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A dual mapping associated to a closed convex set and some subdifferential properties

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Abstract. In this paper we establish some properties of the multivalued mapping $(x,d) \rightrightarrows D_C(x;d)$ that associates to every element x of a linear normed space X the set of linear continuous functionals of norm $d \geq 0$ and which separates the closed ball $B(x;d)$ from a closed convex set $C \subset X$. Using this mapping we give links with other important concepts in convex analysis (ε-approximation element, ε-subdifferential of distance function, duality mapping, polar cone). Thus, we establish a dual characterization of ε-approximation elements with respect to a nonvoid closed convex set as a generalization of a known result of Garkavi. Also, we give some properties of univocity and monotonicity of mapping D_C.

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1. Introduction and preliminaries

Let C be a nonvoid closed convex set in a real linear normed space X and a closed ball $B(x;d)$, $d > 0$ such that $C \cap int\, B(x,d) = \emptyset$. It is well known that those two sets can be separated by closed hyperplanes (see, for instance, [1],[2]).

We denote by

$$d_C(x) = \inf_{u \in C} \|x-u\|, \; x \in X, \tag{1.1}$$

the *distance function* to a set $C \subset X$. Also, let us denote by X^* the *dual* space of X.

In the special case $d = d_C(x)$, $x \notin C$, using separating hyperplane, Garkavi [4] has obtained a well known dual characterization of best approximation elements of $x \in X$ in C.

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We recall that an element $z \in C$ is a *ε-approximation of* x in C if

$$\|x - z\| \leq \|x - u\| + \varepsilon, \text{ for all } u \in C. \tag{1.2}$$

Therefore using, the distance of x to the set C the property (1.2) is equivalent to

$$\|x - z\| \leq d_C(x) + \varepsilon.$$

Obviously, here it is necessary that $\varepsilon \geq 0$.

If $\varepsilon = 0$, then z is a *best approximation of* x in C, that is $\|x - z\| = d_C(x)$ and $z \in C$.

If $\varepsilon > 0$ then the set of ε-approximations of $x \in C$ is always nonvoid, but the set of the best approximations may be void.

Using separating hyperplanes, Garkavi [4] established a well known dual characterization of best approximation elements as follows.

Theorem 1.1. ([4]) An element $z \in C$ is a best approximation element of $x \in X \in C$ if and only if there exists an element $x_0^* \in X^*$ such that

i) $\|x_0^*\| = \|x - z\|$;

ii) $x_0^*(x - u) \geq \|x - z\|^2$ for all $u \in C$.

Here, the property ii) is equivalent with the following two properties:

i') $x_0^*(x - z) = \|x - z\|^2$;

ii') $x_0^*(z) = \sup\{x_0^*(u); u \in C\}$.

Obviously, if $x \in C$, and z is a best approximation element, then $x = z$, and so we take $x_0^* = 0$. Now, if $x \notin C$, then $d_C(x) > 0$ and we consider a closed separating hyperplane (x_0^*, α) for the sets C and $B(x, d_C(x))$ such that $\|x_0^*\| = \|x - z\|$. Conversely, if $z \in C$ has the property i) and ii) it follows that

$$\|x - u\|\|x_0^*\| \geq x_0^*(x - u) \geq \|x - z\|^2,$$

for all $u \in C$ which prove that z is a best approximation element in C for $x \in X$.

Let us denote by $P_C(x)$ the set of all best approximations of x in C. The (multivalued) mapping $x \rightrightarrows P_C(x)$ $x \in X$ is called the *metric projection* associated to the set C. Clearly, $P_C(x) = x$ for any $x \in C$. Also, we can have $P_C(x) = \emptyset$ for certain elements in X. If $P_C(x) \neq \emptyset$ for any $x \in X$ then the set C is called *proximinal* and if $P_C(x) = \emptyset$ for any $x \in X \backslash C$, the set C is called *antiproximinal.* It is well known that in a reflexive space any closed convex set is proximinal.

Given a convex real extended function $f : X \to \overline{\mathbb{R}}$, its *$\varepsilon$-subdifferential* is defined by

$$\partial_\varepsilon f(x) = \{x^* \in X^*; x^*(x - u) \geq f(x) - f(u) - \varepsilon, \text{ for all } u \in X\}, x \in X \tag{1.3}$$

where $\overline{\mathbb{R}} = [-\infty, +\infty]$.

