$ is bounded linear operator and $\|P\| \leq \|\varphi\|_E$. Thus, $\mathcal{D}(H) = \mathcal{D}(L)$ and $L = H + P$. By Theorem 2.2 and Theorem 2.4, the operator H is invertible. Hence, if $\|\varphi\|_E$ is sufficiently small then L is also invertible.

Two evolution families $(U(t,\tau))_{t,\tau\in\mathbb{R}}$ and $(T(t,\tau))_{t,\tau\in\mathbb{R}}$ have the relation as follows:

$$U(t,\tau)x = T(t,\tau)x + \int_\tau^t T(t,s)(B(s) - A(s))U(s,\tau)x\, ds$$

for $t, \tau \in \mathbb{R}$ and $x \in X$. Using Gronwall inequality and the relation above, we easily get

$$\|U(t,\tau)x\| \leq K e^{\alpha|t-\tau| + K\left|\int_\tau^t \varphi(s)ds\right|} \|x\| \quad \text{for} \quad x \in X \ \text{ and } \ t, \tau \in \mathbb{R}.$$

On the other hand,

$$\left|\int_\tau^t \varphi(s)ds\right| \leq \|\Lambda_1\varphi\|_\infty(|t-\tau| + 1) \quad \text{for} \quad t, \tau \in \mathbb{R}.$$

Thus,

$$\|U(t,\tau)x\| \leq K e^{K\|\Lambda_1\varphi\|_\infty} e^{(\alpha + K\|\Lambda_1\varphi\|_\infty)|t-\tau|}\|x\| \quad \text{for} \quad x \in X \ \text{ and } \ t, \tau \in \mathbb{R}.$$

By Theorem 2.4, we deduce that the evolution family $(U(t,\tau))_{t,\tau\in\mathbb{R}}$ of the equation $v' = B(t)v$ has exponential dichotomy on the line. $\square$

3. Stable and unstable manifolds

Let be semi-linear differential equation

$$v'(t) = A(t)v(t) + g(t, v(t)), \quad t \in \mathbb{R} \tag{3.1}$$

in Banach space X, in which A and g satisfy the following assumptions.
Assumption 1: $A : \mathbb{R} \to \mathcal{L}(X)$ is strongly continuous function and generates an evolution family $(T(t,\tau))_{t,\tau\in\mathbb{R}}$ having exponential dichotomy (that means the assertions 1) and 2) in Theorem 2.2 are satisfied).
Assumption 2: $g : \mathbb{R} \times X \to X$ is continuous and satisfies φ-Lipschitz condition, i.e,

(i) $\|g(t, 0)\| \le \varphi(t)$ for $t \in \mathbb{R}$,
(ii) $\|g(t, x) - g(t, y)\| \le \varphi(t)\|x - y\|$ for $t \in \mathbb{R}$ and $x, y \in X$.

Assumption 3: E is admissible Banach function space such that its associate space E' is also admissible Banach function space and $\varphi \in E'$ is exponentially E-invariant (see Definition 1.8).

The these underlying assumptions, we show the existence of stable and unstable manifolds for the Eq. (3.1). Actually, these manifolds include trajectories of continuous solutions lying in the Banach space $\mathcal{E}$ (see Definition 1.3). We easily get the following result.

Lemma 3.1. *A function $v : \mathbb{R} \to X$ is solution of Eq.* (3.1) *if only if it is continuous on $\mathbb{R}$ and satisfies the integral equation*

$$v(t) = T(t, t_0)v(t_0) + \int_{t_0}^{t} T(t,\tau)g(\tau, v(\tau))d\tau, \quad t_0, t \in \mathbb{R}.$$

From now on we shall suppose that Assumption 1, Assumption 2 and Assumption 3 hold. For convenience, we define Green function as follows

$$\mathcal{G}(t,\tau) = \begin{cases} T(t,\tau)P(\tau) & \text{for } \ t > \tau, \\ -T(t,\tau)Q(\tau) & \text{for } \ t < \tau. \end{cases} \tag{3.2}$$

