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Popoviciu type inequalities for $n-$convex functions via extension of weighted Montgomery identity

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Abstract. In this article, we derive the Popoviciu-type inequalities by using the weighted version of the extension of Montgomery's identity and Green functions. By considering the n-convex function, we prove some identities and related inequalities involving sums $\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i)$ and integrals $\int_{\alpha_1}^{\beta_1} \varrho(\chi)\zeta(g(\chi))\, d\chi$. Some results for n-convex functions at a point are also obtained. Besides that, some Ostrowski-type inequalities are also presented, which are interrelated with the obtained inequalities.

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1. Introduction

Pečarić [15] established the following result (see also [18, p.262]):

Proposition 1.1. *The inequality*

$$\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i) \geq 0 \tag{1.1}$$

holds for all convex functions ζ if and only if the $\gamma-$tuples

$$\chi = (\chi_1, \ldots, \chi_\gamma), \quad \varrho = (\varrho_1, \ldots, \varrho_\gamma) \in \mathbb{R}^\gamma$$

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satisfy

$$\sum_{i=1}^{\gamma} \varrho_i = 0 \quad \text{and} \quad \sum_{i=1}^{\gamma} \varrho_i |\chi_i - \chi_\kappa| \geq 0 \ \forall\ \kappa \in \{1, \dots, \gamma\}. \tag{1.2}$$

Since

$$\sum_{i=1}^{\gamma} \varrho_i |\chi_i - \chi_\kappa| = 2 \sum_{i=1}^{\gamma} \varrho_i (\chi_i - \chi_\kappa)_+ - \sum_{i=1}^{\gamma} \varrho_i (\chi_i - \chi_\kappa),$$

where $y_+ = \max(y, 0)$, it is easy to see that condition (1.2) is equivalent to

$$\sum_{i=1}^{\gamma} \varrho_i = 0, \quad \sum_{i=1}^{\gamma} \varrho_i \chi_i = 0 \quad \text{and} \quad \sum_{i=1}^{\gamma} \varrho_i (\chi_i - \chi_\kappa)_+ \geq 0 \text{ for } \kappa \in \{1, \dots, \gamma - 1\}. \tag{1.3}$$

Let $\chi_{(1)} \leq \chi_{(2)} \leq \ldots \leq \chi_{(\gamma)}$ be the sequence χ in ascending order, $w(\chi, \tau) = (\chi - \tau)_+$ and Λ_0 denote the linear operator

$$\Lambda_0(\zeta) = \sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i).$$

Notice that

$$\Lambda(w(\cdot, \chi_\kappa)) = \sum_{i=1}^{\gamma} \varrho_i (\chi_i - \chi_\kappa)_+.$$

For $\tau \in [\chi_{(\kappa)}, \chi_{(\kappa+1)}]$ we have

$$\Lambda(w(\cdot, \tau)) = \Lambda(w(\cdot, \chi_{(\kappa)})) + (\chi_{(\kappa)} - \tau) \sum_{\{i : \chi_i > \chi_{(\kappa)}\}} \varrho_i,$$

so the mapping $\tau \mapsto \Lambda(w(\cdot, \tau))$ is linear on $[\chi_{(\kappa)}, \chi_{(\kappa+1)}]$.
Additionally, $\Lambda(w(\cdot, \chi_{(\gamma)}) = 0$, so condition (1.3) is equivalent to

$$\sum_{i=1}^{\gamma} \varrho_i = 0, \quad \sum_{i=1}^{\gamma} \varrho_i \chi_i = 0 \quad \text{and} \quad \sum_{i=1}^{\gamma} \varrho_i (\chi_i - \tau)_+ \geq 0 \ \forall\ \tau \in [\chi_{(1)}, \chi_{(\gamma-1)}]. \tag{1.4}$$

It comes out that condition (1.4) is suitable for extension of Proposition 1.1 to the integral version and the more general class of n-convex functions (see e.g. [18]).

Definition 1.2. The *nth order divided difference* of a function $\zeta : I \to \mathbb{R}$ at distinct points $\chi_i, \chi_{i+1}, \dots, \chi_{i+n} \in I = [a_1, b_1] \subset \mathbb{R}$ for some $i \in \mathbb{N}$ is defined recursively by:

$$\begin{aligned} [\chi_j; \zeta] &= \zeta(\chi_j), \quad j \in \{i, \dots, i+n\} \\ [\chi_i, \dots, \chi_{i+n}; \zeta] &= \frac{[\chi_{i+1}, \dots, \chi_{i+n}; \zeta] - [\chi_i, \dots, \chi_{i+n-1}; \zeta]}{\chi_{i+n} - \chi_i}. \end{aligned}$$

It may easily be verified that

$$[\chi_i, \dots, \chi_{i+n}; \zeta] = \sum_{\kappa=0}^{n} \frac{\zeta(x_{i+\kappa})}{\prod_{j=i, j \neq i+\kappa}^{i+n} (\chi_{i+\kappa} - \chi_j)}.$$

Remark 1.3. Let us denote $[\chi_i, \dots, \chi_{i+n}; \zeta]$ by $\Delta_{(n)}\zeta(\chi_i)$. The value $[\chi_i, \dots, \chi_{i+n}; \zeta]$ is independent of the order of the points $\chi_i, \chi_{i+1}, \dots, \chi_{i+n}$. This definition can be extended by involving the cases in which two or more points coincide by taking respective limits. □

Definition 1.4. If for all choices of $(n+1)$ distinct points $\chi_i, \dots, \chi_{i+n}$ we have $\Delta_{(n)}\zeta(\chi_i) \geq 0$ then the function $\zeta : I \to \mathbb{R}$ is called *convex of order* n or $n-$*convex.*

If the function is nth order differentiable such that $\zeta^{(n)} \geq 0$ then ζ is n-convex. A function ζ is n-convex for $1 \leq \kappa \leq n-2$, if and only if $\zeta^{(\kappa)}$ exists and is $(n-\kappa)$-convex.

Popoviciu [19], [20] obtained the following result (see also [17, 18, 22]).

Proposition 1.5. *Let $n \geq 2$. Inequality (1.1) is valid for all n-convex functions $\zeta : [a_1, b_1] \to \mathbb{R}$ if and only if the $\gamma-$tuples $\chi \in [a_1, b_1]^\gamma$, $\varrho \in \mathbb{R}^\gamma$ satisfy*

$$\sum_{i=1}^{\gamma} \varrho_i \chi_i^\kappa = 0, \quad \forall \kappa = 0, 1, \dots, n-1 \tag{1.5}$$

$$\sum_{i=1}^{\gamma} \varrho_i (\chi_i - \tau)_+^{n-1} \geq 0, \quad \forall \tau \in [a_1, b_1]. \tag{1.6}$$

Definitely Popoviciu established a significant result - it is adequate to postulate that (1.6) holds $\forall\ \tau \in [\chi_{(1)}, \chi_{(\gamma-n+1)}]$ and then, because of (1.5), it is automatically stated $\forall\ \tau \in [a_1, b_1]$. The integral version is given in the following proposition (see [17, 18, 21]).

Proposition 1.6. *Let $n \geq 2$, $\varrho : [\alpha_1, \beta_1] \to \mathbb{R}$ and $g : [\alpha_1, \beta_1] \to [a_1, b_1]$. Then, the inequality*

$$\int_{\alpha_1}^{\beta_1} \varrho(\chi)\zeta(g(\chi))\, d\chi \geq 0 \tag{1.7}$$

holds for all n-convex functions $\zeta : [a_1, b_1] \to \mathbb{R}$ if and only if

$$\begin{aligned} &\int_{\alpha_1}^{\beta_1} \varrho(\chi) g(\chi)^\kappa\, d\chi = 0, \qquad \forall\ \kappa = 0, 1, \dots, n-1 \\ &\int_{\alpha_1}^{\beta_1} \varrho(\chi) \left(g(\chi) - \tau\right)_+^{n-1}\, d\chi \geq 0, \qquad \forall\ \tau \in [a_1, b_1]. \end{aligned} \tag{1.8}$$

In this article, we would like to establish some inequalities of type (1.1) and (1.7) by using the following extension of Montgomery's identity via Taylor's formula for n-convex functions obtained in [1].

Proposition 1.7. *Let $\zeta : I \to \mathbb{R}$ be such that $\zeta^{(n-1)}$ is absolutely continuous, $n \in \mathbb{N}$, $a_1, b_1 \in I$, $a_1 < b_1$, $I \subset \mathbb{R}$ an open interval, $w : [a_1, b_1] \to [0, \infty)$ is some probability* *density function. Then the following identity holds*

$$\begin{aligned}
\zeta(\chi) &= \int_{a_1}^{b_1} w(\tau)\zeta(\tau)d\tau \\
&+\sum_{\kappa=0}^{n-2}\frac{\zeta^{(\kappa+1)}(a_1)}{(\kappa+1)!}\int_{a_1}^{\chi} w(\varsigma)\left((\chi-a_1)^{\kappa+1}-(\varsigma-a_1)^{\kappa+1}\right)d\varsigma \\
&+\sum_{\kappa=0}^{n-2}\frac{\zeta^{(\kappa+1)}(b_1)}{(\kappa+1)!}\int_{\chi}^{b_1} w(\varsigma)\left((\chi-b_1)^{\kappa+1}-(\varsigma-b_1)^{\kappa+1}\right)d\varsigma \\
&+\frac{1}{(n-1)!}\int_{a_1}^{b_1} T_{w,n}(\chi,\varsigma)\zeta^{(n)}(\varsigma)d\varsigma, \qquad (1.9)
\end{aligned}$$

where

$$T_{w,n}(\chi,\varsigma)=\begin{cases} \displaystyle\int_{\chi}^{\varsigma} w(u)(u-\varsigma)^{n-1}du+W(\chi)(\chi-\varsigma)^{n-1}, \\ \qquad a_1\le\varsigma\le\chi \\ \displaystyle\int_{\chi}^{\varsigma} w(u)(u-\varsigma)^{n-1}du+\left(W(\chi)-1\right)(\chi-\varsigma)^{n-1}, \\ \qquad \chi<\varsigma\le b_1 \end{cases} \qquad (1.10)$$

If we put $w(\tau)=\frac{1}{b_1-a_1}$, $\tau\in[a_1,b_1]$, *the above identity reduces*

$$\begin{aligned}
\zeta(\chi) &= \frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\tau)\,d\tau+\sum_{\kappa=0}^{n-2}\frac{\zeta^{(\kappa+1)}(a_1)}{\kappa!\,(\kappa+2)}\frac{(\chi-a_1)^{\kappa+2}}{b_1-a_1} \\
&- \sum_{\kappa=0}^{n-2}\frac{\zeta^{(n-1)}(b_1)}{\kappa!\,(\kappa+2)}\frac{(\chi-b_1)^{\kappa+2}}{b_1-a_1}+\frac{1}{(n-1)!}\int_{a_1}^{b_1}T_n(\chi,\varsigma)\,\zeta^{(n)}(\varsigma)\,d\varsigma, \qquad (1.11)
\end{aligned}$$

where

$$T_n(\chi,\varsigma)=\begin{cases} -\dfrac{(\chi-\varsigma)^n}{n(b_1-a_1)}+\dfrac{\chi-a_1}{b_1-a_1}(\chi-\varsigma)^{n-1}, & a_1\le\varsigma\le\chi, \\ -\dfrac{(\chi-\varsigma)^n}{n(b_1-a_1)}+\dfrac{\chi-b_1}{b_1-a_1}(\chi-\varsigma)^{n-1}, & \chi<\varsigma\le b_1. \end{cases} \qquad (1.12)$$

In case $n=1$ the sum $\sum_{\kappa=0}^{n-2}\cdots$ is empty, so identity (1.11) encounters to the renowned Montgomery identity (see for instance [13])

$$\zeta(\chi)=\frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\tau)\,d\tau+\int_{a_1}^{b_1}P(\chi,\varsigma)\,\zeta'(\varsigma)\,d\varsigma$$

where $P(\chi, \varsigma)$ is the Peano kernel, given by

$$P(\chi, \varsigma) = \begin{cases} \dfrac{s - a_1}{b_1 - a_1}, & a_1 \leq \varsigma \leq \chi, \\ \dfrac{\varsigma - b_1}{b_1 - a_1}, & \chi < \varsigma \leq b_1. \end{cases}$$