Here, we suppose that f is a *proper function*, that is $f(u) > -\infty$ for all $u \in X$ and there exist elements $\overline{x} \in X$ such that $f(\overline{x}) < \infty$. If $\varepsilon = 0$ we obtain the *subdifferential* of function f in x, denoted by $\partial f(x)$.

The multivalued operator $x \rightrightarrows \partial_\varepsilon f(x)$, $x \in X$, has the following ε-monotonicity property

$$(x_1^* - x_2^*)(x_1 - x_2) \geq -2\varepsilon \text{ for all } x_1^* \in \partial_\varepsilon f(x_1), x_2^* \in \partial_\varepsilon f(x_2). \tag{1.4}$$

Generally, a multivalued operator $A : X \rightrightarrows X^*$ which has the property of ε-monotonicity of type (1.4) is called *ε-monotone*. Some properties of those mappings were given in [13]. This type of monotonicity is different of ε-monotonicity defined in [9].

Also, we recall the definition of *duality mapping* $J : X \Rightarrow X^*$,

$$J(x) = \left\{x^* \in X^*; x^*(x) = \|x^*\|^2 = \|x\|^2\right\}, \ x \in X. \tag{1.5}$$

It is well known that J is the subdifferential of the function $x \mapsto \frac{1}{2}\|x\|^2$, $x \in X$ (see, for instance, [1], [2]).

If A is a subset of X, we denote by A^o the *polar set* of $A \subset X$, that is

$$A^o = \{x^* \in X^*; x^*(a) \le 1 \text{ for all } a \in A\}. \tag{1.6}$$

In this paper we intend to analyze some properties of the (multivalued) mapping $(x, d) \rightrightarrows D_C(x; d)$, $x \in X$, $C \subset X$, $d \ge 0$, where

$$D_C(x; d) = \{x^* \in X^*; x^*(v) \ge x^*(u), \|x^*\| = d, \forall v \in B(x; d), \forall u \in C\}. \tag{1.7}$$

Remark 1.2. Obviously, $D_C(x; 0) = \{0\}$, for all $x \in X$ and $D_C(x; d) = \emptyset$ whenever $d > d_C(x)$.

Geometrically, for each $x \in X$ and $d > 0$, $D_C(x; d)$ coincides with the set of all linear continuous functionals $x^* \in X^*$ such that $\|x^*\| = d$ and for which $x^*(y) = k$, $y \in X$, is a separating hyperplane for the sets C and $B(x; d)$ for a certain $k \in \mathbb{R}$.

Equivalently,

$$D_C(x; d) = \left\{x^* \in X^*; \|x^*\| = d, \ x^*(x - u) \ge d^2, \forall u \in C\right\}. \tag{1.7'}$$

In the special case $d = d_C(x)$ we denote

$$D_C(x) = D_C(x; d_C(x)), \ x \in X. \tag{1.7''}$$

We establish a dual characterization of real number d such that $0 \le d \le d_C(x)$ (Theorem 2.1). Consequently, if $x \notin C$, we obtain the basic properties of elements in $D_C(x; d)$. Using this multivalued mapping naturally generated by the geometric problem of separation of a nonvoid closed convex set and a closed ball we give connections with some important concepts and properties of convex analysis (ε-subdifferentials of distance function, ε-approximation elements, duality mapping, polar cone). For example,

$x^* \in D_C(x; d)$ if and only if $\frac{1}{\|x^*\|}x^* \in \partial_\varepsilon d_C(x) \cap Bd\, B^*(0; 1)$ for $\varepsilon = d_C(x) - d$, where $0 < d \le d_C(x)$. Generally, by BdA we denote the *boundary* of a set $A \subset X$. Also, we denote by $B^*(x_0^*; d)$, $x_0^* \in X^*$, $d \ge 0$, the closed balls in X^*.