By (2.3), we have $\|\mathcal{G}(t,\tau)\| \le Ne^{-\eta|t-\tau|}$ for all $t, \tau \in \mathbb{R}$. Moreover, if a function v has the domain $D(v)$ then it can be extended on $\mathbb{R}$ by the characteristic function $\chi_{D(v)}$ as follows $(\chi_{D(v)}v)(t) = v(t)$ if $t \in D(v)$ and $(\chi_{D(v)}v)(t) = 0$ if otherwise. To construct stable and unstable manifolds we now give characteristic formula denoted solutions of Eq. (3.1) which belong to the Banach space $\mathcal{E}$.

Proposition 3.2. *The following assertions hold.*

i. The function $v \in \mathcal{E}$ is a solution of Eq. (3.1) *on $\mathbb{R}$ if only if it has the form*

$$v(t) = \int_{-\infty}^{\infty} \mathcal{G}(t,\tau)g(\tau, v(\tau))d\tau, \quad t \in \mathbb{R}.$$

ii. For each fixed s, the function $\chi_{[s,\infty)}v \in \mathcal{E}$ is a solution of Eq. (3.1) *on $[s,\infty)$ if only if there is $\nu_0 \in \mathrm{Im}P(s)$ such that*

$$v(t) = T(t, s)\nu_0 + \int_{s}^{\infty} \mathcal{G}(t,\tau)g(\tau, v(\tau))d\tau, \quad t \ge s. \tag{3.3}$$

iii. For each fixed s, the function $\chi_{(-\infty,s]}v \in \mathcal{E}$ *is a solution of Eq.* (3.1) *on* $(-\infty, s]$ *if only if there is* $\mu_0 \in \mathrm{Im}Q(s)$ *such that*

$$v(t) = T(t,s)\mu_0 + \int_{-\infty}^{s} \mathcal{G}(t,\tau)g(\tau, v(\tau))d\tau, \quad t \le s. \tag{3.4}$$

Proof. The sufficient condition in the three assertions above is checked easily by simple computations. So we only prove the necessary condition in the these.

i. Put $y(t) = \displaystyle\int_{-\infty}^{\infty} \mathcal{G}(t,\tau)g(\tau, v(\tau))d\tau,\ t \in \mathbb{R}$. Using Hölder-type inequality (1.2) and Assumption 3, we have

$$\begin{aligned}\|y(t)\| &\le N \int_{-\infty}^{\infty} e^{-\eta|t-\tau|}\varphi(\tau)(1 + \|v(\tau)\|)d\tau \\ &\le N h_{\frac{\eta}{2}}(t)\|e^{-\frac{\eta}{2}|t-\cdot|}\|_E + N h_\eta(t)\|v\|_{\mathcal{E}}.\end{aligned}$$

By iii) in the Definition 1.4, we get $\|e^{-\frac{\eta}{2}|t-\cdot|}\|_E \le \max\{N_1, N_2\}\|e_{\frac{\eta}{2}}\|_E$, in which $e_{\frac{\eta}{2}}(\tau) = e^{-\frac{\eta}{2}|\tau|}$. Therefore,

$$\|y(t)\| \le N \max\{N_1, N_2\}\|e_{\frac{\eta}{2}}\|_E h_{\frac{\eta}{2}}(t) + N\|v\|_{\mathcal{E}} h_\eta(t).$$

Because E is the Banach lattice and $h_{\frac{\eta}{2}}, h_\eta \in E$ so $\|y(\cdot)\| \in E$. Thus, $y \in \mathcal{E}$. On the other hand, y also satisfies the integral equation

$$y(t) = T(t,t_0)y(t_0) + \int_{t_0}^{t} T(t,\tau)g(\tau, v(\tau))d\tau, \quad t_0, t \in \mathbb{R}.$$