The weighted version of Montgomery identity can be found in [14]:

Proposition 1.8. *Let $\zeta \in AC[a_1, b_1]$. Suppose that $w : [a_1, b_1] \to [0, \infty)$ is satisfying*

$$\int_{a_1}^{b_1} w(\tau) d\tau = 1,$$

some probability density function, i.e., it is a positive integrable function and

$$W(\tau) = \begin{cases} 0, & \tau < a_1, \\ \displaystyle\int_{a_1}^{\tau} w(\chi) d\chi, & \tau \in [a_1, b_1], \\ 1, & \tau > b_1. \end{cases}$$

Then

$$\zeta(\chi) = \int_{a_1}^{b_1} w(\tau)\zeta(\tau)\, d\tau + \int_{a_1}^{b_1} P_w(\chi, \tau)\zeta(\tau)\, d\tau,$$

where the weighted Peano kernel is given by

$$P_w(\chi, \tau) = \begin{cases} W(\tau), & a_1 \leq \tau \leq \chi, \\ W(\tau) - 1, & \chi < \tau \leq b_1. \end{cases}$$

Let us denote the Green's function by $G_l : [a_1, b_1] \times [a_1, b_1] \to \mathbb{R}$ with the boundary value problem

$$z''(\lambda) = 0, \; z(a_l) = z(b_l) = 0.$$

The function G_0 is defined as

$$G_0(\tau, \varsigma) = \begin{cases} \dfrac{(\tau - b_1)(\varsigma - a_1)}{b_1 - a_1} & \text{for } a_1 \leq \varsigma \leq \tau, \\ \dfrac{(\varsigma - b_1)(\tau - a_1)}{b_1 - a_1} & \text{for } \tau \leq \varsigma \leq b_1 \end{cases} \tag{1.13}$$

and for any function $\zeta \in C^2[a_1, b_1]$,the following identity induces using integration by parts

$$\zeta(\chi) = \frac{b_1 - \chi}{b_1 - a_1}\zeta(a_1) + \frac{\chi - a_1}{b_1 - a_1}\zeta(b_1) + \int_{b_1}^{a_1} G_0(\chi, \varsigma)\zeta''(\varsigma) d\varsigma. \tag{1.14}$$

The function G_0 is continuous, symmetric and convex with respect to both variables τ and ς.

As a special choice Abel-Gontscharoff polynomial for 'two-point right focal' interpolating polynomial for $n = 2$ can be given as (see [16]):

$$\zeta(\chi) = \zeta(a_1) + (\chi - a_1)\zeta'(b_1) + \int_{a_1}^{b_1} G_1(\chi, \tau)\zeta''(\tau) d\tau. \tag{1.15}$$

where $G_1(\varsigma,\tau)$ is Green's function for two-point right focal problem defined as

$$G_1(\varsigma,\tau)=\begin{cases} a_1-\tau & \text{for } a_1\le\tau\le\varsigma, \\ a_1-\varsigma & \text{for } \varsigma\le\tau\le b_1 \end{cases} \tag{1.16}$$

Motivated by Abel-Gontscharoff identity (1.15) and related Green's function (1.16), we would recall here some new types of Green functions $G_l:[a_1,b_1]\times[a_1,b_1]\longrightarrow\mathbb{R}$ for $l\in 2,3,4$ defined as in [3]:

$$G_2(\varsigma,\tau)=\begin{cases} \varsigma-b_1 & \text{for } a_1\le\tau\le\varsigma, \\ \tau-b_1 & \text{for } \varsigma\le\tau\le b_1 \end{cases} \tag{1.17}$$

$$G_3(\varsigma,\tau)=\begin{cases} \varsigma-a_1 & \text{for } a_1\le\tau\le\varsigma, \\ \tau-a_1 & \text{for } \varsigma\le\tau\le b_1 \end{cases} \tag{1.18}$$

$$G_4(\varsigma,\tau)=\begin{cases} b_1-\tau & \text{for } a_1\le\tau\le\varsigma, \\ b_1-\varsigma & \text{for } \varsigma\le\tau\le b_1 \end{cases} \tag{1.19}$$

In [3] (see also [4], [12]), it is also shown that all four Green functions are symmetric and continuous. These new Green functions enable us to present some new identities, stated as follow

$$\zeta(\chi)=\zeta(b_1)+(b_1-\chi)\zeta'(a_1)+\int_{a_1}^{b_1}G_2(\chi,\tau)\zeta''(\tau)d\tau. \tag{1.20}$$

$$\zeta(\chi)=\zeta(b_1)-(b_1-a_1)\zeta'(b_1)+(\chi-a_1)\zeta'(a_1)+\int_{a_1}^{b_1}G_3(\chi,\tau)\zeta''(\tau)d\tau. \tag{1.21}$$

$$\zeta(\chi)=\zeta(a_1)+(b_1-a_1)\zeta'(a_1)-(b_1-\chi)\zeta'(b_1)+\int_{a_1}^{b_1}G_4(\chi,\varsigma)\zeta''(\tau)d\tau. \tag{1.22}$$

To recall definitions of a generalized convex function and related concepts and results we refer to interested readers following references [11], [6] and [18]. This article is arranged in the following manner. In Section 2 we will obtain inequalities of type (1.1), (1.7) for n-convex functions by using the extension of Montgomery's identity (1.11). In Section 3 we will give some discrete and integral nature identities and corresponding linear inequalities using Green functions and applying extension of weighted Montgomery identity. In both sections, we will discuss a generalization of the class of n-convex functions introduced in [17]. On the basis of this discussion, we will give related inequalities for n-convex functions at a point. we will also provide some Ostrowski-type inequalities by obtaining bounds for the remainders of the identities from obtained results.

We will first prove some results that will have a crucial role in each Section of the paper. Then we will propose some Related Popoviciu type inequalities.

2. Popoviciu type identities and inequalities via extension of weighted Montgomery identity

Theorem 2.1. *Under the assumptions of Proposition* 1.7 *and let $T_{w,n}$ be defined by (1.10). Additionally, let $\chi_i \in [a_1, b_1]$, $\varrho_i \in \mathbb{R}$ for $i \in \{1, 2, \ldots, \gamma\}$ and $\gamma \in \mathbb{N}$ be s.t.*

$$\sum_{i=1}^{\gamma} \varrho_i = 0.$$

Then

$$\begin{aligned}\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i) &= \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{(\kappa+1)!} \sum_{i=1}^{\gamma} \varrho_i \int_{a_1}^{\chi_i} w(\varsigma) \left((\chi_i - a_1)^{\kappa+1} - (\varsigma - a_1)^{\kappa+1}\right) d\varsigma \\ &+ \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{(\kappa+1)!} \sum_{i=1}^{\gamma} \varrho_i \int_{\chi_i}^{b_1} w(\varsigma) \left((\chi_i - b_1)^{\kappa+1} - (\varsigma - b_1)^{\kappa+1}\right) d\varsigma \\ &+ \frac{1}{(n-1)!} \int_{a_1}^{b_1} \left(\sum_{i=1}^{\gamma} \varrho_i T_{w,n}(\chi_i, \varsigma) \right) \zeta^{(n)}(\varsigma) d\varsigma. \end{aligned} \tag{2.1}$$

Proof. Putting in the extension of Montgomery identity (1.9) $\chi_i, i = 1, \ldots, m$, multiplying with ϱ_i and summing all the identities we obtain

$$\begin{aligned}\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i) &= \int_{a_1}^{b_1} w(\varsigma)\zeta(\tau) d\tau \sum_{i=1}^{\gamma} \varrho_i \\ &+ \sum_{i=1}^{\gamma} \varrho_i \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{(\kappa+1)!} \int_{a_1}^{\chi} w(\varsigma) \left((\chi - a_1)^{\kappa+1} - (\varsigma - a_1)^{\kappa+1}\right) d\varsigma \\ &+ \sum_{i=1}^{\gamma} \varrho_i \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{(\kappa+1)!} \int_{\chi}^{b_1} w(\varsigma) \left((\chi - b_1)^{\kappa+1} - (\varsigma - b_1)^{\kappa+1}\right) d\varsigma \\ &+ \frac{1}{(n-1)!} \sum_{i=1}^{\gamma} \varrho_i \int_{a_1}^{b_1} T_{w,n}(\chi, \varsigma)\zeta^{(n)}(\varsigma) d\varsigma, \end{aligned}$$

By simplifying this expressions we obtain (2.1). □

Remark 2.2. If we put $w(\varsigma) = \frac{1}{b_1 - a_1}$, $\varsigma \in [a_1, b_1]$ above identity reduces to Theorem 1 of [8].

Its integral version is as follows.

Theorem 2.3. *Let $\varrho : [\alpha_1, \beta_1] \to \mathbb{R}$ and $g : [\alpha_1, \beta_1] \to [a_1, b_1]$ be integrable functions s.t.*

$$\int_{\alpha_1}^{\beta_1} \varrho(\chi) d\chi = 0.$$

Let $\zeta : I \to \mathbb{R}$ be such that $\zeta^{(n-1)}$ is absolutely continuous, $a_1 < b_1$, $a_1, b_1 \in I$, $I \subset \mathbb{R}$ be an open interval, $n \in \mathbb{N}$, $T_{w,n}$ be given by (1.10). Then

$$\begin{aligned}
&\int_{\alpha_1}^{\beta_1} \varrho(\chi)\,\zeta(g(\chi))\,d\chi \\
&= \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{(\kappa+1)!} \int_{\alpha_1}^{\beta_1} \varrho(\chi) \int_{a_1}^{g(\chi)} w(\varsigma) \left((g(\chi)-a_1)^{\kappa+1} - (\varsigma - a_1)^{\kappa+1}\right) d\varsigma d\chi \\
&+ \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{(\kappa+1)!} \int_{\alpha_1}^{\beta_1} \varrho(\chi) \int_{g(\chi)}^{b_1} w(\varsigma) \left((g(\chi)-b_1)^{\kappa+1} - (\varsigma - b_1)^{\kappa+1}\right) d\varsigma d\chi \\
&+ \frac{1}{(n-1)!} \int_{a_1}^{b_1} \left(\int_{\alpha_1}^{\beta_1} \varrho(\chi) T_{w,n}(g(\chi), \varsigma) d\chi \right) \zeta^{(n)}(\varsigma) d\varsigma,
\end{aligned} \tag{2.2}$$

Proof. We obtain the required result by putting $\chi = g(\chi)$, multiplying with $\varrho(\chi)$, integrating over $[\alpha_1, \beta_1]$, and using some transformations and then using Fubini's theorem in the extension of Montgomery identity (1.9), □

Remark 2.4. If we put $w(\varsigma) = \frac{1}{b_1 - a_1}$, $\varsigma \in [a_1, b_1]$ above identity reduces to Theorem 2 of [8].

Now we present some inequalities which can be derived from the previous identities.

Theorem 2.5. *Under the assumptions of Theorem* 2.1 *with the additional condition*

$$\sum_{i=1}^{\gamma} \varrho_i T_{w,n}(\chi_i, \varsigma) \geq 0, \qquad \forall\, \varsigma \in [a_1, b_1]. \tag{2.3}$$

Then, for every $n-$convex function $\zeta : I \to \mathbb{R}$ the following inequality holds

$$\begin{aligned}
\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i) \geq &\sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{(\kappa+1)!} \sum_{i=1}^{\gamma} \varrho_i \int_{a_1}^{\chi_i} w(\varsigma) \left((\chi_i - a_1)^{\kappa+1} - (\varsigma - a_1)^{\kappa+1}\right) d\varsigma \\
&+ \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{(\kappa+1)!} \sum_{i=1}^{\gamma} \varrho_i \int_{\chi_i}^{b_1} w(\varsigma) \left((\chi_i - b_1)^{\kappa+1} - (\varsigma - b_1)^{\kappa+1}\right) d\varsigma.
\end{aligned} \tag{2.4}$$

If the inequality in (2.3) is reversed, then (2.4) holds with the reversed sign of inequality.