Consequently, we give an explicit formula for $\partial_\varepsilon d_C(x)$ in the case $x \notin C$, but $\varepsilon > d_C(x)$ (Theorem 2.5, ii). The special case $d = d_C(x)$ was considered by Ioffe in [8]. A detailed study of subdifferential of distance function was given by Penot, Ratsimahalo in [10] (see also [3] and [6] if $P(x) \ne \emptyset$). In [5] Hiriart-Urruty (see, also, [6]) has obtained formula for the ε-subdifferential of a marginal function. Particularly, one can be obtained formulas for ε-subdifferential of distance function which is considered either as a marginal function, or as the convolution of the norm and the indicator function of the set C. But, by Theorem 2.5, we establish some explicit properties of

$\partial_\varepsilon d_C(x)$. We remark that we have a special situation if $\varepsilon = d_C(x)$. The assertion iii) in Theorem 2.5 is similar to the one shown in [10] for the subdifferential distance function. We also establish a property of univocity of D_C.

Following Jofre, Luc and Thera ([9]), we define a new type of ε-monotonicity by (3.1), in according with D_C (Theorem 3.1). Some monotonicity properties of D_C are given in Section 3.

2. A dual mapping associated to a closed convex set and an arbitrary positive number

Now, we give a dual characterization of the numbers d such that $d_C(x) \geq d \geq 0$.

Theorem 2.1. Let C be a nonvoid closed convex set in a linear normed space X. If $x \in X$ is a fixed element then $d_C(x) \geq d \geq 0$ if and only if there exists $x^* \in X^*$ such that

i) $\|x^*\| = d$;
ii) $x^*(x-u) \geq d^2$, for all $u \in C$.

Proof. If $d = 0$ then i) and ii) are obviously fulfilled taking $x^* = 0$ and conversely. Hence we can suppose that $d > 0$.

Now, if $0 < d \leq d_C(x)$ it follows that $B(x;d)$ has nonvoid interior set and $C \cap int\, B(x;d) = \emptyset$. Thus, using a separation theorem for sets C and $B(x;d)$ (see, for instance, [1] or [2]), there exists a non null element $y^* \in X^*$ such that $y^*(v) \geq y^*(u)$ for all $u \in C$ and $v \in B(x;d)$. Taking $x_0^* = d\,\|y^*\|^{-1} y^*$ it follows that

$$x_0^*(x - dz) \geq x_0^*(u)$$

for any $z \in B(0;1)$ and $u \in C$, and so $x_0^*(x-u) \geq d\,\|x_0^*\|$ for all $u \in C$. Obviously, $\|x_0^*\| = d$. Therefore, the properties i) and ii) are fulfilled.

Conversely, if i) and ii) hold, then

$$d^2 \leq x_0^*(x-u) \leq \|x_0^*\|\,\|x-u\| \leq d\,\|x-u\|,$$

for all $u \in C$, and so $d \leq d_C(x)$. □

From the proof of Theorem 2.1 in the case $0 < d \leq d_C(x)$ (and, so, $x \notin C$), we see that every x^* which verifies i) and ii) is in $D_C(x;d)$.

Remark 2.2. Given an element $x \in X$, taking $d = \|x - z\|$, where $z \in C$, by Theorem 2.1 it results that $\|x - z\| \leq d_C(x)$ if and only if the properties i) and ii) in Theorem 2.1 are fulfilled. But it is clear that $\|x - z\| \leq d_C(x)$ and $z \in C$ if and only if z is the best approximation of x in C. Therefore, Theorem 2.1 is a slight extension of a famous characterization established by Garkavi [4] concerning the best approximation elements.

Corollary 2.3. Let X be a linear normed space, C a nonvoid closed convex set of X and $\varepsilon \geq 0$. Then $z_\varepsilon \in C$ is an ε-approximation element for $x \notin C$, $\varepsilon < d_C(x)$, if and only if there exists $x^* \in X^*$ such that the properties i) and ii) in Theorem 2.1 are fulfilled for $d = \|x - z_\varepsilon\| - \varepsilon$.

