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Certain class of analytic functions defined by $q-$analogue of Ruscheweyh differential operator

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Abstract. In this paper, we obtain coefficient estimates, distortion theorems, radii of close-to-convexity, starlikeness and convexity for functions belonging to the class $TB_q^{\lambda}(\alpha,\beta)$ of analytic starlike and convex functions defined by $q-$analogue of Ruscheweyh differential operator. Also we find closure theorems, $N_{k,q,\delta}(e,g)$ neighborhood and partial sums for functions in this class.

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1. Introduction

Let $\mathcal{S}$ be the class of analytic and univalent functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \ , z \in \mathbb{U} = \{z : z \in \mathbb{C} : |z| < 1\}. \tag{1.1}$$

Also let $\mathcal{S}^*(\alpha)$ and $C(\alpha)$ denote the subclasses of $\mathcal{S}$ which are, respectively, starlike and convex functions of order $\alpha(0 \leq \alpha < 1)$, satisfying (see Robertson [30])

$$\mathcal{S}^*(\alpha) = \left\{ f : f \in \mathcal{S} \ \text{ and Re} \left(\frac{zf'(z)}{f(z)} \right) > \alpha \right\}, \tag{1.2}$$

and

$$C(\alpha) = \left\{ f : f \in \mathcal{S} \ \text{ and Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \right\}. \tag{1.3}$$

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It readily follows from (1.2) and (1.3) that

$$f(z) \in C(\alpha) \Leftrightarrow zf^{'}(z) \in S^{*}(\alpha).$$

For $0 < q < 1$ the Jackson's $q-$derivative of a function $f(z) \in \mathcal{S}$ is given by [22] (see also [2, 3, 8, 13, 17, 20, 24, 34, 35, 39])

$$D_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z} & \text{for} \quad z \neq 0, \\ f^{'}(0) & \text{for} \quad z = 0, \end{cases} \tag{1.4}$$

For $f(z)$ of the form (1.1), we have

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q \, a_k z^{k-1}, \tag{1.5}$$

where

$$[n]_q = \frac{1-q^n}{1-q} \quad (0 < q < 1;\ n \in \mathbb{N} = \{1, 2, ...\}). \tag{1.6}$$

Kanas and Raducanu [23] (see also Aldweby and Darus [1]) defined the $q-$analogue of Ruscheweyh operator by

$$R_q^{\lambda} f(z) = z + \sum_{k=2}^{\infty} \tfrac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!} a_k z^k \quad (0 < q < 1; \lambda \geq 0), \tag{1.7}$$

where

$$[n]_q! = \begin{cases} [n]_q \, [n-1]_q \, ... \, [1]_q \, , & n \in \mathbb{N}, \\ 1, & n = 0, \end{cases} \tag{1.8}$$

From (1.7) we obtain that

$$R_q^0 f(z) = f(z) \quad \text{ and } R_q^1 f(z) = z D_q f(z),$$

and

$$\lim_{q \longrightarrow 1^-} R_q^{\lambda} f(z) = z + \sum_{k=2}^{\infty} \frac{(k+\lambda-1)!}{\lambda! \, (k-1)!} a_k z^k = R^{\lambda} f(z), \tag{1.9}$$

where R^{λ} is the Ruscheweyh differential operator (see [32] and [4, 7, 10, 14, 18]).

Definition 1.1. For $0 < q < 1, 0 \leq \alpha < 1, \beta \geq 0$ and $\lambda \geq 0$, let $B_q^{\lambda}(\alpha, \beta)$ be the class of functions $f \in \mathcal{S}$ satisfying

$$\operatorname{Re}\left\{ \frac{z D_q (R_q^{\lambda} f(z))}{R_q^{\lambda} f(z)} - \alpha \right\} > \beta \left| \frac{z D_q (R_q^{\lambda} f(z))}{R_q^{\lambda} f(z)} - 1 \right|. \tag{1.10}$$

Let $\mathcal{T} \subset \mathcal{S}$ such that:

$$\mathcal{T} = \left\{ f \in \mathcal{S} : f(z) = z - \sum_{k=2}^{\infty} a_k z^k \ , a_k \geq 0 \right\}, \tag{1.11}$$

and

$$TB_q^{\lambda}(\alpha, \beta) = B_q^{\lambda}(\alpha, \beta) \cap \mathcal{T}. \tag{1.12}$$

Note that

(i) $TB_q^0(\alpha,\beta) = S_p^q(\alpha,\beta) = \left\{ f \in T : \mathrm{Re}\left\{\frac{zD_qf(z)}{f(z)} - \alpha\right\} > \beta\left|\frac{zD_qf(z)}{f(z)} - 1\right|, z \in \mathbb{U}\right\}$;

(ii) $TB_q^0(\alpha,0) = TB_q(\alpha) = \left\{ f \in T : \mathrm{Re}\left\{\frac{zD_qf(z)}{f(z)}\right\} > \alpha\right\}$;

(iii) $\lim\limits_{q\to 1^-} TB_q^0(\alpha,\beta) = S_p(\alpha,\beta) = \{f \in T : \mathrm{Re}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\}$
$> \beta\left\{\frac{zf'(z)}{f(z)} - 1\right\}, z \in \mathbb{U}\}$ (see [29] and [36]);