Proof. By using the fact that function ζ is n-convex, so $\zeta^{(n)} \geq 0$ and (2.3) in (2.1), we can easily derive our required result. □

Remark 2.6. If reverse inequality holds in (2.3) then reverse inequality holds in (2.4).

Remark 2.7. If we put $w(\varsigma) = \frac{1}{b_1 - a_1}$, $\varsigma \in [a_1, b_1]$ above identity reduces to Theorem 3 of [8].

Now we discuss a major consequence.

Theorem 2.8. *Under the assumptions of Theorem 2.1 and let $w(\chi) \in C^n[a_1, b_1]$, $\chi = (x_1, \dots, x_m) \in [a_1, b_1]^\gamma$, $\varrho = (\varrho_1, \dots, \varrho_m) \in \mathbb{R}^\gamma$ satisfy (1.5) and (1.6) with n replaced by j where $j \in \mathbb{N}$, $2 \le j \le n$. If ζ is n-convex and $n - j$ is even, then*

$$\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i) \ge \sum_{\kappa=j-2}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{(\kappa+1)!} \sum_{i=1}^{\gamma} \varrho_i \int_{a_1}^{\chi_i} w(\varsigma) \left((\chi_i - a_1)^{\kappa+1} - (\varsigma - a_1)^{\kappa+1} \right) d\varsigma$$
$$+ \sum_{\kappa=j-2}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{(\kappa+1)!} \sum_{i=1}^{\gamma} \varrho_i \int_{\chi_i}^{b_1} w(\varsigma) \left((\chi_i - b_1)^{\kappa+1} - (\varsigma - b_1)^{\kappa+1} \right) d\varsigma. \quad (2.5)$$

Proof. Let $\varsigma \in [a_1, b_1]$ be fixed. Notice that

$$T_{w,n}(x, \varsigma) = L_{w,n}(\chi) + (\chi - \varsigma)_+^{n-1}, \quad (2.6)$$

where

$$L_{w,n}(\chi) = \int_{\chi}^{\varsigma} w(u)(u - \varsigma)^{n-1} du + (W(\chi) - 1)(\chi - \varsigma)^{n-1}$$

Using the Leibnitz theorem we have

$$L_{w,n}^{(j)}(\chi) = (n-1) \sum_{i=0}^{j-1} \binom{j-1}{i} \left[\frac{d^{j-1-i}}{d\chi^{j-1-i}} (\chi - \varsigma)^{n-2} \right] \left[\frac{d^i}{d\chi^i} \int_{b_1}^{\chi} w(u) du \right]. \quad (2.7)$$

Therefore, (2.6) and (2.7) for $\varsigma < x \le b_1$ yield

$$\frac{d^j}{d\chi^j} T_{w,n}(\chi, \varsigma) = L_{wn}^{(j)}(\chi) + (n-1)_j (\chi - \varsigma)^{n-j-1}$$
$$= (n-1) \sum_{i=0}^{j-1} \binom{j-1}{i} \left[\frac{d^{j-1-i}}{d\chi^{j-1-i}} (\chi - \varsigma)^{n-2} \right] \left[\frac{d^i}{d\chi^i} \int_{b_1}^{\chi} w(u) du \right]$$
$$+ (n-1)_j (\chi - \varsigma)^{n-j-1} \ge 0, \quad (2.8)$$

while for $a_1 \le \chi < \varsigma$ we have

$$\frac{d^j}{d\chi^j} T_{w,n}(\chi, \varsigma)$$
$$= (-1)^{n-2} (n-1) \sum_{i=0}^{j-1} \binom{j-1}{i} \left[\frac{d^{j-1-i}}{d\chi^{j-1-i}} (\varsigma - \chi)^{n-2} \right] \left[\frac{d^i}{d\chi^i} \int_{b_1}^{\chi} w(u) du \right] \ge 0. \quad (2.9)$$

From (2.6) it is clear that for $j \le n-2$, $\chi \mapsto \frac{d^j}{d\chi^j} T_{w,n}(\chi, \varsigma)$ is continuous. Hence, if $n - j$ is even and $j \le n - 2$, from (2.8) and (2.9) we can conclude that the function $\chi \mapsto T_{w,n}(\chi, \varsigma)$ is j-convex. Furthermore, the conclusion extends towards the case $j = n$, *i. e.* the mapping $\chi \mapsto T_{w,n}(\chi, \varsigma)$ is n-convex, since the mapping $x \mapsto \frac{d^{n-2}}{d\chi^{n-2}} T_n(\chi, \varsigma)$ is 2-convex.

Now, by Proposition 1.5, we see that assumption (2.3) is satisfied, so inequality (2.4) holds. Moreover, due to the assumption (1.5), $\sum_{i=1}^{\gamma} \varrho_i(\chi_i) = 0$ for every polynomial P of degree $\le j - 1$, so the first $j - 2$ terms in the inner sum in (2.4) vanish, *i. e.*, the right hand side of (2.4) under the assumptions of this theorem is equal to the right hand side of (2.5). $\square$

Remark 2.9. If we put $w(\varsigma)=\frac{1}{b_1-a_1}$, $\varsigma\in[a_1,b_1]$ above identity reduces to Theorem 4 of [8].

Corollary 2.10. *Under the assumptions of Theorem 2.5 we denote*

$$\begin{aligned} H(\chi) = & \sum_{\kappa=j-2}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{(\kappa+1)!} \int_{a_1}^{\chi_i} w(\varsigma)\left((\chi_i-a_1)^{\kappa+1}-(\varsigma-a_1)^{\kappa+1}\right) d\varsigma \\ & + \sum_{\kappa=j-2}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{(\kappa+1)!} \int_{\chi_i}^{b_1} w(\varsigma)\left((\chi_i-b_1)^{\kappa+1}-(\varsigma-b_1)^{\kappa+1}\right) d\varsigma. \end{aligned} \tag{2.10}$$

If H is j-convex on $[a_1,b_1]$ and $n-j$ is even, then

$$\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i) \geq 0.$$

Proof. Applying Proposition 1.5 we conclude that the right hand side of (2.5) is nonnegative for the j-convex function H. □

Remark 2.11. If we put $w(\varsigma)=\frac{1}{b_1-a_1}$, $\varsigma\in[a_1,b_1]$ above identity reduces to Corollary 1 of [8].

The rest of this section will present integral versions of the previous results. We will skip the details because the proofs are identical to the discrete case.

Theorem 2.12. *Under the assumptions of Theorem* 2.3 *with the additional condition*

$$\int_{\alpha_1}^{\beta_1} \varrho(\chi)\, T_{w,n}(g(\chi),\varsigma)\, d\chi \geq 0, \qquad \forall \varsigma\in[a_1,b_1].$$

Then, for every $n-$convex function $\zeta: I\to\mathbb{R}$ the following inequality holds

$$\begin{aligned} & \int_{\alpha_1}^{\beta_1} \varrho(\chi)\,\zeta(g(\chi))\, d\chi \geq \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{(\kappa+1)!} \int_{\alpha_1}^{\beta_1} \varrho(\chi) \times \\ & \int_{a_1}^{g(\chi)} w(\varsigma)\left((g(\chi)-a_1)^{\kappa+1}-(\varsigma-a_1)^{\kappa+1}\right) d\varsigma d\chi \\ & + \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{(\kappa+1)!} \int_{\alpha_1}^{\beta_1} \varrho(\chi) \times \\ & \int_{g(\chi)}^{b_1} w(\varsigma)\left((g(\chi)-b_1)^{\kappa+1}-(\varsigma-b_1)^{\kappa+1}\right) d\varsigma d\chi. \end{aligned} \tag{2.11}$$

Remark 2.13. If we put $w(\varsigma)=\frac{1}{b_1-a_1}$, $\varsigma\in[a_1,b_1]$ above identity reduces to Theorem 5 of [8].

Theorem 2.14. *Let all the assumptions from Theorem 2.3 be valid. Moreover, let $w(\chi)\in C^n[a_1,b_1]$, let $\varrho:[\alpha_1,\beta_1]\to\mathbb{R}$ and $g:[\alpha_1,\beta_1]\to[a_1,b_1]$ satisfy (1.8) with n*

replaced by j where $j \in \mathbb{N}$, $2 \le j \le n$. If ζ is n-convex and $n-j$ is even, then

$$\begin{aligned}
&\int_{\alpha_1}^{\beta_1} \varrho(\chi)\,\zeta(g(\chi))\,d\chi \ge \sum_{\kappa=j-2}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{(\kappa+1)!} \int_{\alpha_1}^{\beta_1} \varrho(\chi) \times \\
&\int_{a_1}^{g(\chi)} w(\varsigma)\left((g(\chi)-a_1)^{\kappa+1} - (\varsigma - a_1)^{\kappa+1}\right) d\varsigma d\chi \\
&+ \sum_{\kappa=j-2}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{(\kappa+1)!} \int_{\alpha_1}^{\beta_1} \varrho(\chi) \times \\
&\int_{g(\chi)}^{b_1} w(\varsigma)\left((g(\chi)-b_1)^{\kappa+1} - (\varsigma - b_1)^{\kappa+1}\right) d\varsigma d\chi. \qquad (2.12)
\end{aligned}$$

Remark 2.15. If we put $w(\varsigma) = \frac{1}{b_1-a_1}$, $\varsigma \in [a_1, b_1]$ above identity reduces to Theorem 6 of [8].

Corollary 2.16. *Let n, ϱ, ζ, j and g be as in Theorem 2.14 and let H be given by (2.10). If $n-j$ is even and H is j-convex, then*

$$\int_{\alpha_1}^{\beta_1} \varrho(\chi)\zeta(g(\chi))\,d\chi \ge 0.$$

2.1. Inequalities related to n-convex functions at a point

Throughout this section, we will discuss related results obtained in [17] for the class of n-convex functions at a point.

Definition 2.17. Let $n \in \mathbb{N}$, c_1 a point in the interior of I and I be an interval in $\mathbb{R}$. If there exists a constant K such that

$$F_1(\chi) = \zeta(\chi) - \frac{K}{(n-1)!}\chi^{n-1} \qquad (2.13)$$

where the function $\zeta : I \to \mathbb{R}$ is said to be n-convex at point c_1 and $(n-1)$-concave on $I \cap (-\infty, c_1]$ and $(n-1)$-convex on $I \cap [c_1, \infty)$. If the function $-\zeta$ is n-convex at point $c_1]$ then ζ is called n-concave at point c_1. For more details, we refer the readers to see [2, 17].

In [17], authors discussed sufficient conditions on two linear functionals $\Lambda : C([a_1, c_1]) \to \mathbb{R}$ and $\Xi : C([c_1, b_1]) \to \mathbb{R}$ so that the inequality $\Lambda(\zeta) \le \Xi(\zeta)$ holds for every function ζ that is n-convex at c_1.