Proof. According to (1.1) and (1.2) $z_\varepsilon \in C$ is an ε-approximation element for $x \in X$ if and only if $\|x - z_\varepsilon\| - \varepsilon \le d_C(x)$.Therefore it is sufficient to apply Theorem 2.1 taking $d = \|x - z_\varepsilon\| - \varepsilon$. □

Now, we intent to characterize $x^* \in D_C(x; d)$ using the set $\partial_\varepsilon d_C(x)$, where $\varepsilon = d_C(x) - d \ge 0$, whenever $x \notin C$.

Proposition 2.4. If $x^* \in \partial_\varepsilon d_C(x)$ and $\varepsilon > 0$ then:

$$\|x^*\| \le 1; \tag{2.1}$$

$$\|x^*\| \ge 1 - \frac{\varepsilon}{d_C(x)}, \text{ for all } x \notin C. \tag{2.2}$$

whenever C is a nonvoid closed convex set in X.

Proof. If $x^* \in \partial_\varepsilon d_C(x)$ then $x^*(x - y) \ge d_C(x) - d_C(y) - \varepsilon$ for any $y \in X$. Taking $y = x + tz$, $t > 0$ and $z \in X$ it follows that

$$tx^*(z) + d_C(x) - \varepsilon \le d_C(x + tz) \le \|x + tz - \overline{u}\| \le t\|z\| + \|x - \overline{u}\|,$$

for a given $\overline{u} \in C$. Therefore, $x^*(z) - \|z\| \le \frac{1}{t}(\|x - \overline{u}\| - d_C(x))$, for any $t > 0$ and $z \in X$, and so, for $t \to \infty$ we obtain that $x^*(z) \le \|z\|, z \in X$.

If $x \in C$ and $x^* \in \partial_\varepsilon d_C(x)$ we take $y = x + tz$, $z \in X$, $t < 0$ in inequality

$$x^*(x - y) \ge d_C(x) - d_C(y) - \varepsilon$$

and we obtain $x^*(x - y) \ge d_C(x) - \varepsilon$, so $\|x^*\|\,\|x - y\| \ge d_C(x) - \varepsilon$, equivalently

$$\|x^*\| d_C(x) \ge d_C(x) - \varepsilon.$$

Therefore, if $x \notin C$ then $d_c(x) > 0$. Thus, we obtain the inequality (2.2). □

We recall that if X is a linear normed space, the *conic polar* A^+ of a set $A \subset X$ is defined by

$$A^+ = \{x^* \in X^*; x^*(a) \ge 0 \text{ for all } a \in A\}.$$

If A is a cone, then $A^+ = -A^0$.

In the next result we establish some special properties of ε-subdifferential distance function.

Theorem 2.5. Suppose that X is a real normed space, $x \in X$ and $C \subset X$ is a nonvoid closed convex set.

i) If $x \notin C$, $0 < d \le d_C(x)$ and $\varepsilon = d_C(x) - d$, then

$$\partial_\varepsilon d_C(x) \cap Bd\ B^*(0; 1) = \frac{1}{d} D_C(x; d);$$

ii) If $x \notin C$ and $\varepsilon > d_C(x)$ then

$$\partial_\varepsilon d_C(x) = (\varepsilon - d_C(x))(C - x)^o \cap B^*(0; 1);$$

iii) If $x \notin C$, and $\varepsilon = d_C(x)$ then $\partial_\varepsilon d_C(x) = (x - C)^+ \cap B^*(0; 1)$.

iv) If $x \in C$ then $\partial_\varepsilon d_C(x) = \varepsilon (C - x)^o \cap B^*(0; 1)$ for every $\varepsilon > 0$.

Proof. i) Using (1.3) it follows that $z^* \in \partial_\varepsilon d_C(x)$ if and only if

$$z^*(x-y) \geq d - d_C(y) \text{ for all } y \in X.$$

If $y \in C$ then $z^*(x-y) \geq d$, which implies $z^* \in \frac{1}{d} D_C(x;d)$ whenever $\|z^*\| = 1$.

Conversely, suppose $z^* \in \frac{1}{d} D_C(x;d)$. Then $\|z^*\| = 1$ and $z^*(x-y) \geq d$ for any $y \in C$, so $z^*(x-y) \geq d - d_C(y)$ for all $y \in C$.

Now, consider $y \in X \backslash C$ and some $u \in C$.