(iv) $TB_q^1(\alpha,\beta) = UCS_p^q(\alpha,\beta) = \{f \in T : \mathrm{Re}\left\{\frac{D_q(zD_qf(z))}{D_qf(z)} - \alpha\right\}$
$> \beta\left|\frac{D_q(zD_qf(z))}{D_qf(z)} - 1\right|, z \in \mathbb{U}\}$;

(v) $TB_q^1(\alpha,0) = C_q(\alpha) = \{f \in T : \mathrm{Re}\left\{\frac{D_q(zD_qf(z))}{D_qf(z)}\right\} > \alpha, z \in \mathbb{U}\}$;

(vi) $\lim\limits_{q\to 1^-} TB_q^1(\alpha,\beta) = UCS_p(\alpha,\beta) = \{f \in T : \mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\}$
$> \beta\left|\frac{zf''(z)}{f'(z)}\right|, z \in \mathbb{U}\}$ (see [29]);

(vii) $\lim\limits_{q\to 1^-} TB_q^\lambda(\alpha,\beta) = S_p^\lambda(\alpha,\beta)$ (see Rosy et al. [31]).

2. Coefficient estimates

Unless indicated, we assume that $0 \le \alpha < 1$, $\beta \ge 0, \lambda \ge 0$, $0 < q < 1$ and $f(z) \in \mathcal{T}$.

Theorem 2.1. *A function $f(z) \in TB_q^\lambda(\alpha,\beta)$ if and only if*

$$\sum_{k=2}^{\infty}\left[[k]_q(1+\beta) - (\alpha+\beta)\right]\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_k \le 1-\alpha. \tag{2.1}$$

Proof. Assume that (2.1) holds. Then it is suffices to show that

$$\beta\left|\frac{zD_q(R_q^\lambda f(z))}{R_q^\lambda f(z)} - 1\right| - \mathrm{Re}\left\{\frac{zD_q(R_q^\lambda f(z))}{R_q^\lambda f(z)} - 1\right\} \le 1-\alpha.$$

We have

$$\begin{aligned} & \beta\left|\frac{zD_q(R_q^\lambda f(z))}{R_q^\lambda f(z)} - 1\right| - \mathrm{Re}\left\{\frac{zD_q(R_q^\lambda f(z))}{R_q^\lambda f(z)} - 1\right\} \\ \le\ & (1+\beta)\left|\frac{zD_q(R_q^\lambda f(z))}{R_q^\lambda f(z)} - 1\right| \\ \le\ & \frac{(1+\beta)\sum\limits_{k=2}^{\infty}\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}\left([k]_q-1\right)a_k}{1-\sum\limits_{k=2}^{\infty}\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_k}. \end{aligned}$$

This last expression is bounded above by $(1-\alpha)$ since (2.1) holds.

Conversely if $f(z) \in TB_q^{\lambda}(\alpha,\beta)$ and z is real, then

$$\operatorname{Re}\left\{ \frac{1-\sum_{k=2}^{\infty}\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}[k]_q a_k z^{k-1}}{1-\sum_{k=2}^{\infty}\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_k z^{k-1}} - \alpha \right\} \geq \beta \left| \frac{\sum_{k=2}^{\infty}\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}([k]_q-1)a_k z^{k-1}}{1-\sum_{k=2}^{\infty}\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_k z^{k-1}} \right|.$$

Letting $z \to 1^-$ along the real axis, we obtain (2.1). Hence the proof is completed. □

Corollary 2.2. *For* $f(z) \in TB_q^{\lambda}(\alpha,\beta)$,

$$a_k \leq \frac{1-\alpha}{\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}} \quad (k \geq 2) \tag{2.2}$$

and

$$f(z) = z - \frac{1-\alpha}{\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}} z^k \quad (k \geq 2), \tag{2.3}$$

gives the sharpness.

Remark 2.1. Letting $q \to 1^-$ in the results of Section 2, we get the results of Section 2 for the class $S_p^{\lambda}(\alpha,\beta)$ studied by Rosy et al. [31].

3. Growth and distortion theorems

Theorem 3.1. *For* $f(z) \in TB_q^{\lambda}(\alpha,\beta)$ *and* $|z| = r < 1$, *we have*

$$|f(z)| \geq r - \frac{1-\alpha}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][1+\lambda]_q} r^2, \tag{3.1}$$

and

$$|f(z)| \leq r + \frac{1-\alpha}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][1+\lambda]_q} r^2. \tag{3.2}$$

Equalities hold for

$$f(z) = z - \frac{1-\alpha}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][1+\lambda]_q} z^2, \tag{3.3}$$

at $z = r$ *and* $z = re^{i(2k+1)\pi}$ $(k \geq 2)$.