Thus,

$$v(t) - y(t) = T(t,t_0)(v(t_0) - y(t_0)).$$

Because of $v - y \in \mathcal{E}$ so we obtain $v(t_0) = y(t_0)$. This deduces $v = y$ on $\mathbb{R}$.

ii. Put $y_2(t) = \displaystyle\int_{s}^{\infty} \mathcal{G}(t,\tau)g(\tau, v(\tau))d\tau,\ t \ge s$. The similar argumentation as above, we have

$$\begin{aligned}\|y_2(t)\| &\le N \int_{s}^{\infty} e^{-\eta|t-\tau|}\varphi(\tau)(1 + \|v(\tau)\|)d\tau \\ &\le N \int_{-\infty}^{\infty} e^{-\eta|t-\tau|}\varphi(\tau)(1 + \|(\chi_{[s,\infty)}v)(\tau)\|)d\tau \\ &\le N \max\{N_1, N_2\}\|e_{\frac{\eta}{2}}\|_E h_{\frac{\eta}{2}}(t) + N h_\eta(t)\|\chi_{[s,\infty)}v\|_{\mathcal{E}}.\end{aligned}$$

Thus, $\chi_{[s,\infty)}y_2 \in \mathcal{E}$. On the other hand, y_2 also satisfies the integral equation

$$y_2(t) = T(t,s)y_2(s) + \int_{s}^{t} T(t,\tau)g(\tau, v(\tau))d\tau, \quad t \ge s.$$

Therefore, $v(t) - y_2(t) = T(t,s)(v(s) - y_2(s))$. Because of $\chi_{[s,\infty)}v - \chi_{[s,\infty)}y_2 \in \mathcal{E}$ so we obtain $v(s) - y_2(s) \in \mathrm{Im}P(s)$. So, there exists $\nu_0 \in \mathrm{Im}P(s)$ such that $v(t) = T(t,s)\nu_0 + y_2(t)$ with $t \ge s$. The last assertion is proved similarly. □

Using Proposition 3.2 and Banach fixed-point theorem we get the existence of solutions of Eq. (3.1) in the Banach space $\mathcal{E}$. The proof is basic, so we omit here.

Theorem 3.3. *Assume that* $N\|h_\eta\|_E < 1$. *Then:*

a) The Eq. (3.1) *has a unique solution in the Banach space* $\mathcal{E}$ *and this solution takes the form*

$$v^*(t) = \int_{-\infty}^{\infty} \mathcal{G}(t,\tau)g(\tau, v^*(\tau))d\tau, \quad t \in \mathbb{R}.$$

b) For each fixed s *and* $\nu_0 \in \mathrm{Im}P(s)$, *the Eq.* (3.1) *has a unique solution* v *on* $[s,\infty)$ *such that* $\chi_{[s,\infty)}v \in \mathcal{E}$ *and this solution is represented by the formula* (3.3).

c) For each fixed s *and* $\mu_0 \in \mathrm{Im}Q(s)$, *the Eq.* (3.1) *has a unique solution* v *on* $(-\infty, s]$ *such that* $\chi_{(-\infty,s]}v \in \mathcal{E}$ *and this solution is represented by the formula* (3.4).

The next, we show the existence of stable and unstable manifolds for the Eq. (3.1). These manifolds like bundles in $\mathbb{R}\times X$ space, each a fiber of these manifolds is a submanifold in X space. In precisely, it is graph of a Lipschitz map.

Theorem 3.4. *Assume that* $N^2\max\{N_1,N_2\}\|e_\eta\|_E\|\varphi\|_{E'} + N\|h_\eta\|_E < 1$, *in which* $e_\eta(\tau) = e^{-\eta|\tau|}$. *Then, there exist an invariant stable manifold* $S = \bigsqcup_{s\in\mathbb{R}} S_s$ *and an invariant unstable manifold* $U = \bigsqcup_{s\in\mathbb{R}} U_s$ *of Eq.* (3.1). *Moreover, the stable manifold has the following properties*