Then $z^*(x-y) = z^*(x-u) - z^*(y-u) \geq d - z^*(y-u)$.

But $z^*(y-u) \leq \|z^*\| \|y-u\| = \|y-u\|$. So, $z^*(x-y) \geq d - \|y-u\|$, for any $u \in C$. Passing to the $\sup\limits_{u \in C}$ in this inequality we obtain $z^* \in \partial_\varepsilon d_C(x) \cap B^*(0;1)$.

ii) If $\varepsilon > d_C(x)$ denote $\eta = \varepsilon - d_C(x) > 0$. Let x^* be an element of $\partial_\varepsilon d_C(x)$. Then $x^*(x-y) \geq -\eta - d_C(y)$, for all $y \in X$. Taking $y \in C$ it results $x^*(y-x) \leq \eta$ for any $y \in C$, that is $x^* \in \left(\frac{C-x}{\eta}\right)^o \cap B^*(0;1) = \eta (C-x)^o \cap B^*(0;1)$ according to (2.1).

Now, if $x^* \in \eta(C-x)^o \cap B^*(0;1)$ then $x^*(u-x) \leq \varepsilon - d_C(x)$ for all $u \in C$. If $y \notin C$ then

$$\begin{aligned} x^*(x-y) &= x^*(x-u) + x^*(u-y) \geq d_C(x) - \varepsilon + x^*(u-y) \\ &\geq d_C(x) - \varepsilon - \|x^*\| \|u-y\| \geq d_C(x) - \varepsilon - \|u-y\| \end{aligned}$$

for all $u \in C$.

Using (1.1) it follows that $x^* \in \partial_\varepsilon d_C(x)$.

iii) Let x^* be an element in $\partial_\varepsilon d_C(x)$. Taking $\varepsilon = d_C(x)$ in the definition of ε-subdifferential of d_C and arbitrary $y \in C$ one obtains $x^*(y-x) \leq 0$, so $x^* \in (x-C)^+$. Now, using (2.1), the conclusion follows.

iv) Let $y \in X$ be arbitrary and $x \in C$. If $\varepsilon > 0$ and $x^* \in \partial_\varepsilon d_C(x)$ then $x^*(x-y) \geq -d_C(y) - \varepsilon$, so $x^*(y-x) \leq \varepsilon$, whenever $y \in C$. Hence $x^* \in \varepsilon(C-x)^o$.

Also, from (2.1) we have $\|x^*\| \leq 1$.

Conversely, for $x^* \in \varepsilon(C-x)^o \cap B(0;1)$ and $y \in X$ we have $x^*(y-u) \leq \|y-u\|$ for all $u \in C$. We deduce

$$x^*(x-y) = x^*(x-u) + x^*(u-y) \geq -\varepsilon - \|y-u\|.$$

Passing to the infimum for $u \in C$ it results $x^*(x-y) \geq -\varepsilon - d_C(y)$ for all $y \in X$ as claimed. □

Corollary 2.6. Let X be a linear normed space. Then:

i) $\frac{1}{d} D_{\{0\}}(x;d) = \partial_\varepsilon \|\cdot\|(x) \cap Bd\, B(0;1)$ where $\varepsilon = \|x\| - d > 0$, $d > 0$;

ii) $D_{\{0\}}(x; \|x\|) = J(x)$.

Proof. i) Observe that $d_C(x) = \|x\|$ if $C = \{0\}$. Now, we apply Theorem 2.5, i).

ii) Consider $x^* \in D_{\{0\}}(x; \|x\|)$, that is $\|x^*\| = \|x\|$ and $x^*(x) \geq \|x\|^2$. But $x^*(x) \leq \|x\|^2$ and so $x^*(x) = \|x\|^2$. According to (1.5) we obtain that $x^* \in J(x)$. □

Remark 2.7. The assertion iii) of Theorem 2.5 has obtained by Hiriart-Urruty in [5] (see also, [6], [7]). The special case $\varepsilon = 0$ was studied by Penot and Ratsimahalo [10].

Remark 2.8. Theorem 2.5, i), can be reformulated as

$$\frac{1}{d} D_C(x;d) = \partial_\lambda (d \cdot d_C(x)) \cap Bd\ B(0;1),$$

where $\lambda = d(d_C(x) - d)$, $0 < d \le d_C(x)$.