Proof. Since for $k \geq 2$,

$$[[2]_q(1+\beta)-(\alpha+\beta)][1+\lambda]_q \sum_{k=2}^{\infty} a_k \leq \sum_{k=2}^{\infty}[[k]_q(1+\beta)-(\alpha+\beta)]\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_k \leq 1-\alpha, \tag{3.4}$$

then

$$\sum_{k=2}^{\infty} a_k \leq \frac{1-\alpha}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][1+\lambda]_q}. \tag{3.5}$$

From (1.12) and (3.5), we have

$$|f(z)| \geq r - r^2 \sum_{k=2}^{\infty} a_k \geq r - \frac{1-\alpha}{\left[[2]_q (1+\beta) - (\alpha+\beta)\right][1+\lambda]_q} r^2 \tag{3.6}$$

and

$$|f(z)| \leq r + r^2 \sum_{k=2}^{\infty} a_k \leq r + \frac{1-\alpha}{\left[[2]_q (1+\beta) - (\alpha+\beta)\right][1+\lambda]_q} r^2. \tag{3.7}$$

This completes the proof. □

Letting $q \to 1^-$ in Theorem 3.1, we have

Corollary 3.2. *For $f(z) \in S_p^{\lambda}(\alpha, \beta)$, then*

$$|f(z)| \geq r - \frac{1-\alpha}{(2+\beta-\alpha)(1+\lambda)} r^2, \tag{3.8}$$

and

$$|f(z)| \leq r + \frac{1-\alpha}{(2+\beta-\alpha)(1+\lambda)} r^2. \tag{3.9}$$

Equalities hold for

$$f(z) = z - \frac{1-\alpha}{(2+\beta-\alpha)(1+\lambda)} z^2, \tag{3.10}$$

at $z = r$ and $z = re^{i(2k+1)\pi}$ $(k \geq 2)$.

Proof. Letting $q \to 1^-$ in Theorem 3.1, we can show (3.8) and (3.9). □

Theorem 3.3. *Let $f(z) \in TB_q^{\lambda}(\alpha, \beta)$. Then for $|z| = r < 1$,*

$$\left|f'(z)\right| \geq 1 - \frac{2(1-\alpha)}{\left[[2]_q (1+\beta) - (\alpha+\beta)\right][1+\lambda]_q} r, \tag{3.11}$$

and

$$\left|f'(z)\right| \leq 1 + \frac{2(1-\alpha)}{\left[[2]_q (1+\beta) - (\alpha+\beta)\right][1+\lambda]_q} r. \tag{3.12}$$

The sharpness are attained for $f(z)$ given by (3.3).

Proof. For $k \geq 2$, we have

$$\left|f'(z)\right| \leq 1 - r \sum_{k=2}^{\infty} k a_k.$$

We find from (2.1) and (3.5) that

$$\begin{aligned}
[2]_q (1+\beta) [\lambda+1]_q \sum_{k=2}^{\infty} k a_k &\leq 2(1-\alpha) + 2(\alpha+\beta) [\lambda+1]_q \sum_{k=2}^{\infty} a_k \\
&\leq 2(1-\alpha) + \frac{2(\alpha+\beta)(1-\alpha)}{\left[[2]_q (1+\beta) - (\alpha+\beta)\right]} \\
&\leq \frac{2[2]_q (1+\beta)(1-\alpha)}{\left[[2]_q (1+\beta) - (\alpha+\beta)\right]},
\end{aligned}$$

that is, that

$$\sum_{k=2}^{\infty} k a_k \leq \frac{2(1-\alpha)}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][\lambda+1]_q}. \tag{3.13}$$

From (3.11) and (3.12) that

$$\left|f^{'}(z)\right| \geq 1 - r\sum_{k=2}^{\infty} k a_k \geq 1 - \frac{2(1-\alpha)}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][1+\lambda]_q} r \tag{3.14}$$

and

$$\left|f^{'}(z)\right| \leq 1 + r\sum_{k=2}^{\infty} k a_k \leq 1 + \frac{2(1-\alpha)}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][1+\lambda]_q} r. \tag{3.15}$$

This completes the proof. □

Theorem 3.4. *For* $f(z) \in TB_q^{\lambda}(\alpha,\beta)$ *and* $|z| = r < 1$,

$$|D_q f(z)| \geq 1 - \frac{[2]_q(1-\alpha)}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][1+\lambda]_q} r, \tag{3.16}$$

and

$$|D_q f(z)| \leq 1 + \frac{[2]_q(1-\alpha)}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][1+\lambda]_q} r. \tag{3.17}$$

The sharpness are attained for $f(z)$ given by (3.3).

Proof. For $k \geq 2$, we have

$$|D_q f(z)| \leq 1 - r\sum_{k=2}^{\infty} [k]_q a_k.$$

We find from (2.1) and (3.5) that

$$\begin{aligned}(1+\beta)[\lambda+1]_q \sum_{k=2}^{\infty}[k]_q a_k &\leq (1-\alpha) + (\alpha+\beta)[\lambda+1]_q \sum_{k=2}^{\infty} a_k \\ &\leq (1-\alpha) + \frac{[2]_q(\alpha+\beta)(1-\alpha)}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right]} \\ &\leq \frac{[2]_q(1+\beta)(1-\alpha)}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right]},\end{aligned}$$

that is, that

$$\sum_{k=2}^{\infty}[k]_q a_k \leq \frac{[2]_q(1-\alpha)}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][\lambda+1]_q}, \tag{3.18}$$

From (3.16) and (3.17) that

$$|D_q f(z)| \geq 1 - r\sum_{k=2}^{\infty}[k]_q a_k \geq 1 - \frac{[2]_q(1-\alpha)}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][1+\lambda]_q} r \tag{3.19}$$

and

$$|D_q f(z)| \leq 1 + r\sum_{k=2}^{\infty} [k]_q \, a_k \leq 1 + \tfrac{[2]_q(1-\alpha)}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][1+\lambda]_q} r. \tag{3.20}$$

This completes the proof. □

Letting $q \to 1^-$ in Theorem 3.4, we have

Corollary 3.5. *For $f(z) \in S_p^{\lambda}(\alpha,\beta)$, then*

$$\left|f^{'}(z)\right| \geq 1 - \tfrac{2(1-\alpha)}{(2+\beta-\alpha)(1+\lambda)} r, \tag{3.21}$$

and

$$\left|f^{'}(z)\right| \leq 1 + \tfrac{2(1-\alpha)}{(2+\beta-\alpha)(1+\lambda)} r. \tag{3.22}$$

The sharpness are attained for $f(z)$ given by (3.10).