This section will provide inequalities of this type for specific linear functionals that connect to the inequalities derived in the preceding section. Let e_i denote the monomials $e_i(\chi) = \chi^i$, $i \in \mathbb{N}_0$. More specifically, let $T_{w,n}^{[a_1,c_1]}$ and $T_{w,n}^{[c_1,b_1]}$ represent the same as (1.10) on these intervals, *i. e.*,

$$T_{w,n}^{[a_1,c_1]}(\chi,\varsigma)=\begin{cases}\displaystyle\int_{\chi}^{\varsigma} w(u)(u-\varsigma)^{n-1}du+W(\chi)(\chi-\varsigma)^{n-1}, \\ \qquad a_1\le\varsigma\le\chi \\ \displaystyle\int_{\chi}^{\varsigma} w(u)(u-\varsigma)^{n-1}du+\left(W(\chi)-1\right)(\chi-\varsigma)^{n-1}, \\ \qquad \chi<\varsigma\le c_1,\end{cases} \tag{2.14}$$

$$T_{w,n}^{[c_1,b_1]}(\chi,\varsigma)=\begin{cases}\displaystyle\int_{\chi}^{\varsigma} w(u)(u-\varsigma)^{n-1}du+W(\chi)(\chi-\varsigma)^{n-1}, \\ \qquad c_1\le\varsigma\le\chi \\ \displaystyle\int_{\chi}^{\varsigma} w(u)(u-\varsigma)^{n-1}du+\left(W(\chi)-1\right)(\chi-\varsigma)^{n-1} \\ \qquad \chi<\varsigma\le b_1,\end{cases} \tag{2.15}$$

Let $\chi\in[a_1,c_1]^{\gamma}$, $\varrho\in\mathbb{R}^{\gamma}$, $\mathbf{y}\in[c_1,b_1]^{\ell}$ and $\mathbf{q}\in\mathbb{R}^{\ell}$ and denote

$$\begin{aligned}\Lambda(\zeta)=&\sum_{i=1}^{\gamma}\varrho_i\zeta\left(\chi_i\right)\\ &-\sum_{\kappa=0}^{n-2}\frac{\zeta^{\kappa+1}(a_1)}{(\kappa+1)!}\sum_{i=1}^{\gamma}\varrho_i\int_{a_1}^{\chi_i} w(\varsigma)\left((\chi_i-a_1)^{\kappa+1}-(\varsigma-a_1)^{\kappa+1}\right)d\varsigma\\ &-\sum_{\kappa=0}^{n-2}\frac{\zeta^{\kappa+1}(c_1)}{(\kappa+1)!}\sum_{i=1}^{\gamma}\varrho_i\int_{\chi_i}^{c_1} w(\varsigma)\left((\chi_i-c_1)^{\kappa+1}-(\varsigma-c_1)^{\kappa+1}\right)d\varsigma,\end{aligned} \tag{2.16}$$

$$\begin{aligned}\Xi(\zeta)=&\sum_{i=1}^{\ell}q_i\zeta\left(y_i\right)\\ &-\sum_{\kappa=0}^{n-2}\frac{\zeta^{\kappa+1}(c_1)}{(\kappa+1)!}\sum_{i=1}^{\ell}q_i\int_{c_1}^{y_i} w(\varsigma)\left((y_i-c_1)^{\kappa+1}-(\varsigma-c_1)^{\kappa+1}\right)d\varsigma\\ &-\sum_{\kappa=0}^{n-2}\frac{\zeta^{\kappa+1}(b_1)}{(\kappa+1)!}\sum_{i=1}^{\ell}q_i\int_{y_i}^{b_1} w(\varsigma)\left((y_i-b_1)^{\kappa+1}-(\varsigma-b_1)^{\kappa+1}\right)d\varsigma.\end{aligned} \tag{2.17}$$

Identity (2.1) applied to the intervals $[a_1,c_1]$ and $[c_1,b_1]$ and by using the functionals Λ and Ξ can be written as

$$\Lambda(\zeta)=\frac{1}{(n-1)!}\int_{a_1}^{c_1}\left(\sum_{i=1}^{\gamma}\varrho_i T_{w,n}^{[a_1,c_1]}\left(\chi_i,\varsigma\right)\right)\zeta^{(n)}\left(\varsigma\right)\,d\varsigma, \tag{2.18}$$

$$\Xi(\zeta)=\frac{1}{(n-1)!}\int_{c_1}^{b}\left(\sum_{i=1}^{\ell}q_i T_{w,n}^{[c_1,b_1]}\left(y_i,\varsigma\right)\right)\zeta^{(n)}\left(\varsigma\right)\,d\varsigma. \tag{2.19}$$

Theorem 2.18. *Let $\chi \in [a_1, c_1]^\gamma$, $\varrho \in \mathbb{R}^\gamma$, $\mathbf{y} \in [c_1, b_1]^\ell$ and $\mathbf{q} \in \mathbb{R}^\ell$ be such that*

$$\sum_{i=1}^{\gamma} \varrho_i T_{w,n}^{[a_1,c_1]}(\chi_i, \varsigma) \geq 0, \quad \textit{for every } \varsigma \in [a_1, c_1], \tag{2.20}$$

$$\sum_{i=1}^{\ell} q_i T_{w,n}^{[c_1,b_1]}(y_i, \varsigma) \geq 0, \quad \textit{for every } \varsigma \in [c_1, b_1], \tag{2.21}$$

$$\int_{a_1}^{c_1} \left(\sum_{i=1}^{\gamma} \varrho_i T_{w,n}^{[a_1,c_1]}(\chi_i, \varsigma) \right) d\varsigma = \int_{c_1}^{b_1} \left(\sum_{i=1}^{\ell} q_i T_{w,n}^{[c_1,b_1]}(y_i, \varsigma) \right) d\varsigma, \tag{2.22}$$

where $T_{w,n}^{[a_1,c_1]}$, $T_{w,n}^{[c_1,b_1]}$, Λ and Ξ are given by (2.14), (2.15), (2.16) and (2.17) respectively. If $\zeta : [a_1, b_1] \to \mathbb{R}$ is $(n+1)$-convex at point c_1, then

$$\Lambda(\zeta) \leq \Xi(\zeta). \tag{2.23}$$

If the inequalities in (2.20) *and* (2.21) *are reversed, then* (2.23) *holds with the reversed sign of inequality.*

Proof. Let the function $F_1 = \zeta - \frac{K}{n!} e_n$ is n-concave on $[a_1, c_1]$ and n-convex on $[c_1, b_1]$(see Definition 2.17). Applying Theorem 2.5 to F_1 on the intervals $[a_1, c_1]$ and $[c_1, b_1]$ respectively we have

$$0 \geq \Lambda(F_1) = \Lambda(\zeta) - \frac{K}{n!} \Lambda(e_n) \tag{2.24}$$

$$0 \leq \Xi(F_1) = \Xi(\zeta) - \frac{K}{n!} \Xi(e_n). \tag{2.25}$$

Identities (2.18) and (2.19) applied to the function e_n yield

$$\Lambda(e_n) = n \int_{a_1}^{c_1} \left(\sum_{i=1}^{\gamma} \varrho_i T_{w,n}^{[a_1,c_1]}(\chi_i, \varsigma) \right) d\varsigma,$$

$$\Xi(e_n) = n \int_{c_1}^{b} \left(\sum_{i=1}^{\ell} q_i T_{w,n}^{[c_1,b_1]}(y_i, \varsigma) \right) d\varsigma.$$

Therefore, assumption (2.22) is equivalent to $\Lambda(e_n) = \Xi(e_n)$. Now, from (2.24) and (2.25) we obtain the stated inequality. □

Remark 2.19. If we put $w(u) = \frac{1}{b_1 - a_1}$, $u \in [a_1, b_1]$ above identity reduces to Theorem 7 of [8].

Remark 2.20. In the Theorem 2.18 we have proved that

$$\Lambda(\zeta) \leq \frac{K}{n!} \Lambda(e_n) = \frac{K}{n!} \Xi(e_n) \leq \Xi(\zeta).$$

Inequality (2.23) still holds if we substitute assumption (2.22) with the weaker assumption that $K\left(\Xi(e_n) - \Lambda(e_n)\right) \geq 0$.

Corollary 2.21. *Let $n, j_1, j_2 \in \mathbb{N}$, $\leq j_1, j_2 \leq n$, let $\zeta : [a_1, b_1] \to \mathbb{R}$ be $(n+1)$-convex at point c_1, let $\varrho \in \mathbb{R}^\gamma$ and γ-tuples $\chi \in [a_1, c_1]^\gamma$ satisfy (1.5) and (1.6) with n replaced by j_1, let $\mathbf{q} \in \mathbb{R}^\ell$ and ℓ-tuples $\mathbf{y} \in [c_1, b_1]^\ell$ satisfy*

$$\sum_{i=1}^{\ell} q_i y_i^\kappa = 0, \qquad \forall\ \kappa = 0, 1, \ldots, j_2 - 1$$

$$\sum_{i=1}^{\ell} q_i (y_i - \tau)_+^{j_2-1} \geq 0, \qquad \forall\ \tau \in [y_{(1)}, y_{(\ell-n+1)}]$$

and let (2.22) holds. If $n - j_1$ and $n - j_2$ are even, then

$$\Lambda(\zeta) \leq \Xi(\zeta).$$

Proof. Same as the proof of Theorem 2.8. □

Remark 2.22. Similar results can also be stated for integral versions as well by defining new functionals using identity (2.2).

2.2. Bounds for identities related to the Popoviciu-type inequalities

Let $\zeta_1, \zeta_2 : [a_1, b_1] \to \mathbb{R}$ be two Lebesgue integrable functions. We consider the Čebyšev functional

$$\begin{aligned} T(\zeta_1, \zeta_2) = {} & \frac{1}{b_1 - a_1} \int_{a_1}^{b_1} \zeta_1(\chi)\zeta_2(\chi) d\chi \\ & - \left(\frac{1}{b_1 - a_1} \int_{a_1}^{b_1} \zeta_1(\chi) d\chi \right) \left(\frac{1}{b_1 - a_1} \int_{a_1}^{b_1} \zeta_2(\chi) d\chi \right). \end{aligned} \tag{2.26}$$

The symbol $L_p\,[a_1, b_1]$ $(1 \leq p < \infty)$ denotes the space of p-power integrable functions on the interval $[a_1, b_1]$ equipped with the norm

$$\|\zeta_1\|_p = \left(\int_{a_1}^{b_1} |\zeta_1(\tau)|^p \, d\tau \right)^{\frac{1}{p}}$$

and $L_\infty\,[a_1, b_1]$ denotes the space of essentially bounded functions on $[a_1, b_1]$ with the norm

$$\|\zeta_1\|_\infty = \operatorname{ess} \sup_{\tau \in [a_1, b_1]} |\zeta_1(\tau)|.$$

The following results can be found in [5].

Proposition 2.23. *Let $\zeta 1 : [a_1, b_1] \to \mathbb{R}$ be a Lebesgue integrable function and $\zeta_2 : [a_1, b_1] \to \mathbb{R}$ be an absolutely continuous function with $(\cdot - a_1)(b_1 - \cdot)[\zeta_2']^2 \in L[a_1, b_1]$. Then we have the inequality*

$$|T(\zeta_1, \zeta_2)| \leq \frac{1}{\sqrt{2}} \left(\frac{1}{b_1 - a_1} |T(\zeta_1, \zeta_1)| \int_{a_1}^{b_1} (\chi - a_1)(b_1 - \chi)[\zeta_2'(\chi)]^2 d\chi \right)^{1/2}. \tag{2.27}$$

The constant $\frac{1}{\sqrt{2}}$ in (2.27) is the best possible.

Proposition 2.24. *Let $\zeta_2 : [a_1, b_1] \to \mathbb{R}$ be a monotonic nondecreasing function and let $\zeta_1 : [a_1, b_1] \to \mathbb{R}$ be an absolutely continuous function such that $\zeta_1' \in L_\infty[a_1, b_1]$. Then we have the inequality*

$$|T(\zeta_1, \zeta_2)| \leq \frac{1}{2(b_1 - a_1)} \|\zeta_1'\|_\infty \int_{a_1}^{b_1} (\chi - a_1)(b_1 - \chi) d\zeta_2(\chi). \tag{2.28}$$

The constant $\frac{1}{2}$ in (2.28) is the best possible.

Under the assumptions of Theorems 2.1 and 2.3 we denote the following functions. For $\gamma-$tuples $\varrho = (\varrho_1, \ldots, \varrho_\gamma)$, $\chi = (\chi_1, \ldots, \chi_\gamma)$ with $\chi_i \in [a_1, b_1]$, $\varrho_i \in \mathbb{R}$ $(i = 1, \ldots, \gamma)$ such that $\sum_{i=0}^{\gamma} \varrho_i = 0$ and the function $T_{w,n}$ defined as in (1.10), denote

$$\Psi_1(\varsigma) = \sum_{i=1}^{\gamma} \varrho_i T_{w,n}(\chi_i, \varsigma), \quad \text{ for } \varsigma \in [a_1, b_1]. \tag{2.29}$$

Similarly for functions $g : [\alpha_1, \beta_1] \to [a_1, b_1]$ and $\varrho : [\alpha_1, \beta_1] \to \mathbb{R}$ such that $\int_{\alpha_1}^{\beta_1} \varrho(\chi) d\chi = 0$, denote

$$\Psi_2(\varsigma) = \int_{\alpha_1}^{\beta_1} \varrho\,(\chi)\, T_{w,n}\,(g(\chi), \varsigma)\; d\chi, \quad \text{ for } \varsigma \in [a_1, b_1]. \tag{2.30}$$

Now, we are ready to state bounds for the integral remainders of identities obtained in Section 2.