(i) $S_s = \{\nu_0 + g_s^{st}(\nu_0) : \nu_0 \in \mathrm{Im}P(s)\}$, *where* $g_s^{st} : \mathrm{Im}P(s) \to \mathrm{Im}Q(s)$ *is a Lipschitz map having Lipschitz coefficient*

$$\mathrm{Lip}(g_s^{st}) \le \frac{N^2\max\{N_1,N_2\}\|e_\eta\|_E\|\varphi\|_{E'}}{1 - N\|h_\eta\|_E} < 1$$

for all $s \in \mathbb{R}$;

(ii) S_s *is homeomorphic to* $\mathrm{Im}P(s)$ *for all* $s \in \mathbb{R}$;

(iii) to each $x_0 \in S_s$, *the Eq.* (3.1) *has a unique solution* v *on* $[s,\infty)$ *such that* $\chi_{[s,\infty)}v \in \mathcal{E}$, *and* $v(t) \in S_t$ *for all* $t \ge s$;

(iv) if $N(N_1+N_2)\|\Lambda_1\varphi\|_\infty < 1$ *then the solution* v^* *attracts other solutions on* S *in the sense there exist* $\mu, C_\mu > 0$ *such that*

$$\|v(t) - v^*(t)\| \le C_\mu e^{-\mu(t-s)}\|P(s)v(s) - P(s)v^*(s)\| \quad \text{for all } t \ge s,\ v(s) \in S_s;$$

and the unstable manifold has the following properties

(i) $U_s = \{\mu_0 + g_s^{un}(\mu_0) : \mu_0 \in \mathrm{Im}Q(s)\}$, *where* $g_s^{un} : \mathrm{Im}Q(s) \to \mathrm{Im}P(s)$ *is a Lipschitz map having Lipschitz coefficient*

$$\mathrm{Lip}(g_s^{un}) \le \frac{N^2\max\{N_1,N_2\}\|e_\eta\|_E\|\varphi\|_{E'}}{1 - N\|h_\eta\|_E} < 1$$

for all $s \in \mathbb{R}$;

(ii) U_s *is homeomorphic to* $\mathrm{Im}Q(s)$ *for all* $s \in \mathbb{R}$;

(iii) to each $x_0 \in U_s$, *the Eq.* (3.1) *has a unique solution* v *on* $(-\infty, s]$ *such that* $\chi_{(-\infty,s]}v \in \mathcal{E}$, *and* $v(t) \in U_t$ *for all* $t \le s$;

(iv) if $N(N_1+N_2)\|\Lambda_1\varphi\|_\infty < 1$ *then the solution* v^* *attracts other solutions on* U *in the sense there exist* $\mu, C_\mu > 0$ *such that*

$$\|v(t) - v^*(t)\| \le C_\mu e^{\mu(t-s)}\|Q(s)v(s) - Q(s)v^*(s)\| \quad \text{for all } t \le s,\ v(s) \in U_s.$$

Proof. We shall prove the existence of stable manifold and its properties, the unstable manifold is done similarly.

By Theorem 3.3, for each $\nu_0 \in \mathrm{Im}P(s)$ then the Eq. (3.1) has a unique solution v on $[s,\infty)$ such that $\chi_{[s,\infty)}v \in \mathcal{E}$. So we define the map $g_s^{st} : \mathrm{Im}P(s) \to \mathrm{Im}Q(s)$ as follows

$$g_s^{st}(\nu_0) = \int_s^{\infty} \mathcal{G}(s,\tau)g(\tau, v(\tau))d\tau, \tag{3.5}$$

where $\mathcal{G}(s,\tau)$ is the Green function defined by (3.2). For $\nu_1, \nu_2 \in \mathrm{Im}P(s)$ we have