Proof. Letting $q \to 1^-$ in Theorem 3.4, we can show (3.21) and (3.22). Then Corollary 3.5 corresponds to Theorem 3.3 when $q \to 1^-$. □

4. Closure theorems

Let $f_j(z)$ be defined, for $j = 1, 2, ..., m$, by

$$f_j(z) = z - \sum_{k=2}^{\infty} a_{k,j} z^k \quad (a_{k,j} \geq 0, \ z \in \mathbb{U}). \tag{4.1}$$

Theorem 4.1. *Let $f_j(z) \in TB_q^{\lambda}(\alpha,\beta)$ for $j = 1, 2, ..., m$. Then*

$$g(z) = \sum_{j=1}^{m} c_j f_j(z), \tag{4.2}$$

is also in the same class, where $c_j \geq 0$, $\sum\limits_{j=1}^{m} c_j = 1$.

Proof. According to (4.2), we can write

$$g(z) = z - \sum_{k=2}^{\infty} \left(\sum_{j=1}^{m} c_j a_{k,j} \right) z^k. \tag{4.3}$$

Further, since $f_j(z) \in TB_q^{\lambda}(\alpha,\beta)$, we get

$$\sum_{k=2}^{\infty} \left[[k]_q (1+\beta) - (\alpha+\beta) \right] \tfrac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!} a_{k,j} \leq 1 - \alpha. \tag{4.4}$$

Hence

$$\sum_{k=2}^{\infty}[[k]_q(1+\beta)-(\alpha+\beta)]\tfrac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}(\sum_{j=1}^{m}c_j a_{k,j})$$
$$=\sum_{j=1}^{m}c_j[\sum_{k=2}^{\infty}[[k]_q(1+\beta)-(\alpha+\beta)]\tfrac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_{k,j}]$$
$$\leq\left(\sum_{j=1}^{m}c_j\right)(1-\alpha)=1-\alpha, \tag{4.5}$$

which implies that $g(z)\in TB_q^{\lambda}(\alpha,\beta)$. Thus we have the theorem. □

Corollary 4.2. *The class $TB_q^{\lambda}(\alpha,\beta)$ is closed under convex linear combination.*

Proof. Let $f_j(z)\in TB_q^{\lambda}(\alpha,\beta)$ $(j=1,2)$ and

$$g(z)=\mu f_1(z)+(1-\mu)f_2(z)\quad(0\leq\mu\leq 1), \tag{4.6}$$

Then by, taking $m=2$, $c_1=\mu$ and $c_2=1-\mu$ in Theorem 5, we have $g(z)\in TB_q^{\lambda}(\alpha,\beta)$. □

Theorem 4.3. *Let $f_1(z)=z$ and*

$$f_k(z)=z-\frac{1-\alpha}{\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}}z^k\quad(k\geq 2). \tag{4.7}$$

Then $f(z)\in TB_q^{\lambda}(\alpha,\beta)$ if and only if

$$f(z)=\sum_{k=1}^{\infty}\mu_k f_k(z), \tag{4.8}$$

where $\mu_k\geq 0$ $(k\geq 1)$ and $\sum_{k=1}^{\infty}\mu_k=1$.

Proof. Suppose that

$$f(z)=\sum_{k=1}^{\infty}\mu_k f_k(z)=z-\sum_{k=2}^{\infty}\frac{1-\alpha}{\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}}\mu_k z^k. \tag{4.9}$$

Then it follows that

$$\sum_{k=2}^{\infty}\frac{\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}}{1-\alpha}\cdot\frac{1-\alpha}{\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}}\mu_k=\sum_{k=2}^{\infty}\mu_k=1-\mu_1\leq 1. \tag{4.10}$$

So by Theorem 2.1, $f(z)\in TB_q^{\lambda}(\alpha,\beta)$. Conversely, assume that $f(z)\in TB_q^{\lambda}(\alpha,\beta)$. Then

$$a_k\leq\frac{1-\alpha}{\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}}\quad(k\geq 2). \tag{4.11}$$

Setting

$$\mu_k = \frac{\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}}{1-\alpha} a_k \quad (k \geq 2), \tag{4.12}$$

and

$$\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k, \tag{4.13}$$

we see that $f(z)$ can be expressed in the form (4.8). This completes the proof. □

Corollary 4.4. *The extreme points of $TB_q^\lambda(\alpha,\beta)$ are $f_k(z)$ $(k \geq 1)$ given by Theorem 4.3.*

5. Some radii of the class $TB_q^\lambda(\alpha,\beta)$

Theorem 5.1. *Let $f(z) \in TB_q^\lambda(\alpha,\beta)$. Then for $0 \leq \rho < 1, k \geq 2, f(z)$ is*