Theorem 2.25. *Let $n \in \mathbb{N}$, $\zeta : [a_1, b_1] \to \mathbb{R}$ be such that $\zeta^{(n)}$ is an absolutely continuous function with $(\cdot - a_1)(b_1 - \cdot)[\zeta^{(n+1)}]^2 \in L[a_1, b_1]$, $\chi_i \in [a_1, b_1]$ and $\varrho_i \in \mathbb{R}$ $(i \in \{1, \ldots, \gamma\})$ such that $\sum_{i=0}^{\gamma} \varrho_i = 0$ and let the functions $T_{w,n}$, T and Ψ_1 be defined in* (1.10), (2.26) *and* (2.29) *respectively. Then*

$$\begin{aligned}
\sum_{i=1}^{\gamma} \varrho_i \zeta\,(\chi_i) &= \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{(\kappa+1)!} \sum_{i=1}^{\gamma} \varrho_i \int_{a_1}^{\chi_i} w(\varsigma) \left((\chi_i - a_1)^{\kappa+1} - (\varsigma - a_1)^{\kappa+1}\right) d\varsigma \\
&+ \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{(\kappa+1)!} \sum_{i=1}^{\gamma} \varrho_i \int_{\chi_i}^{b_1} w(\varsigma) \left((\chi_i - b_1)^{\kappa+1} - (\varsigma - b_1)^{\kappa+1}\right) d\varsigma \\
&+ \frac{\left[\zeta^{(n-1)}(b_1) - \zeta^{(n-1)}(a_1)\right]}{(n-1)!(b_1 - a_1)} \int_{a_1}^{b_1} \Psi_1(\varsigma) d\varsigma + R_n^1(\zeta; a_1, b_1),
\end{aligned} \tag{2.31}$$

where the remainder $R_n^1(\zeta; a_1, b_1)$ satisfies the estimation

$$\begin{aligned}
&|R_n^1(\zeta; a_1, b_1)| \\
&\leq \frac{1}{(n-1)!} \left(\frac{b_1 - a_1}{2} \left| T(\Psi_1, \Psi_1) \int_{a_1}^{b_1} (\varsigma - a_1)(b_1 - \varsigma)[\zeta^{(n+1)}(\varsigma)]^2 d\varsigma \right| \right)^{1/2}
\end{aligned} \tag{2.32}$$

Proof. If we apply Proposition 2.23 for $\zeta_1 \to \Psi_1$ and $\zeta_2 \to \zeta^{(n)}$, then we obtain

$$\begin{aligned}
&\Bigg| \frac{1}{b_1-a_1}\int_{a_1}^{b_1} \Psi_1(\varsigma)\zeta^{(n)}(\varsigma)d\varsigma \\
&\qquad - \left(\frac{1}{b_1-a_1}\int_{a_1}^{b_1} \Psi_1(\varsigma)d\varsigma\right)\left(\frac{1}{b_1-a_1}\int_{a_1}^{b_1} \zeta^{(n)}(\varsigma)d\varsigma\right)\Bigg| \\
&\qquad \le \frac{1}{\sqrt{2}}\left(\frac{1}{b_1-a_1}|T(\Psi_1,\Psi_1)|\int_{a_1}^{b_1}(\varsigma-a_1)(b_1-\varsigma)[\zeta^{(n+1)}(\varsigma)]^2 d\varsigma\right)^{1/2}.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\frac{1}{(n-1)!}\int_{a_1}^{b_1} \Psi_1(\varsigma)\zeta^{(n)}(\varsigma)d\varsigma &= \frac{\left[\zeta^{(n-1)}(b_1)-\zeta^{(n-1)}(a_1)\right]}{(n-1)!(b_1-a_1)}\int_{a_1}^{b_1}\Psi_1(\varsigma)d\varsigma \\
&+ R_n^1(\zeta;a_1,b_1).
\end{aligned}$$

where $R_n^1(\zeta;a_1,b_1)$ satisfies inequality (2.32). Now from identity (2.1) we obtain (2.31). □

Remark 2.26. If we put $w(u)=\frac{1}{b_1-a_1}$, $u\in[a_1,b_1]$ above identity reduces to Theorem 8 of [8].

Here we state the integral version of the previous theorem.

Theorem 2.27. *Let $n\in\mathbb{N}$, $\zeta:[a_1,b_1]\to\mathbb{R}$ be such that $\zeta^{(n)}$ is an absolutely continuous function with $(\cdot-a_1)(b_1-\cdot)[\zeta^{(n+1)}]^2\in L[a_1,b_1]$, let $g:[\alpha_1,\beta_1]\to[a_1,b_1]$ and $\varrho:[\alpha_1,\beta_1]\to\mathbb{R}$ be functions such that $\int_{\alpha_1}^{\beta_1}\varrho(\chi)d\chi=0$ and let the functions $T_{w,n}$, T and Ψ_2 be defined in* (1.10), (2.26) *and* (2.30) *respectively. Then*

$$\begin{aligned}
&\int_{\alpha_1}^{\beta_1}\varrho(\chi)\,\zeta(g(\chi))\,d\chi \\
&=\sum_{\kappa=0}^{n-2}\frac{\zeta^{(\kappa+1)}(a_1)}{(\kappa+1)!}\int_{\alpha_1}^{\beta_1}\varrho(\chi)\int_{a_1}^{g(\chi)} w(\varsigma)\left((g(\chi)-a_1)^{\kappa+1}-(\varsigma-a_1)^{\kappa+1}\right)d\varsigma d\chi \\
&+\sum_{\kappa=0}^{n-2}\frac{\zeta^{(\kappa+1)}(b_1)}{(\kappa+1)!}\int_{\alpha_1}^{\beta_1}\varrho(\chi)\int_{g(\chi)}^{b_1} w(\varsigma)\left((g(\chi)-b_1)^{\kappa+1}-(\varsigma-b_1)^{\kappa+1}\right)d\varsigma d\chi \\
&+\frac{\left[\zeta^{(n-1)}(b_1)-\zeta^{(n-1)}(a_1)\right]}{(n-1)!(b_1-a_1)}\int_{a_1}^{b_1}\Psi_2(\varsigma)d\varsigma+R_n^2(\zeta;a_1,b_1), \qquad (2.33)
\end{aligned}$$

where the remainder $R_n^2(\zeta;a_1,b_2)$ satisfies the estimation

$$\begin{aligned}
&\left|R_n^2(\zeta;a_1,b_1)\right| \\
&\le \frac{1}{(n-1)!}\left(\frac{b_1-a_1}{2}\,|T(\Psi_2,\Psi_2)|\int_{a_1}^{b_2}(\varsigma-a_1)(b_1-\varsigma)[\zeta^{(n+1)}(\varsigma)]^2 d\varsigma\right)^{1/2}. \qquad (2.34)
\end{aligned}$$

Proof. This result easily follows by proceeding as in the proof of the previous theorem and replacing (2.1) with (2.2). □

Remark 2.28. If we put $w(u) = \frac{1}{b_1 - a_1}$, $u \in [a_1, b_1]$ above identity reduces to Theorem 9 of [8].

By using Proposition 2.24, we obtain the following Grüss type inequality.

Theorem 2.29. *Let $n \in \mathbb{N}$, $\zeta : [a_1, b_1] \to \mathbb{R}$ be such that $\zeta^{(n)}$ is an absolutely continuous function with $\zeta^{(n+1)} \geq 0$ on $[a_1, b_1]$, $\chi_i \in [a_1, b_1]$ and $\varrho_i \in \mathbb{R}$ $(i \in \{1, \ldots, \gamma\})$ such that*

$$\sum_{i=0}^{\gamma} \varrho_i = 0.$$

Also, let the functions T and Ψ_1 be defined in (2.26) *and* (2.29) *respectively. Then we have representation* (2.31) *and the remainder $R_n^1(\zeta; a_1, b_1)$ satisfies the following estimation*

$$|R_n^1(\zeta; a_1, b_1)| \leq \frac{1}{(n-1)!} \|\Psi_1'\|_\infty \left[\frac{b_1 - a_1}{2} \left[\zeta^{(n-1)}(b_1) + \zeta^{(n-1)}(a_1) \right] \right.$$
$$\left. - \left[\zeta^{(n-2)}(b_1) - \zeta^{(n-2)}(a_1) \right] \right]. \tag{2.35}$$

Proof. If we apply Proposition 2.24 for $\zeta_1 \to \Psi_1$ and $\zeta_2 \to \zeta^{(n)}$, then we obtain

$$\left| \frac{1}{b_1 - a_1} \int_{a_1}^{b_1} \Psi_1(\varsigma) \zeta^{(n)}(\varsigma) d\varsigma \right.$$
$$\left. - \left(\frac{1}{b_1 - a_1} \int_{a_1}^{b_1} \Psi_1(\varsigma) d\varsigma \right) \left(\frac{1}{b_1 - a_1} \int_{a_1}^{b_1} \zeta^{(n)}(\varsigma) d\varsigma \right) \right|$$
$$\leq \frac{1}{2(b_1 - a_1)} \|\Psi_1'\|_\infty \int_{a_1}^{b_1} (\varsigma - a_1)(b_1 - \varsigma) \zeta^{(n+1)}(\varsigma) d\varsigma.$$

Since

$$\int_{a_1}^{b_1} (\varsigma - a_1)(b_1 - \varsigma) \zeta^{(n+1)}(\varsigma) d\varsigma$$
$$= \int_{a_1}^{b_1} (2\varsigma - a_1 - b_1) \zeta^{(n)}(\varsigma) d\varsigma$$
$$= (b_1 - a_1) \left[\zeta^{(n-1)}(b_1) + \zeta^{(n-1)}(a_1) \right] - 2 \left[\zeta^{(n-2)}(b_1) - \zeta^{(n-2)}(a_1) \right], \tag{2.36}$$

by using the identities (2.1) and (2.36) we deduce (2.35). □

Remark 2.30. If we put $w(u) = \frac{1}{b_1 - a_1}$, $u \in [a_1, b_1]$ above identity reduces to Theorem 10 of [8].

Here we give the integral version of the above theorem.

Theorem 2.31. *Let $n \in \mathbb{N}$, $\zeta : [a_1, b_1] \to \mathbb{R}$ be such that $\zeta^{(n)}$ is an absolutely continuous function with $\zeta^{(n+1)} \geq 0$ on $[a_1, b_1]$, let $g : [\alpha_1, \beta_1] \to [a_1, b_1]$ and $\varrho : [\alpha_1, \beta_1] \to \mathbb{R}$ be functions such that $\int_{\alpha_1}^{\beta_1} \varrho(\chi) d\chi = 0$. Also, let the functions T and Ψ_2 be defined in*

(2.26) and (2.30) respectively. Then we have representation (2.33) and the remainder $R_n^2(\zeta; a_1, b_1)$ satisfies the following estimation

$$|R_n^2(\zeta; a_1, b_1)| \leq \frac{1}{(n-1)!} \|\Psi_2'\|_\infty \left[\frac{b_1 - a_1}{2} \left[\zeta^{(n-1)}(b_1) + \zeta^{(n-1)}(a_1) \right] - \left[\zeta^{(n-2)}(b_1) - \zeta^{(n-2)}(a_1) \right] \right]. \tag{2.37}$$

Remark 2.32. If we put $w(u) = \frac{1}{b_1 - a_1}$, $u \in [a_1, b_1]$ above identity reduces to Theorem 11 of [8].

2.3. Ostrowski type inequalities via extension of Montgomery identity

Here we present some Ostrowski-type inequalities related to the generalized linear inequalities. Throughout the section, we use the following functions Ψ_1 and Ψ_2 defined as in (2.29) and (2.30).