$$\begin{aligned}
\|g_s^{st}(\nu_1) - g_s^{st}(\nu_2)\| &\le N \int_s^{\infty} e^{-\eta|s-\tau|}\varphi(\tau)\|v_1(\tau) - v_2(\tau)\|d\tau \\
&\le N \int_{-\infty}^{\infty} e^{-\eta|s-\tau|}\varphi(\tau)\|(\chi_{[s,\infty)}v_1)(\tau) - (\chi_{[s,\infty)}v_2)(\tau)\|d\tau \\
&\le N \int_{-\infty}^{\infty} \varphi(\tau)\|(\chi_{[s,\infty)}v_1)(\tau) - (\chi_{[s,\infty)}v_2)(\tau)\|d\tau \\
&\le N\|\varphi\|_{E'}\|\chi_{[s,\infty)}v_1 - \chi_{[s,\infty)}v_2\|_{\mathcal{E}} \quad (\text{ by } (1.2)).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|v_1(t) - v_2(t)\| &\le Ne^{-\eta(t-s)}\|\nu_1 - \nu_2\| \\
&\quad + N\int_s^{\infty} e^{-\eta|t-\tau|}\varphi(\tau)\|v_1(\tau) - v_2(\tau)\|d\tau \\
&\le Ne^{-\eta|t-s|}\|\nu_1 - \nu_2\| + Nh_\eta(t)\|\chi_{[s,\infty)}v_1 - \chi_{[s,\infty)}v_2\|_{\mathcal{E}}
\end{aligned}$$

for $t \ge s$, and $\|e^{-\eta|\cdot - s|}\|_E \le \max\{N_1, N_2\}\|e_\eta\|_E$. Therefore, by the Banach lattice property of E we get

$$\begin{aligned}
\|\chi_{[s,\infty)}v_1 - \chi_{[s,\infty)}v_2\|_{\mathcal{E}} &\le N\max\{N_1, N_2\}\|e_\eta\|_E\|\nu_1 - \nu_2\| \\
&\quad + N\|h_\eta\|_E\|\chi_{[s,\infty)}v_1 - \chi_{[s,\infty)}v_2\|_{\mathcal{E}}.
\end{aligned}$$

This implies

$$\|\chi_{[s,\infty)}v_1 - \chi_{[s,\infty)}v_2\|_{\mathcal{E}} \le \frac{N\max\{N_1, N_2\}\|e_\eta\|_E}{1 - N\|h_\eta\|_E}\|\nu_1 - \nu_2\|.$$

So that

$$\|g_s^{st}(\nu_1) - g_s^{st}(\nu_2)\| \le \frac{N^2\max\{N_1, N_2\}\|e_\eta\|_E\|\varphi\|_{E'}}{1 - N\|h_\eta\|_E}\|\nu_1 - \nu_2\|.$$

Thus, g_s^{st} is a Lipschitz map with Lipschitz coefficient

$$\mathrm{Lip}(g_s^{st}) \le \frac{N^2\max\{N_1, N_2\}\|e_\eta\|_E\|\varphi\|_{E'}}{1 - N\|h_\eta\|_E} < 1$$

for all $s \in \mathbb{R}$. This also leads to that S_s is homeomorphic to $\mathrm{Im}P(s)$ for all $s \in \mathbb{R}$.

From the definition of S_s and Theorem 3.3, the solution v^* lies in the stable manifold S and the Eq. (3.1) has a unique solution v on $[s,\infty)$ such that $\chi_{[s,\infty)}v \in \mathcal{E}$ for each $x_0 \in S_s$. By the composition property of solution flows, we get $v(t) \in S_t$ for all $t \ge s$.