(i) *close -to- convex of order ρ in $|z| < r_1$, where*

$$r_1 = r_1(q,\alpha,\beta,\lambda,\rho) := \inf_k \left[\frac{(1-\rho)\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}}{k(1-\alpha)}\right]^{\frac{1}{(k-1)}}. \tag{5.1}$$

(ii) *starlike of order ρ in $|z| < r_2$, where*

$$r_2 = r_2(q,\alpha,\beta,\lambda,\rho) := \inf_k \left[\frac{(1-\rho)\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}}{(k-\rho)(1-\alpha)}\right]^{\frac{1}{(k-1)}}. \tag{5.2}$$

(iii) *convex of order ρ in $|z| < r_3$, where*

$$r_3 = r_3(q,\alpha,\beta,\lambda,\rho) := \inf_k \left[\frac{(1-\rho)\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}}{k(k-\rho)(1-\alpha)}\right]^{\frac{1}{(k-1)}}. \tag{5.3}$$

The result is sharp for $f(z)$ is given by (2.3).

Proof. To prove (i) we must show that

$$\left|f'(z) - 1\right| \leq 1-\rho \quad for \quad |z| < r_1(q,\alpha,\beta,\rho).$$

From (1.12), we have

$$\left|f'(z) - 1\right| \leq \sum_{k=2}^{\infty} k a_k |z|^{k-1}.$$

Thus

$$\left|f'(z) - 1\right| \leq 1-\rho,$$

if

$$\sum_{k=2}^{\infty} \left(\frac{k}{1-\rho}\right) a_k |z|^{k-1} \leq 1. \tag{5.4}$$

But, by Theorem 2.1, (5.4) will be true if

$$\left(\frac{k}{1-\rho}\right)|z|^{k-1} \leq \frac{\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}}{1-\alpha},$$

that is, if

$$|z| \leq \left[\frac{(1-\rho)\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}}{k(1-\alpha)}\right]^{\frac{1}{(k-1)}} \quad (k \geq 2), \tag{5.5}$$

which gives (5.1).

To prove (ii) and (iii) it is suffices to show

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1-\rho \ \text{ for } \ |z| < r_2, \tag{5.6}$$

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq 1-\rho \ \text{ for } \ |z| < r_3, \tag{5.7}$$

respectively, by using arguments as in proving (i), we have the results. □

6. Inclusion relations involving $N_{k,q,\delta}(e)$

In this section following the works of Goodman [21] and Ruscheweyh [33] (see also [5], [6], [9], [16], [26] and [28]) defined the k,δ neighborhood of function $f(z) \in T$ by

$$N_{k,\delta}(f;g) = \left\{ g \in T : g(z) = z - \sum_{k=2}^{\infty} b_k z^k \quad \text{and} \quad \sum_{k=2}^{\infty} k\,|a_k - b_k| \leq \delta \right\}. \tag{6.1}$$

In particular, for the identity function $e(z) = z$, we have

$$N_{k,\delta}(e;g) = \left\{ g \in T : g(z) = z - \sum_{k=2}^{\infty} b_k z^k \quad \text{and} \quad \sum_{k=2}^{\infty} k\,|b_k| \leq \delta \right\}. \tag{6.2}$$

Aouf et al. [12] defined the k,q,δ neighborhood of function $f(z) \in T$ by

$$N_{k,q,\delta}(f;g) = \left\{ g \in T : g(z) = z - \sum_{k=2}^{\infty} b_k z^k \quad \text{and} \sum_{k=2}^{\infty} [k]_q\,|a_k - b_k| \leq \delta_q \right\}. \tag{6.3}$$

In particular, for the identity function $e(z) = z$, we have

$$N_{k,q,\delta}(e;g) = \left\{ g \in T : g(z) = z - \sum_{k=2}^{\infty} b_k z^k \quad \text{and} \quad \sum_{k=2}^{\infty} [k]_q\,|b_k| \leq \delta_q \right\}. \tag{6.4}$$

Theorem 6.1. *Let*

$$\delta_q = \frac{(1-\alpha)}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][\lambda+1]_q}. \tag{6.5}$$

Then $TB_q^{\lambda}(\alpha,\beta) \subset N_{k,q,\delta}(e)$.

Proof. For $f \in TB_q^{\lambda}(\alpha,\beta)$, Theorem 2.1, (3.5) and (3.18), and in view of the (6.4), Theorem 6.1 follows. $\square$

A function $f \in T$ is in the class $TB_q^{\lambda}(\alpha,\beta,\xi)$ if there exists a function $g \in TB_q^{\lambda}(\alpha,\beta)$ such that

$$\left|\frac{f(z)}{g(z)}-1\right| < 1-\xi_q \quad (z \in \mathbb{U},\ 0 \leq \xi_q < 1). \tag{6.6}$$

Now we determine the neighborhood for the class $TB_q^{\lambda}(\alpha,\beta,\xi)$.

Theorem 6.2. *If $g \in TB_q^{\lambda}(\alpha,\beta)$ and*

$$\xi_q = 1-\frac{\delta_q\left[[2]_q(1+\beta)-(\alpha+\beta)\right][\lambda+1]_q}{2\left\{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][\lambda+1]_q-(1-\alpha)\right\}}, \tag{6.7}$$

where

$$\delta_q \leq \frac{2\left\{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][\lambda+1]_q-(1-\alpha)\right\}}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][\lambda+1]_q}.$$

Then $N_{k,q,\delta}(g) \subset TB_q^{\lambda}(\alpha,\beta,\xi)$.