Theorem 2.33. *Let all the assumptions of Theorem 2.1 hold. Additionally, let $\zeta^{(n)} \in L_q[a_1, b_1]$, $1 \leq q, r \leq \infty$, $\frac{1}{q} + \frac{1}{r} = 1$, $n \geq 2$, $n \in \mathbb{N}$ and let $\chi \in [a_1, b_1]^\gamma$ and $\varrho \in \mathbb{R}^\gamma$ satisfy*

$$\sum_{i=1}^{\gamma} \varrho_i = 0 \text{ and } \sum_{i=1}^{\gamma} \varrho_i \chi_i = 0.$$

Then

$$\begin{aligned} \Bigg| \sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i) & - \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{(\kappa+1)!} \sum_{i=1}^{\gamma} \varrho_i \int_{a_1}^{\chi_i} w(\varsigma) \left((\chi_i - a_1)^{\kappa+1} - (\varsigma - a_1)^{\kappa+1} \right) d\varsigma \\ & - \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{(\kappa+1)!} \sum_{i=1}^{\gamma} \varrho_i \int_{\chi_i}^{b_1} w(\varsigma) \left((\chi_i - b_1)^{\kappa+1} - (\varsigma - b_1)^{\kappa+1} \right) d\varsigma \Bigg| \\ & \leq \frac{1}{(n-1)!} \|\zeta^{(n)}\|_q \left\|\Psi_1\right\|_r. \end{aligned} \tag{2.38}$$

The constant on the right hand sides of (2.38) is the best possible for $q = 1$ and sharp for $1 < q \leq \infty$.

Proof. Let us denote

$$\mu(\varsigma) = \frac{1}{(n-1)!} \Psi_1(\varsigma).$$

By using Hölder's inequality on identity (2.1) we obtain inequality (2.38), i. e.

$$\text{L.H.S.} \leq \|\zeta^{(n)}\|_q \left\|\mu\right\|_r. \tag{2.39}$$

Let us find a function ζ for the proof of the sharpness of the constant

$$\left(\int_{a_1}^{b_1} |\mu(\varsigma)|^r \, dt \right)^{1/r},$$

for which the equality in (2.39) is obtained.

For $1 < q < \infty$ take ζ to be s.t.

$$\zeta^{(n)}(\varsigma) = sgn\mu(\varsigma) \cdot |\mu(\varsigma)|^{1/(q-1)}.$$

For $q = \infty$, take ζ s.t.

$$\zeta^{(n)}(\varsigma) = sgn\mu(\varsigma).$$

Finally, for $q = 1$, we prove that

$$\left| \int_{a_1}^{b_1} \mu(\varsigma)\zeta^{(n)}(\varsigma)d\varsigma \right| \leq \max_{\varsigma \in [a_1, b_1]} |\mu(\varsigma)| \int_{a_1}^{b_1} \zeta^{(n)}(\varsigma)d\varsigma \tag{2.40}$$

is the best possible inequality.

Suppose that $|\mu(\varsigma)|$ attains its maximum at $\varsigma_0 \in [a_1, b_1]$. First we consider the case $\mu(\varsigma_0) > 0$. For δ small enough we define $\zeta_{1\delta}(\varsigma)$ by

$$\zeta_{1\delta}(\varsigma) = \begin{cases} 0 \quad , & a_1 \leq \varsigma \leq \varsigma_0, \\ \dfrac{1}{\delta n!}(\varsigma - \varsigma_0)^n \quad , & \varsigma_0 \leq \varsigma \leq \varsigma_0 + \delta, \\ \dfrac{1}{(n-1)!}(\varsigma - \varsigma_0)^{n-1} \quad , & \varsigma_0 + \delta_1 \leq \varsigma \leq b_1. \end{cases}$$

So, we have

$$\left| \int_{a_1}^{b_1} \mu(\varsigma)\zeta_{1\delta}^{(n)}(\varsigma)d\varsigma \right| = \left| \int_{\varsigma_0}^{\varsigma_0+\delta} \mu(\varsigma)\frac{1}{\delta}d\varsigma \right| = \frac{1}{\delta}\int_{\varsigma_0}^{\varsigma_0+\delta} \mu(\varsigma)d\varsigma$$

Now from inequality (2.40) we have

$$\frac{1}{\delta}\int_{\varsigma_0}^{\varsigma_0+\delta} \mu(\varsigma)d\varsigma \leq \mu(\varsigma_0)\frac{1}{\delta}\int_{\varsigma_0}^{\varsigma_0+\delta} d\varsigma = \mu(\varsigma_0)$$

Since

$$\lim_{\delta \to 0} \frac{1}{\delta}\int_{\varsigma_0}^{\varsigma_0+\delta} \mu(\varsigma)d\varsigma = \mu(\varsigma_0)$$

the statement follows.

In the case $\mu(\varsigma_0) < 0$, we define $\zeta_{1\delta}(\varsigma)$ by

$$\zeta_{1\delta}(\varsigma) = \begin{cases} \dfrac{1}{(n-1)!}(\varsigma - \varsigma_0 - \delta)^{n-1} \quad , & a \leq \varsigma \leq \varsigma_0, \\ -\dfrac{1}{\delta n!}(\varsigma - \varsigma_0 - \delta)^n \quad , & \varsigma_0 \leq \varsigma \leq \varsigma_0 + \delta, \\ 0 \quad , & \varsigma_0 + \delta \leq \varsigma \leq b_1. \end{cases}$$

and the rest of the proof is the same as above. □

Remark 2.34. If we put $w(\varsigma) = \frac{1}{b_1 - a_1}$ in Theorem 2.33, we capture Theorem 12 of [8].

At the end of this section, we will present the integral version of the above Theorem. We will skip the details because the proof is identical.

Theorem 2.35. *Let all the assumptions of Theorem* 2.3 *hold. Additionally,* $\zeta^{(n)} \in L_q[a_1, b_1]$, $1 \leq q, r \leq \infty$, $\frac{1}{q} + \frac{1}{r} = 1$, $n \geq 2$, $n \in \mathbb{N}$ *and let* $g : [\alpha_1, \beta_1] \to [a_1, b_1]$ *and* $\varrho : [\alpha_1, \beta_1] \to \mathbb{R}$ *satisfy*

$$\int_{\alpha_1}^{\beta_1} \varrho(\chi) d\chi = 0 \text{ and } \int_{\alpha_1}^{\beta_1} \varrho(\chi) g(\chi) d\chi = 0.$$

Then

$$\Bigg| \int_{\alpha_1}^{\beta_1} \varrho(\chi) \zeta\left(g(\chi)\right) - \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{(\kappa+1)!} \int_{\alpha_1}^{\beta_1} \varrho(\chi) \int_{a_1}^{g(\chi)} w(\varsigma) \left((g(\chi) - a_1)^{\kappa+1} - (\varsigma - a_1)^{\kappa+1}\right) d\varsigma d\chi - \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{(\kappa+1)!} \int_{\alpha_1}^{\beta_1} \varrho(\chi) \int_{g(\chi)}^{b_1} w(\varsigma) \left((g(\chi) - b_1)^{\kappa+1} - (\varsigma - b_1)^{\kappa+1}\right) d\varsigma d\chi \Bigg| \leq \frac{1}{(n-1)!} \|\zeta^{(n)}\|_q \left\|\Psi_2\right\|_r. \quad (2.41)$$

The constant on the right hand side of (2.41) *is the best possible for* $q = 1$ *and sharp for* $1 < q \leq \infty$.

Remark 2.36. If we put $w(\varsigma) = \frac{1}{b_1 - a_1}$ in Theorem 2.35, we capture Theorem 13 of [8].

3. Popoviciu type identities and inequalities via extension of weighted Montgomery identity using Green Functions

In the present section, we obtain some discrete and integral identities and corresponding linear inequalities using Green functions and apply the extension of weighted Montgomery identity. We'll start by proving a few identities that will play a crucial role in the rest of the article.

Theorem 3.1. *Let* $\zeta : I \to \mathbb{R}$ *be such that* $\zeta^{(n-1)}$ *is absolutely continuous,* $n \geq 3$, $n \in \mathbb{N}$, $a_1 < b_1$, $a_1, b_1 \in I$, $I \subset \mathbb{R}$ *an open interval,* $w : [a_1, b_1] \to [0, \infty)$ *is some probability density function. Let* $\varrho \in \mathbb{R}^\gamma$ *satisfy*

$$\sum_{i=1}^{\gamma} \varrho_i = 0 \text{ and } \sum_{i=1}^{\gamma} \varrho_i \chi_i = 0$$

and $\chi \in [a_1, b_1]^\gamma$, G_l *are as given by* (1.13), (1.16) – (1.19). *Then*

$$\begin{aligned}\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i) &= \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{\kappa!} \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma) \\ &\times \left[w(\varsigma)(\varsigma - a_1)^\kappa + \kappa \int_{a_1}^{\varsigma} w(u)(\varsigma - a_1)^{\kappa-1} du \right] d\varsigma \\ &+ \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{\kappa!} \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma) \\ &\times \left[-w(\varsigma)(\varsigma - b_1)^\kappa + \kappa \int_{\varsigma}^{b_1} w(u)(\varsigma - b_1)^{\kappa-1} du \right] d\varsigma \\ &+ \frac{1}{(n-3)!} \int_{a_1}^{b_1} \left(\int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma) \tilde{T}_{w,n-2}(\varsigma, u) d\varsigma \right) \zeta^{(n)}(u) du, \end{aligned} \tag{3.1}$$

where

$$\tilde{T}_{w,n-2}(\varsigma, u) = \begin{cases} \dfrac{w(\varsigma)(\varsigma - u)^{n-2}}{(n-2)} + W(\varsigma)(\varsigma - u)^{n-3}, & a_1 \le u \le \varsigma \\ \dfrac{w(\varsigma)(\varsigma - u)^{n-2}}{(n-2)} + (W(\varsigma) - 1)(\varsigma - u)^{n-3}, & \varsigma < u \le b_1. \end{cases} \tag{3.2}$$

Moreover, the following identity holds

$$\begin{aligned}\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i) &= \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma) \left(\int_{a_1}^{b_1} w(\tau) \zeta''(\tau) d\tau \right) d\varsigma \\ &+ \sum_{\kappa=3}^{n-1} \frac{1}{(\kappa-2)!} \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma) \\ &\times \left[\zeta^{(\kappa)}(a_1) \int_{a_1}^{\varsigma} w(u) \left((\varsigma - a_1)^{\kappa-2} - (u - a_1)^{\kappa-2} \right) du \right. \\ &\left. + \zeta^{(\kappa)}(b_1) \int_{\varsigma}^{b_1} w(u) \left((\varsigma - b_1)^{\kappa-2} - (u - b_1)^{\kappa-2} \right) du \right] d\varsigma \\ &+ \frac{1}{(n-3)!} \int_{a_1}^{b_1} \zeta^{(n)}(u) \left(\int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma) T_{w,n-2}(\varsigma, u) d\varsigma \right) du, \end{aligned} \tag{3.3}$$

where $T_{w,n}$ *is as defined in* (1.10).

Proof. Using (1.14) in $\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i)$ and the fact that $\sum_{i=1}^{\gamma} \varrho_i = 0$ and $\sum_{i=1}^{\gamma} \varrho_i \chi_i = 0$ we get

$$\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i) = \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma) \zeta''(\varsigma) d\varsigma. \tag{3.4}$$

Differentiating the function f in (1.9) twice gives

$$\begin{aligned}\zeta''(\varsigma) &= \sum_{\kappa=0}^{n-2} \frac{f^{(\kappa+1)}(a_1)}{\kappa!} \left[w(\varsigma)(\varsigma - a_1)^{\kappa} + \kappa \int_{a_1}^{\varsigma} w(u)(\varsigma - a_1)^{\kappa-1} du \right] \\ &+ \sum_{\kappa=0}^{n-2} \frac{f^{(\kappa+1)}(b_1)}{\kappa!} \left[-w(\varsigma)(\varsigma - b_1)^{\kappa} + \kappa \int_{\varsigma}^{b_1} w(u)(\varsigma - b_1)^{\kappa-1} du \right] \\ &+ \frac{1}{(n-3)!} \int_{a_1}^{b_1} \tilde{T}_{w,n-2}(\varsigma, u)\zeta^{(n)}(u) du. \end{aligned} \tag{3.5}$$

Inserting (3.5) in (3.4) yields

$$\begin{aligned}\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i) &= \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{\kappa!} \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma) \\ &\times \left[w(\varsigma)(\varsigma - a_1)^{\kappa} + \kappa \int_{a_1}^{\varsigma} w(u)(\varsigma - a_1)^{\kappa-1} du \right] d\varsigma \\ &+ \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{\kappa!} \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma) \\ &\times \left[-w(\varsigma)(\varsigma - b_1)^{\kappa} + \kappa \int_{\varsigma}^{b_1} w(u)(\varsigma - b_1)^{\kappa-1} du \right] d\varsigma \\ &+ \frac{1}{(n-3)!} \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma) \left(\int_{a_1}^{b_1} \tilde{T}_{w,n-2}(\varsigma, u)\zeta^{(n)}(u) du \right) d\varsigma. \end{aligned}$$

and in the last term, by applying the Fubini's theorem we get (3.1).