For $v(s) \in S_s$, the Eq. (3.1) has a unique solution v on $[s, \infty)$ such that $\chi_{[s,\infty)}v \in \mathcal{E}$ and this solution takes the form

$$v(t) = T(t,s)P(s)v(s) + \int_s^\infty \mathcal{G}(t,\tau)g(\tau, v(\tau))d\tau, \quad t \geq s.$$

Therefore,

$$\begin{aligned}\|v(t) - v^*(t)\| &\leq Ne^{-\eta(t-s)}\|P(s)v(s) - P(s)v^*(s)\| \\ &\quad + N\int_s^\infty e^{-\eta|t-\tau|}\varphi(\tau)\|v(\tau) - v^*(\tau)\|d\tau, \quad t \geq s.\end{aligned}$$

Put $w(t) = e^{\mu(t-s)}\|v(t) - v^*(t)\|$ for $t \geq s$ and $\mu \in (0, \eta)$. Then, w satisfies the integral equation

$$\begin{aligned}w(t) &\leq Ne^{-(\eta-\mu)(t-s)}\|P(s)v(s) - P(s)v^*(s)\| \\ &\quad + N\int_s^\infty e^{-\eta|t-\tau|+\mu(t-\tau)}\varphi(\tau)w(\tau)d\tau, \quad t \geq s.\end{aligned}$$

We shall find w in $C_b([s, \infty))$, consider the linear operator A on $C_b([s, \infty))$ as follows

$$(A\phi)(t) = N\int_s^\infty e^{-\eta|t-\tau|+\mu(t-\tau)}\varphi(\tau)\phi(\tau)d\tau, \quad t \geq s.$$

Then, $A\phi \in C_b([s, \infty))$ and $\|A\phi\|_\infty \leq N(N_1 + N_2)(1 - e^{-(\eta-\mu)})^{-1}\|\Lambda_1\varphi\|_\infty\|\phi\|_\infty$ by the property (a) in Proposition 1.6. So, we have

$$w(t) \leq z(t) + (Aw)(t), \ t \geq s \text{ and } z(t) = Ne^{-(\eta-\mu)(t-s)}\|P(s)v(s) - P(s)v^*(s)\|.$$

Take $\mu < \eta + \ln(1 - N(N_1 + N_2)\|\Lambda_1\varphi\|_\infty)$, we get

$$\|A\| \leq N(N_1 + N_2)(1 - e^{-(\eta-\mu)})^{-1}\|\Lambda_1\varphi\|_\infty < 1.$$

Therefore, by cone inequality theorem in Banach space (see [5, Chap. I, Theorem 9.3]) there exists $\phi \in C_b([s, \infty))$ such that $w(t) \leq \phi(t)$ for all $t \geq s$ and ϕ is a unique solution of the equation $\phi = z + A\phi$ in $C_b([s, \infty))$. Thus,

$$\begin{aligned}\|\phi\|_\infty = \|(I - A)^{-1}z\|_\infty &\leq \frac{1}{1 - \|A\|}\|z\|_\infty \\ &\leq \frac{N\|P(s)v(s) - P(s)v^*(s)\|}{1 - N(N_1 + N_2)(1 - e^{-(\eta-\mu)})^{-1}\|\Lambda_1\varphi\|_\infty}.\end{aligned}$$

So, there exist $\mu, C_\mu > 0$ such that

$$\|v(t) - v^*(t)\| \leq C_\mu e^{-\mu(t-s)}\|P(s)v(s) - P(s)v^*(s)\| \quad \text{for all } t \geq s, \ v(s) \in S_s. \quad \square$$

Remark 3.5. By the properties of stable and unstable manifolds, we get $S_s \cap U_s = \{v^*(s)\}$ for all $s \in \mathbb{R}$. Moreover, in Theorem 3.4 if we assume $g(t, 0) = 0$ for all $t \in \mathbb{R}$ then $v^* \equiv 0$. Therefore, $\lim_{t\to\infty} v(t) = 0$ for all $v(s) \in S_s$ and $\lim_{t\to-\infty} v(t) = 0$ for all $v(s) \in U_s$.

When the map $g(t,\cdot)$ is smooth on X for each fixed t then each a fiber of stable and unstable manifolds is also smooth in the sense the map determining this fiber is smooth.