We recall that X is a *smooth space* (see [1], [3]) if there is exactly one supporting hyperplane through each boundary point of closed unit ball.

Generally, closed convex set $A \subset X$ is called *smooth at a point* x_0 if there exists only one closed hyperplane which separates x_0 at A. Obviously, it is necessary that $x_0 \in Bd\ A$.

Theorem 2.9. Let C be a nonvoid closed convex set in X and a fixed element $x \in X$. Then, for any $d \in [0, d_C(x)]$ we have:

i) $D_C(x;d) = \{0\}$ if and only if $d = 0$;

ii) $Dom\ D_C = (X \times \{0\}) \cup \{(x,d); x \notin C, d \in (0; d_C(x)]\}$;

iii) If $D_C(x;d)$ is a singleton then $d = 0$ or $d = d_C(x)$.

iv) $D_C(x; d_C(x))$ is a singleton if and only if the set $C - B(x; d_C(x))$ is smooth at origin.

Proof. The properties i), ii) are obvious.

Also, in the sequel we can suppose that $x \notin C$, and so $d_C(x) > 0$.

Now, we prove properties (iii) and (iv): if $d = 0$ then $D_C(x;0) = \{0\}$ is a singleton. Let us consider an arbitrary element $x \notin C$ and $d \in (0, d_c(x)]$. But, if $d < d_C(x)$ then C and $B(x;d)$ are strongly separated, that is there exists many parallel separating hyperplanes (see, for example, [1], Remark 1.46). Therefore, $D_C(x;d)$ is not a singleton. If $d = d_C(x)$ there exists a unique hyperplane which separates C and $B(x; d_C(x))$ if and only if there exists a unique hyperplane which separates the origin and $C - B(x; d_C(x))$, that is $C - B(x; d_C(x))$ is smooth at the origin. □

Remark 2.10. In the spacial case when $P_C(x) \neq \emptyset$, the property iii) was established by Garkavi ([4]).

Now, if $P_C(x) \neq \emptyset$, we have

$$D_C(x) = \left\{x^* \in X^*; \|x^*\| = \|x - z\|,\ x^*(x-u) \ge \|x - z\|^2\ \ \forall u \in C\right\}, \quad z \in P_C(x) \tag{2.3}$$

since $d_C(x) = \|x - z\|$ for any $z \in P_C(x)$.

In the sequel we prove that the mapping D_C can be equivalently defined using a min-max property. Since $B^*(x;d)$ is a convex w*-compact set in X^* and the function $F_x(x^*, u) = x^*(x-u)$, $(u, x^*) \in X \times X^*$ is convex-concave, using a min-max result (see, for instance, [1], [11] and [12]), it implies the following equality:

$$\max_{x^* \in B^*(0;d)} \inf_{u \in C} x^*(x-u) = \inf_{u \in C} \max_{x^* \in B^*(0;d)} x^*(x-u) \text{ for all } x \in X,\ d > 0. \tag{2.4}$$

Here, by "max", we mean that "sup" is attained. The elements $x_0^* \in B^*(0;d)$, where "*max*" is attained in the left hand of (2.4) and make valid the equality (2.4) are called *the solutions of the max-inf problem (2.4).*

Proposition 2.11. Given an element $x \in X$ and a nonvoid convex, closed set $C \subset X$, then $x^* \in D_C(x)$ if and only if x^* is a solution of max-inf problem (2.4), where $d = d_C(x)$, that is

$$D_C(x) = \left\{ x^* \in B^*(0; d_C(x)); \inf_{u \in C} x^*(x-u) = d_C^2(x) \right\}. \tag{2.5}$$

Proof. We remark that the saddle value of (2.4) is equal to $d_C(x)\,d$. Consequently, for $d = d_C(x)$, the properties i), ii) in Theorem 2.1 are equivalent to the assertion that x^* is a solution of max-inf problem (2.4). □

Remark 2.12. If in the equality (2.4) "inf" is also attained, these elements of C are even the best approximation elements of x in C. Therefore, if $P_C(x) \neq \emptyset$ and $d = d_C(x)$, then the set of all saddle elements of max-min problem associated to (2.4) is $D_C(x) \times P_C(x)$.