Furthermore, in (1.9) by replacing $\zeta \longrightarrow \zeta''$ and $n \longrightarrow n-2$ respectively, and after some rearrangements we get

$$\begin{aligned}\zeta''(\varsigma) &= \int_{a_1}^{b_1} w(\tau)\zeta''(\tau) d\tau \\ &+ \sum_{\kappa=3}^{n-1} \frac{1}{(\kappa-2)!} \left[\zeta^{(\kappa)}(a_1) \int_{a_1}^{\varsigma} w(u) \left((\varsigma - a_1)^{\kappa-2} - (u - a_1)^{\kappa-2} \right) du \right. \\ &\left. + \zeta^{(\kappa)}(b_1) \int_{\varsigma}^{b_1} w(u) \left((\varsigma - b_1)^{\kappa-2} - (u - b_1)^{\kappa-2} \right) du \right] \\ &+ \frac{1}{(n-3)!} \int_{a_1}^{b_1} T_{w,n-2}(\varsigma, u)\zeta^{(n)}(u) du. \end{aligned} \tag{3.6}$$

Similarly, using (3.6) in (3.4) and applying Fubini's Theorem we get (3.3). □

Remark 3.2. If we put $w(\tau) = \frac{1}{b_1 - a_1}$ in Theorem 3.1, we capture Theorem 2.1 of [9].

Now we will discuss some inequalities that can be obtained from the previous identities.

Theorem 3.3. *Under the assumptions of Theorem* 3.1 *with the additional condition*

$$\int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma) \tilde{T}_{w,n-2}(\varsigma, u) d\varsigma \geq 0, \quad \forall\, u \in [a_1, b_1], \tag{3.7}$$

where G_l and $\tilde{T}_{w,n-2}$ are given in $(1.13), (1.16) - (1.19)$ *and* (3.2). *If ζ is $n-$convex, then the following inequality holds*

$$\begin{aligned}
\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i) &- \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{\kappa!} \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma) \\
&\times \left[w(\varsigma)(\varsigma - a_1)^{\kappa} + \kappa \int_{a_1}^{\varsigma} w(u)(\varsigma - a_1)^{\kappa-1} du \right] d\varsigma \\
&- \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{\kappa!} \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma) \\
&\times \left[-w(\varsigma)(\varsigma - b_1)^{\kappa} + \kappa \int_{\varsigma}^{b_1} w(u)(\varsigma - b_1)^{\kappa-1} du \right] d\varsigma \geq 0.
\end{aligned} \tag{3.8}$$

Proof. Using the fact that function ζ is $n-$convex, we have $\zeta^{(n)} \geq 0$ and (3.7) in (3.1) we obtain our required result. □

Remark 3.4. If we put $w(\tau) = \frac{1}{b_1 - a_1}$ in Theorem 3.3, we capture Theorem 2.2 of [9].

Theorem 3.5. *Under the assumptions of Theorem* 3.1 *with the additional condition*

$$\int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma) T_{w,n-2}(\varsigma, u) d\varsigma \geq 0, \quad \forall\, u \in [a_1, b_1], \tag{3.9}$$

where G_l and $T_{w,n}$ are defined in $(1.13), (1.16) - (1.19)$ *and* (1.10). *If ζ is $n-$convex, then the following inequality holds*

$$\begin{aligned}
\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i) &- \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma) \left(\int_{a_1}^{b_1} w(\tau) \zeta''(\tau) d\tau \right) d\varsigma \\
&- \sum_{\kappa=3}^{n-1} \frac{1}{(\kappa-2)!} \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma) \\
&\times \left[\zeta^{(\kappa)}(a_1) \int_{a_1}^{\varsigma} w(u) \left((\varsigma - a_1)^{\kappa-2} - (u - a_1)^{\kappa-2} \right) du \right. \\
&\left. + \zeta^{(\kappa)}(b_1) \int_{\varsigma}^{b_1} w(u) \left((\varsigma - b_1)^{\kappa-2} - (u - b_1)^{\kappa-2} \right) du \right] d\varsigma \geq 0.
\end{aligned} \tag{3.10}$$

Proof. Using the fact that the function ζ is $n-$convex, we have $\zeta^{(n)} \geq 0$ and (3.9) in (3.3), we easily arrive at our required result. □

Remark 3.6. If we put $w(\tau) = \frac{1}{b_1 - a_1}$ in Theorem 3.5, we capture Theorem 2.3 of [9].

Here we discuss a major consequence.

Theorem 3.7. *Under the assumptions of Theorem 3.1 and additionally,*

$$\sum_{i=1}^{\gamma} \varrho_i = 0 \text{ and } \sum_{i=1}^{\gamma} \varrho_i |\chi_i - \chi_k| \geq 0$$

for $\kappa \in \{1, \ldots, \gamma\}$. If n is even and ζ is $n-$convex, then inequalities (3.8) *and* (3.10) *hold.*

Proof. The Green's function $G_l(\varsigma, \tau)$ is convex w.r.t τ $\forall$ $\varsigma \in [a_1, b_1]$. Therefore, from Proposition 1.5, with conditions (1.5) and (1.6) replaced by (1.4) as in [15], we have

$$\sum_{i=1}^{\gamma} \varrho_i G(\chi_i, \varsigma) \geq 0 \quad \forall \quad \varsigma \in [a_1, b_1]. \tag{3.11}$$

Here $\tilde{T}_{w,n-2}(\varsigma, \tau) \geq 0$ and $T_{w,n-2}(\varsigma, \tau) \geq 0$ because n is even. By combining this fact with (3.11) we get inequalities (3.7) and (3.9). As ζ is $n-$convex, the results follow from Theorems 3.3 and 3.5. $\square$

Remark 3.8. If we put $w(\tau) = \frac{1}{b_1 - a_1}$ in Theorem 3.7, we capture Theorem 2.4 of [9]

Following that, we will present the integral versions of our main findings. We will skip the details because the proofs are identical to discrete version.

Theorem 3.9. *Let $\zeta : I \to \mathbb{R}$ be a function such that $\zeta^{(n-1)}$ is absolutely continuous, $n \geq 3$, $n \in \mathbb{N}$, $a_1 < b_1$, $a_1, b_1 \in I$, $I \subset \mathbb{R}$ an open interval, $w : [a_1, b_1] \to [0, \infty)$ is some probability density function. Additionally, let $\varrho : [\alpha_1, \beta_1] \to \mathbb{R}$ satisfy $\int_{\alpha_1}^{\beta_1} \varrho(\chi) d\chi = 0$ and $g : [\alpha_1, \beta_1] \to [a_1, b_1]$, $\int_{\alpha_1}^{\beta_1} \varrho(\chi) g(\chi) d\chi = 0$, and let G_l, $\tilde{T}_{w,n}$ and $T_{w,n}$ be given by* (1.13), (1.16) – (1.19), (3.2) *and* (1.10)*. Then the following two identities hold:*

$$\begin{aligned}
\int_{\alpha_1}^{\beta_1} \varrho(\chi) \zeta(g(\chi)) d\chi &= \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{\kappa!} \int_{a_1}^{b_1} \int_{\alpha_1}^{\beta_1} \varrho(\chi) G_l(g(\chi), \varsigma) \\
&\times \left[w(\varsigma)(\varsigma - a_1)^{\kappa} + \kappa \int_{a_1}^{\varsigma} w(u)(\varsigma - a_1)^{\kappa-1} du \right] d\chi d\varsigma \\
&+ \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{\kappa!} \int_{a_1}^{b_1} \int_{\alpha_1}^{\beta_1} \varrho(\chi) G_l(g(\chi), \varsigma) \\
&\times \left[-w(\varsigma)(\varsigma - b_1)^{\kappa} + \kappa \int_{\varsigma}^{b_1} w(u)(\varsigma - b_1)^{\kappa-1} du \right] d\chi d\varsigma \\
&+ \frac{1}{(n-3)!} \int_{a_1}^{b_1} \left(\left(\int_{\alpha_1}^{\beta_1} \varrho(\chi) G_l(g(\chi), \varsigma) d\chi \right) \tilde{T}_{w,n-2}(\varsigma, u) d\varsigma \right) \zeta^{(n)}(u) du.
\end{aligned} \tag{3.12}$$

and

$$\int_{\alpha_1}^{\beta_1} \varrho(\chi)\zeta(g(\chi))d\chi = \int_{a_1}^{b_1}\int_{\alpha_1}^{\beta_1} \varrho(\chi)G_l(g(\chi),\varsigma)d\chi \left(\int_{a_1}^{b_1} w(\tau)\zeta''(\tau)d\tau\right) d\varsigma$$

$$+\sum_{\kappa=3}^{n-1}\frac{1}{(\kappa-2)!}\int_{a_1}^{b_1}\left(\int_{\alpha_1}^{\beta_1}\varrho(\chi)G_l(g(\chi),\varsigma)d\chi\right)\times$$

$$\left[\zeta^{(\kappa)}(a_1)\int_{a_1}^{\varsigma} w(u)\left((\varsigma-a_1)^{\kappa-2}-(u-a_1)^{\kappa-2}\right)du\right.$$

$$\left.+\zeta^{(\kappa)}(b_1)\int_{\varsigma}^{b_1} w(u)\left((\varsigma-b_1)^{\kappa-2}-(u-b_1)^{\kappa-2}\right)du\right]d\varsigma$$

$$+\frac{1}{(n-3)!}\int_{a_1}^{b_1}\zeta^{(n)}(u)\left(\int_{a_1}^{b_1}\left(\int_{\alpha_1}^{\beta_1}\varrho(\chi)G_l(g(\chi),\varsigma)d\chi\right)T_{w,n-2}(\varsigma,u)d\varsigma\right)du. \quad (3.13)$$

Remark 3.10. If we put $w(\tau)=\frac{1}{b_1-a_1}$ in Theorem 3.9, we capture Theorem 2.5 of [9].

Theorem 3.11. *Under the assumptions of Theorem* 2.3 *with the additional condition*

$$\int_{a_1}^{b_1}\int_{\alpha_1}^{\beta_1}\varrho\,(\chi)\,G_l(g(\chi),\varsigma)\,\tilde{T}_{w,n-2}(\varsigma,u)\,d\chi\,d\varsigma\geq 0,\quad \forall\, u\in[a_1,b_1] \quad (3.14)$$

where G_l *is defined in* $(1.13),(1.16)-(1.19)$ *and* $\tilde{T}_{w,n}$ *is given in* (3.2)*. If* ζ *is* $n-$*convex, then the following inequality holds*

$$\int_{\alpha_1}^{\beta_1}\varrho\,(\chi)\,\zeta(g(\chi))\,d\chi-\sum_{\kappa=0}^{n-2}\frac{\zeta^{(\kappa+1)}(a_1)}{\kappa!}\int_{a_1}^{b_1}\int_{\alpha_1}^{\beta_1}\varrho(\chi)G_l(g(\chi),\varsigma)$$

$$\times\left[w(\varsigma)(\varsigma-a_1)^{\kappa}+\kappa\int_{a_1}^{\varsigma}w(u)(\varsigma-a_1)^{\kappa-1}du\right]d\chi d\varsigma$$

$$-\sum_{\kappa=0}^{n-2}\frac{\zeta^{(\kappa+1)}(b_1)}{\kappa!}\int_{a_1}^{b_1}\int_{\alpha_1}^{\beta_1}\varrho(\chi)G_l(g(\chi),\varsigma)$$

$$\times\left[-w(\varsigma)(\varsigma-b_1)^{\kappa}+\kappa\int_{\varsigma}^{b_1}w(u)(\varsigma-b_1)^{\kappa-1}du\right]d\chi d\varsigma\geq 0. \quad (3.15)$$

Remark 3.12. If we put $w(\tau)=\frac{1}{b_1-a_1}$ in Theorem 3.11, we capture Theorem 2.6 of [9].