Proof. Suppose that $f \in N_{k,q,\delta}(g)$ then

$$\sum_{k=2}^{\infty}[k]_q\,|a_k-b_k| \leq \delta_q,$$

where δ_q is given by (6.5), which implies that the coefficient inequality

$$\sum_{k=2}^{\infty}|a_k-b_k| \leq \frac{\delta_q}{[2]_q}.$$

Next, since $g \in TB_q^{\lambda}(\alpha,\beta)$, we have

$$\sum_{k=2}^{\infty} b_k \leq \frac{1-\alpha}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][\lambda+1]_q},$$

so that

$$\left|\frac{f(z)}{g(z)}-1\right| < \frac{\sum_{k=2}^{\infty}|a_k-b_k|}{1-\sum_{k=2}^{\infty}b_k} \leq \frac{\delta_q}{[2]_q} \times \frac{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][\lambda+1]_q}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][\lambda+1]_q-(1-\alpha)} \leq 1-\xi_q.$$

Provided that ξ_q is given precisely by (6.7). Thus, by definition, $f \in TB_q^{\lambda}(\alpha,\beta,\xi)$, which completes the proof. $\square$

7. Partial sums

For $f(z)$ of the form (1.1), the sequence of partial sums is given by

$$f_m(z) = z + \sum_{k=2}^{m} a_k z^k \quad (m \in \mathbb{N} \setminus \{1\}).$$

Now following the work of [38] and also the works cited in [11, 15, 19, 25, 27, 31, 37] on partial sums of analytic functions, to obtain our results. Let

$$\Phi_{q,k}^{\lambda} = \Phi_q^{\lambda}(k,\alpha,\beta) = \left[[k]_q (1+\beta) - (\alpha+\beta)\right] \tfrac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}. \tag{7.1}$$

Theorem 7.1. *If $f \in \mathcal{S}$, satisfies the condition (2.1), then*

$$\operatorname{Re}\left(\frac{f(z)}{f_m(z)}\right) \geq \frac{\Phi_{q,m+1}^{\lambda} - 1 + \alpha}{\Phi_{q,m+1}^{\lambda}}, \tag{7.2}$$

where

$$\Phi_{q,k}^{\lambda} \geq \begin{cases} 1-\alpha, & \text{if} \quad k = 2,3,...,m \\ \Phi_{q,m+1}^{\lambda}, & \text{if} \quad k = m+1, m+2, ... \end{cases} \tag{7.3}$$

The result (7.2) is sharp for

$$f(z) = z + \frac{1-\alpha}{\Phi_{q,m+1}^{\lambda}} z^{m+1}. \tag{7.4}$$

Proof. Define $g(z)$ by

$$\tfrac{1+g(z)}{1-g(z)} = \tfrac{\Phi_{q,m+1}^{\lambda}}{1-\alpha}\left[\tfrac{f(z)}{f_m(z)} - \tfrac{\Phi_{q,m+1}^{\lambda}-1+\alpha}{\Phi_{q,m+1}^{\lambda}}\right] = \frac{1+\sum\limits_{k=2}^{m} a_k z^{k-1} + \left(\frac{\Phi_{q,m+1}^{\lambda}}{1-\alpha}\right)\sum\limits_{k=m+1}^{\infty} a_k z^{k-1}}{1+\sum\limits_{k=2}^{m} a_k z^{k-1}}. \tag{7.5}$$

It suffices to show that $|g(z)| \leq 1$. Now from (7.5) we have

$$g(z) = \frac{\left(\frac{\Phi_{q,m+1}^{\lambda}}{1-\alpha}\right)\sum\limits_{k=m+1}^{\infty} a_k z^{k-1}}{2+2\sum\limits_{k=2}^{m} a_k z^{k-1} + \left(\frac{\Phi_{q,m+1}^{\lambda}}{1-\alpha}\right)\sum\limits_{k=m+1}^{\infty} a_k z^{k-1}}.$$

Hence we obtain

$$|g(z)| \leq \frac{\left(\frac{\Phi_{q,m+1}^{\lambda}}{1-\alpha}\right)\sum\limits_{k=m+1}^{\infty} |a_k|}{2-2\sum\limits_{k=2}^{m} |a_k| - \left(\frac{\Phi_{q,m+1}^{\lambda}}{1-\alpha}\right)\sum\limits_{k=m+1}^{\infty} |a_k|}.$$

Now $|g(z)| \leq 1$ if and only if

$$2\left(\frac{\Phi_{q,m+1}^{\lambda}}{1-\alpha}\right)\sum_{k=m+1}^{\infty} |a_k| \leq 2 - 2\sum_{k=2}^{m} |a_k|,$$

or, equivalently,

$$\sum_{k=2}^{m} |a_k| + \sum_{k=m+1}^{\infty} \frac{\Phi_{q,m+1}^{\lambda}}{1-\alpha} |a_k| \leq 1.$$