Now, if we return to the dual characterization of the best approximation elements, we observe that in the special case $P_C(x) \neq \emptyset$, we have a conection with the duality map J. Firstly, we remark that if we put in equality (1.7) $d = \|x - z\|$ it results that $D_C(X)$ is exactly the set of all $x^* \in X^*$ with the properties of Garkavi Theorem 1.1. But, the properties i) and i′) in Theorem 1.1 prove that $x_0^* \in J(z-x)$. Also, ii′) say that $x^* \in (x-C)^*$. Consequently we have the following equality

$$D_C(x) = J(x-z) \cap (C-x)^+ \ \text{ whenever } z \in P_C(x) \ \text{ and } x \in X.$$

3. Properties of monotonicity

It is well known the relationship between the subdifferentials of convex functions and their property of monotonicity ([9]). Also, the ε-subdifferentials are ε-monotone in the sense of definition (1.4) and they have some good properties (see, for e.g., [13]).

Because the multivalued mapping $x \rightrightarrows D_C(x;d)$ is expressed using the ε-subdifferential of $d_C(\cdot)$ (Theorem 2.5, i)), it is expected to have an ε-monotonicity property.

Now, we establish two special monotonicity properties of D_C.

Theorem 3.1. The mapping $(x,d) \rightrightarrows D_C(x;d)$ is monotone in the following sense:

$\forall x_i \in X \backslash C$, $0 < d_i \leq d_C(x_i)$, $\varepsilon_i = d_C(x_i) - d_i$ and $\forall x_i^* \in D_C(x_i; d_i)$, $i = 1, 2$, then

$$(x_1^* - x_2^*)(x_1 - x_2) \geq -\varepsilon_2 d_1 - \varepsilon_1 d_2. \tag{3.1}$$

Proof. Let us consider $x_i^* \in D_C(x_i, d_i)$, $i = 1, 2$. By property ii) in Theorem 2.1 and the definition of D_C we have $(x_i^*, x_i - u_i) \geq d_i^2$ for any $u_i \in C$, $i = 1, 2$. Therefore it follows that

$$\begin{aligned}(x_1^* - x_2^*)(x_1 - x_2) &= x_1^*(x_1 - u_1) + x_2^*(x_2 - u_2) - x_1^*(x_2 - u_1) - x_2^*(x_1 - u_2) \\ &\geq d_1^2 + d_2^2 - x_1^*(x_2 - u_1) - x_2^*(x_1 - u_2) \\ &\geq d_1^2 + d_2^2 - d_1 \|x_2 - u_1\| - d_2 \|x_1 - u_2\|.\end{aligned}$$

Since u_1, u_2 are arbitrary elements in C we get

$$(x_1^* - x_2^*)(x_1 - x_2) \geq d_1^2 + d_2^2 - d_1 d_C(x_2) - d_2 d_C(x_1) = -d_1\varepsilon_2 - d_2\varepsilon_1,$$

as claimed. □

Also, the mapping D_C has a property of monotonicity with respect to corresponding best approximation elements.

Proposition 3.2. If $x_i^* \in D_C(x_i; d_i)$ and $z_i \in P_C(x_i)$, $i = 1, 2$, then

$$(x_1^* - x_2^*)(z_1 - z_2) \geq 0.$$

Proof. Taking $u_1 = z_2$ and $u_2 = z_1$ in Theorem 2.1, we have

$$\begin{aligned}(x_1^* - x_2^*)(z_1 - z_2) &= x_1^*(x_1 - z_2) + x_2^*(x_2 - z_1) - x_1^*(x_1 - z_1) - x_2^*(x_2 - z_2) \\ &\geq d_1^2 + d_2^2 - x_1^*(x_1 - z_1) - x_2^*(x_2 - z_2).\end{aligned}$$

By properties i) and ii) in Theorem 1.1 it follows that

$$(x_1^* - x_2^*)(z_1 - z_2) \geq d_1^2 + d_2^2 - d_1 \|x_1 - z_1\| - d_2 \|x_2 - z_2\| = 0.$$ □

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