Theorem 3.6. *Assume that*

$$\max\{N^2\max\{N_1,N_2\}\|e_\eta\|_E\|\varphi\|_{E'}+N\|h_\eta\|_E, N(N_1+N_2)\|\Lambda_1\varphi\|_\infty\}<1$$

and the map $g(t,\cdot)$ is continuously differentiable on X for each fixed $t\in\mathbb{R}$ such that $D_xg(t,v^*(t))=0$ for all $t\in\mathbb{R}$. Then, S_s and U_s are differentiable submanifolds of class C^1 and tangent to $v^*(s)+\mathrm{Im}P(s)$ and $v^*(s)+\mathrm{Im}Q(s)$ respectively at $v^*(s)$ for all $s\in\mathbb{R}$.

Proof. We need prove that the map g_s^{st} (see (3.5)) is continuously differentiable on closed subspace ImP(s). Because g satisfies φ-Lipschitz condition and $g(t,\cdot)$ is continuously differentiable on X so

$$\|D_xg(t,a)\|\le\varphi(t)\quad\text{ for all } t\in\mathbb{R}\text{ and } a\in X. \tag{3.6}$$

For $\nu_0,h\in$ ImP(s), we have

$$\begin{aligned}&\frac{g_s^{st}(\nu_0+h)-g_s^{st}(\nu_0)}{\|h\|}-\frac{1}{\|h\|}\int_s^\infty\mathcal{G}(s,\tau)D_xg(\tau,v(\tau))hd\tau\\&=\int_s^\infty\mathcal{G}(s,\tau)\Big(\frac{g(\tau,v_1(\tau))-g(\tau,v(\tau))-D_xg(\tau,v(\tau))h}{\|h\|}\Big)d\tau,\end{aligned}$$

in which v_1 and v are solutions of Eq. (3.1) on $[s,\infty)$ corresponding to ν_0+h and ν_0, and by (3.6) then $\int_s^\infty\mathcal{G}(s,\tau)D_xg(\tau,v(\tau))d\tau$ is absolutely convergent in $\mathcal{L}(X)$. By the attractive property of stable manifold S, we have

$$\|v_1(\tau)-v(\tau)\|\le2C_\mu\|h\|$$

for all $\tau\ge s$. Therefore,

$$\lim_{h\to0}\mathcal{G}(s,\tau)\Big(\frac{g(\tau,v_1(\tau))-g(\tau,v(\tau))-D_xg(\tau,v(\tau))h}{\|h\|}\Big)=0$$

for all $\tau\ge s$. On the other hand,

$$\begin{aligned}&\Big\|\mathcal{G}(s,\tau)\Big(\frac{g(\tau,v_1(\tau))-g(\tau,v(\tau))-D_xg(\tau,v(\tau))h}{\|h\|}\Big)\Big\|\\&\le N(2C_\mu+1)e^{-\eta|s-\tau|}\varphi(\tau),\quad\tau\ge s.\end{aligned}$$

According to Lebesgue's dominated convergence theorem, g_s^{st} is differentiable at ν_0 and

$$Dg_s^{st}(\nu_0)=\int_s^\infty\mathcal{G}(s,\tau)D_xg(\tau,v(\tau))d\tau.$$

From here deduces $Dg_s^{st}(P(s)v^*(s))=0$. By (3.6) and Lebesgue's dominated convergence theorem, Dg_s^{st} is continuous on $\mathrm{Im}P(s)$. So, S_s is differentiable submanifold of class C^1 and tangent to $v^*(s)+\mathrm{Im}P(s)$ at $v^*(s)$. Similarly, U_s is differentiable submanifold of class C^1 and tangent to $v^*(s)+\mathrm{Im}Q(s)$ at $v^*(s)$. □

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Trinh Viet Duoc
Faculty of Mathematics, Mechanics, and Informatics,
University of Science, Vietnam National University,
334 Nguyen Trai, Hanoi, Vietnam
e-mail: tvduoc@gmail.com, duoctv@vnu.edu.vn

Nguyen Ngoc Huy
Department of Mathematics, Thuyloi University,
175 Tay Son, Dong Da, Hanoi, Vietnam
e-mail: huynn@tlu.edu.vn