From (2.1), it is sufficient to show that

$$\sum_{k=2}^{m} |a_k| + \sum_{k=m+1}^{\infty} \frac{\Phi_{q,m+1}^{\lambda}}{1-\alpha} |a_k| \le \sum_{k=2}^{\infty} \frac{\Phi_{q,k}^{\lambda}}{1-\alpha} |a_k|,$$

which is equivalent to

$$\sum_{k=2}^{m} \left(\frac{\Phi_{q,k}^{\lambda}-1+\alpha}{1-\alpha} \right) |a_k| + \sum_{k=m+1}^{\infty} \left(\frac{\Phi_{q,k}^{\lambda}-\Phi_{q,m+1}^{\lambda}}{1-\alpha} \right) |a_k| \ge 0. \tag{7.6}$$

For $z = re^{i\pi/m}$ we have

$$\frac{f(z)}{f_m(z)} = 1 + \frac{1-\alpha}{\Phi_{q,m+1}^{\lambda}} z^k \to 1 - \frac{1-\alpha}{\Phi_{q,m+1}^{\lambda}} = \frac{\Phi_{q,m+1}^{\lambda}-1+\alpha}{\Phi_{q,m+1}^{\lambda}} \quad \text{where } r \to 1^-,$$

which shows that $f(z)$ is given by (7.4) gives the sharpness. □

Remark 7.1. (i) Putting $\lambda = 0$ and (ii) $\lambda = 1$ in Theorem 7.1, we obtain the following results, respectively.

Corollary 7.2. *If $f \in \mathcal{S}$, satisfies the condition (2.1) and $\frac{f(z)}{z} \neq 0 (0 < |z| < 1)$, then*

$$\operatorname{Re}\left(\frac{f(z)}{f_m(z)} \right) \ge \frac{\left[[m+1]_q(1+\beta)-(\alpha+\beta)\right]-1+\alpha}{\left[[m+1]_q(1+\beta)-(\alpha+\beta)\right]}. \tag{7.7}$$

The result is sharp for

$$f(z) = z + \frac{1-\alpha}{\left[[m+1]_q(1+\beta)-(\alpha+\beta)\right]} z^{m+1}. \tag{7.8}$$

Corollary 7.3. *If $f \in \mathcal{S}$, satisfies the condition (2.1) and $\frac{f(z)}{z} \neq 0 (0 < |z| < 1)$, then*

$$\operatorname{Re}\left(\frac{f(z)}{f_m(z)} \right) \ge 1 - \frac{1-\alpha}{[m+1]_q\left[[m+1]_q(1+\beta)-(\alpha+\beta)\right]}. \tag{7.9}$$

The result is sharp for

$$f(z) = z + \frac{1-\alpha}{[m+1]_q\left[[m+1]_q(1+\beta)-(\alpha+\beta)\right]} z^{m+1}. \tag{7.10}$$

Theorem 7.4. *If $f \in \mathcal{S}$, satisfies the condition (2.1), then*

$$\operatorname{Re}\left(\frac{f_m(z)}{f(z)} \right) \ge \frac{\Phi_{q,m+1}^{\lambda}}{\Phi_{q,m+1}^{\lambda}+1-\alpha}, \tag{7.11}$$

where $\Phi_{q,m+1}^{\lambda}$ is defined by (7.1) and satisfies (7.3) and $f(z)$ given by (7.4) gives the sharpness.

Proof. The proof follows by defining

$$\frac{1+g(z)}{1-g(z)} = \frac{\Phi_{q,m+1}^{\lambda}+1-\alpha}{1-\alpha} \left[\frac{f_m(z)}{f(z)} - \frac{\Phi_{q,m+1}^{\lambda}}{\Phi_{q,m+1}^{\lambda}+1-\alpha} \right]$$

and much akin are to similar arguments in Theorem 7.1. So, we omit it. □

Remark 7.2. (i) Putting $\lambda = 0$ and (ii) $\lambda = 1$ in Theorem 7.4, we obtain the following sharp results, respectively.

Corollary 7.5. *If $f \in \mathcal{S}$, satisfies the condition (2.1) and $\frac{f(z)}{z} \neq 0 (0 < |z| < 1)$, then*

$$\operatorname{Re}\left(\frac{f_m(z)}{f(z)}\right) \geq \frac{[m+1]_q(1+\beta)-(\alpha+\beta)}{[m+1]_q(1+\beta)-(\alpha+\beta)+1-\alpha}. \tag{7.12}$$

Corollary 7.6. *If $f \in \mathcal{S}$, satisfies the condition (2.1) and $\frac{f(z)}{z} \neq 0 (0 < |z| < 1)$, then*

$$\operatorname{Re}\left(\frac{f_m(z)}{f(z)}\right) \geq \frac{[m+1]_q\left[[m+1]_q(1+\beta)-(\alpha+\beta)\right]}{[m+1]_q\left[[m+1]_q(1+\beta)-(\alpha+\beta)\right]+1-\alpha}. \tag{7.13}$$

Theorem 7.7. *If $f \in \mathcal{S}$, satisfies the condition (2.1), then*

$$\operatorname{Re}\left(\frac{f'(z)}{f'_m(z)}\right) \geq \frac{\Phi^{\lambda}_{q,m+1}-(m+1)(1-\alpha)}{\Phi^{\lambda}_{q,m+1}}, \tag{7.14}$$

and

$$\operatorname{Re}\left(\frac{f'_m(z)}{f'(z)}\right) \geq \frac{\Phi^{\lambda}_{q,m+1}}{\Phi^{\lambda}_{q,m+1}+(m+1)(1-\alpha)}, \tag{7.15}$$

where $\Phi^{\lambda}_{q,m+1} \geq (m+1)(1-\alpha)$ and

$$\Phi^{\lambda}_{q,k} \geq \begin{cases} k(1-\alpha), & \text{if} \quad k = 2,3,...,m \\ k\left(\frac{\Phi^{\lambda}_{q,m+1}}{(m+1)}\right), & \text{if} \quad k = m+1, m+2, ... \end{cases} \tag{7.16}$$

$f(z)$ is given by (7.4) gives the sharpness.

Proof. We write

$$\frac{1+g(z)}{1-g(z)} = \frac{\Phi^{\lambda}_{q,m+1}}{(m+1)(1-\alpha)}\left[\frac{f'(z)}{f'_m(z)} - \left(\frac{\Phi^{\lambda}_{q,m+1}-(m+1)(1-\alpha)}{\Phi^{\lambda}_{q,m+1}}\right)\right],$$

where

$$g(z) = \frac{\left(\frac{\Phi^{\lambda}_{q,m+1}}{(m+1)(1-\alpha)}\right)\sum\limits_{k=m+1}^{\infty} k a_k z^{k-1}}{2+2\sum\limits_{k=2}^{m} k a_k z^{k-1}+\left(\frac{\Phi^{\lambda}_{q,m+1}}{(m+1)(1-\alpha)}\right)\sum\limits_{k=m+1}^{\infty} k a_k z^{k-1}}.$$

Now $|g(z)| \leq 1$ if and only if

$$\sum_{k=2}^{m} k|a_k| + \left(\frac{\Phi^{\lambda}_{q,m+1}}{(m+1)(1-\alpha)}\right)\sum_{k=m+1}^{\infty} k|a_k| \leq 1.$$

From (2.1), it is sufficient to show that

$$\sum_{k=2}^{m} k|a_k| + \left(\frac{\Phi^{\lambda}_{q,m+1}}{(m+1)(1-\alpha)}\right)\sum_{k=m+1}^{\infty} k|a_k| \leq \sum_{k=2}^{\infty}\frac{\Phi^{\lambda}_{q,k}}{1-\alpha}|a_k|,$$

which is equivalent to

$$\sum_{k=2}^{m}\left(\frac{\Phi^{\lambda}_{q,k}-k(1-\alpha)}{1-\alpha}\right)|a_k| + \sum_{k=m+1}^{\infty}\left(\frac{(m+1)\Phi^{\lambda}_{q,k}-k\Phi^{\lambda}_{q,m+1}}{(m+1)(1-\alpha)}\right)|a_k| \geq 0.$$

To prove the result (7.15), define the function $g(z)$ by

$$\frac{1+g(z)}{1-g(z)} = \frac{(m+1)(1-\alpha)+\Phi^{\lambda}_{q,m+1}}{(m+1)(1-\alpha)}\left[\frac{f'_m(z)}{f'(z)} - \frac{\Phi^{\lambda}_{q,m+1}}{(m+1)(1-\alpha)+\Phi^{\lambda}_{q,m+1}}\right],$$

and by similar arguments in first part we get desired result. □

Remark 7.3. (*i*) Putting $\lambda = 0$ and (*ii*) $\lambda = 1$ in Theorem 7.7, we obtain the following sharp results, respectively.

Corollary 7.8. *If $f \in \mathcal{S}$, satisfies the condition (2.1) and $\frac{f(z)}{z} \neq 0 (0 < |z| < 1)$, then*

$$\operatorname{Re}\left(\frac{f'(z)}{f'_m(z)}\right) \geq 1 - \frac{(m+1)(1-\alpha)}{[m+1]_q(1+\beta)-(\alpha+\beta)}, \tag{7.17}$$

and

$$\operatorname{Re}\left(\frac{f'_m(z)}{f'(z)}\right) \geq \frac{[m+1]_q(1+\beta)-(\alpha+\beta)}{[m+1]_q(1+\beta)-(\alpha+\beta)+(m+1)(1-\alpha)}. \tag{7.18}$$

Corollary 7.9. *If $f \in \mathcal{S}$, satisfies the condition (2.1) and $\frac{f(z)}{z} \neq 0 (0 < |z| < 1)$, then*

$$\operatorname{Re}\left(\frac{f'(z)}{f'_m(z)}\right) \geq 1 - \frac{(m+1)(1-\alpha)}{[m+1]_q[(m+1)(1+\beta)-(\alpha+\beta)]}, \tag{7.19}$$

and

$$\operatorname{Re}\left(\frac{f'_m(z)}{f'(z)}\right) \geq \frac{[m+1]_q[(m+1)(1+\beta)-(\alpha+\beta)]}{[m+1]_q[(m+1)(1+\beta)-(\alpha+\beta)]+(m+1)(1-\alpha)}. \tag{7.20}$$

Remark 7.4. Letting $q \to 1^-$ in Theorems 7.1, 7.4 and 7.7, respectively, we get Theorems 4.1 and 4.2, respectively, for the class $S^{\lambda}_q(\alpha, \beta)$ studied by Rosy et al. [31].

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