Theorem 3.13. *Under the assumptions of Theorem* 2.3 *with the additional condition*

$$\int_{a_1}^{b_1}\int_{\alpha_1}^{\beta_1}\varrho\,(\chi)\,G_l(g(\chi),\varsigma)\,T_{w,n-2}(\varsigma,u)d\chi\,d\varsigma\geq 0,\quad \forall\, u\in[a_1,b_1], \quad (3.16)$$

where G_l is defined in (1.13), (1.16) − (1.19) *and $T_{w,n}$ is given in* (1.10). *If ζ is $n-$convex, then the following inequality holds*

$$\begin{aligned}\int_{\alpha_1}^{\beta_1} \varrho(\chi)\zeta(g(\chi))d\chi - \int_{a_1}^{b_1}\int_{\alpha_1}^{\beta_1} \varrho(\chi)G_l(g(\chi),\varsigma)d\chi\left(\int_{a_1}^{b_1} w(\tau)\zeta''(\tau)d\tau\right)d\varsigma \\ -\sum_{\kappa=3}^{n-1}\frac{1}{(\kappa-2)!}\int_{a_1}^{b_1}\left(\int_{\alpha_1}^{\beta_1}\varrho(\chi)G_l(g(\chi),\varsigma)d\chi\right) \\ \times\left[\zeta^{(\kappa)}(a_1)\int_{a_1}^{\varsigma} w(u)\left((\varsigma-a_1)^{\kappa-2}-(u-a_1)^{\kappa-2}\right)du\right. \\ \left.-\zeta^{(\kappa)}(b_1)\int_{\varsigma}^{b_1} w(u)\left((\varsigma-b_1)^{\kappa-2}-(u-b_1)^{\kappa-2}\right)du\right]d\varsigma \geq 0. \quad (3.17)\end{aligned}$$

Remark 3.14. If we put $w(\tau)=\frac{1}{b_1-a_1}$ in Theorem 3.13, we capture Theorem 2.7 of [9].

Theorem 3.15. *Under the assumptions of Theorem* 3.9 *and additionally let $g : [\alpha_1,\beta_1] \to [a_1,b_1]$ and $\varrho : [\alpha_1,\beta_1] \to \mathbb{R}$ satisfy (1.8). If n is even and ζ is $n-$convex, then inequalities* (3.15) *and* (3.17) *hold.*

Remark 3.16. If we put $w(\tau)=\frac{1}{b_1-a_1}$ in above, we capture Theorem 2.8 of [9]

3.1. Inequalities related to n-convex functions at a point

In the present subsection, we would like to discuss some results related to the Green function following the definition of convexity at a point (Definition 2.17 of subsection 2.1). Here we improve results from previous subsection. More specifically, let $T_{w,n}^{[a_1,c_1]}$ and $T_{w,n}^{[c_1,b_1]}$ represent the same as (1.10) on these intervals, *i.e.*,

$$T_{w,n}^{[a_1,c_1]}(\chi,\varsigma)=\begin{cases}\displaystyle\int_{\chi}^{\varsigma} w(u)(u-\varsigma)^{n-1}du+W(\chi)(\chi-\varsigma)^{n-1}, \\ \qquad a_1\leq\varsigma\leq\chi; \\ \displaystyle\int_{\chi}^{\varsigma} w(u)(u-\varsigma)^{n-1}du+(W(\chi)-1)(\chi-\varsigma)^{n-1}, \\ \qquad \chi<\varsigma\leq c_1;\end{cases} \quad (3.18)$$

$$T_{w,n}^{[c_1,b_1]}(\chi,\varsigma)=\begin{cases}\displaystyle\int_{\chi}^{\varsigma} w(u)(u-\varsigma)^{n-1}du+W(\chi)(\chi-\varsigma)^{n-1}, \\ \qquad c_1\leq\varsigma\leq\chi; \\ \displaystyle\int_{\chi}^{\varsigma} w(u)(u-\varsigma)^{n-1}du+(W(\chi)-1)(\chi-\varsigma)^{n-1}, \\ \qquad \chi<\varsigma\leq b_1.\end{cases} \quad (3.19)$$

Similarly, $\tilde{T}_{w,n-2}^{[a_1,c_1]}$ and $\tilde{T}_{w,n-2}^{[c_1,b_1]}$ denote equivalent of (3.2) on these intervals, i.e.,

$$\tilde{T}_{w,n-2}^{[a_1,c_1]}(\varsigma,u)=\begin{cases}\dfrac{w(\varsigma)(\varsigma-u)^{n-2}}{(n-2)}+W(\varsigma)(\varsigma-u)^{n-3}, & a_1\le u\le\varsigma;\\ \dfrac{w(\varsigma)(\varsigma-u)^{n-2}}{(n-2)}+\left(W(\varsigma)-1\right)(\varsigma-u)^{n-3}, & \varsigma<u\le b_1;\end{cases}\tag{3.20}$$

$$\tilde{T}_{w,n-2}^{[c_1,b_1]}(\varsigma,u)=\begin{cases}\dfrac{w(\varsigma)(\varsigma-u)^{n-2}}{(n-2)}+W(\varsigma)(\varsigma-u)^{n-3}, & c_1\le u\le\varsigma;\\ \dfrac{w(\varsigma)(\varsigma-u)^{n-2}}{(n-2)}+\left(W(\varsigma)-1\right)(\varsigma-u)^{n-3}, & \varsigma<u\le b_1.\end{cases}\tag{3.21}$$

Let $\chi\in[a_1,c_1]^\gamma$, $\varrho\in\mathbb{R}^\gamma$, $\mathbf{y}\in[c_1,b_1]^\ell$ and $\mathbf{q}\in\mathbb{R}^\ell$ and denote

$$\begin{aligned}\Lambda_1(\zeta)=&\sum_{i=1}^{\gamma}\varrho_i\zeta(\chi_i)-\sum_{\kappa=0}^{n-2}\frac{\zeta^{(\kappa+1)}(a_1)}{\kappa!}\int_{a_1}^{c_1}\sum_{i=1}^{\gamma}\varrho_iG(\chi_i,\varsigma)\\&\times\left[w(\varsigma)(\varsigma-a_1)^\kappa+\kappa\int_{a_1}^{\varsigma}w(u)(\varsigma-a_1)^{\kappa-1}du\right]d\varsigma\\&-\sum_{\kappa=0}^{n-2}\frac{\zeta^{(\kappa+1)}(c_1)}{\kappa!}\int_{a_1}^{c_1}\sum_{i=1}^{\gamma}\varrho_iG(\chi_i,\varsigma)\\&\times\left[-w(\varsigma)(\varsigma-c_1)^\kappa+\kappa\int_{\varsigma}^{c_1}w(u)(\varsigma-c_1)^{\kappa-1}du\right]d\varsigma,\end{aligned}\tag{3.22}$$

$$\begin{aligned}\Xi_1(\zeta)=&\sum_{i=1}^{\ell}q_i\zeta(y)-\sum_{\kappa=0}^{n-2}\frac{\zeta^{(\kappa+1)}(c_1)}{\kappa!}\int_{c_1}^{b_1}\sum_{i=1}^{\ell}q_iG(y_i,\varsigma)\\&\times\left[w(\varsigma)(\varsigma-c_1)^\kappa+\kappa\int_{c_1}^{\varsigma}w(u)(\varsigma-c_1)^{\kappa-1}du\right]d\varsigma\\&-\sum_{\kappa=0}^{n-2}\frac{\zeta^{(\kappa+1)}(b_1)}{\kappa!}\int_{c_1}^{b_1}\sum_{i=1}^{\ell}q_iG(y_i,\varsigma)\\&\times\left[-w(\varsigma)(\varsigma-b_1)^\kappa+\kappa\int_{\varsigma}^{b_1}w(u)(\varsigma-b_1)^{\kappa-1}du\right]d\varsigma.\end{aligned}\tag{3.23}$$

Identity (3.1) applied to the intervals $[a_1,c_1]$ and $[c_1,b_1]$ and by using the functionals Λ_1 and Ξ_1 can be written as

$$\Lambda_1(\zeta)=\frac{1}{(n-3)!}\int_{a_1}^{c_1}\left(\int_{a_1}^{c_1}\sum_{i=1}^{\gamma}\varrho_iG_l(\chi_i,\varsigma)\tilde{T}_{w,n-2}^{[a_1,c_1]}(\varsigma,u)d\varsigma\right)\zeta^{(n)}(u)du,\tag{3.24}$$

$$\Xi_1(\zeta)=\frac{1}{(n-3)!}\int_{c_1}^{b_1}\left(\int_{c_1}^{b_1}\sum_{i=1}^{\ell}q_iG_l(y_i,\varsigma)\tilde{T}_{w,n-2}^{[c_1,b_1]}(\varsigma,u)d\varsigma\right)\zeta^{(n)}(u)du.\tag{3.25}$$

In the same manner, we can introduce further functionals namely

$$\Lambda_2(\zeta) = \frac{1}{(n-3)!} \int_{a_1}^{c_1} \left(\int_{a_1}^{c_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma) T_{w,n-2}^{[a_1,c_1]}(\varsigma, u) d\varsigma \right) \zeta^{(n)}(u) du, \tag{3.26}$$

$$\Xi_2(\zeta) = \frac{1}{(n-3)!} \int_{c_1}^{b_1} \left(\int_{c_1}^{b_1} \sum_{i=1}^{\ell} q_i G_l(y_i, \varsigma) T_{w,n-2}^{[c_1,b_1]}(\varsigma, u) d\varsigma \right) \zeta^{(n)}(u) du, \tag{3.27}$$

$$\Lambda_3(\zeta) = \frac{1}{(n-3)!} \int_{a_1}^{c_1} \left(\int_{a_1}^{c_1} \left(\int_{\alpha_1}^{\beta_1} \varrho(\chi) G_l(g(\chi), \varsigma) d\chi \right) \right.$$
$$\left. \times \tilde{T}_{w,n-2}^{[a_1,c_1]}(\varsigma, u) d\varsigma \right) \zeta^{(n)}(u) du, \tag{3.28}$$

$$\Xi_3(\zeta) = \frac{1}{(n-3)!} \int_{c_1}^{b_1} \left(\int_{c_1}^{b_1} \left(\int_{\alpha_1}^{\beta_1} q(y) G_l(g(y), \varsigma) dy \right) \right.$$
$$\left. \times \tilde{T}_{w,n-2}^{[c_1,b_1]}(\varsigma, u) d\varsigma \right) \zeta^{(n)}(u) du, \tag{3.29}$$

$$\Lambda_4(\zeta) = \frac{1}{(n-3)!} \int_{a_1}^{c_1} \left(\int_{a_1}^{c_1} \left(\int_{\alpha_1}^{\beta_1} \varrho(\chi) G_l(g(\chi), \varsigma) d\chi \right) \right.$$
$$\left. \times\, T_{w,n-2}^{[a_1,c_1]}(\varsigma, u) d\varsigma \right) \zeta^{(n)}(u) du, \tag{3.30}$$

$$\Xi_4(\zeta) = \frac{1}{(n-3)!} \int_{c_1}^{b_1} \left(\int_{c_1}^{b_1} \left(\int_{\alpha_1}^{\beta_1} q(y) G_l(g(y), \varsigma) dy \right) \right.$$
$$\left. \times T_{w,n-2}^{[c_1,b_1]}(\varsigma, u) d\varsigma \right) \zeta^{(n)}(u) du. \tag{3.31}$$

Theorem 3.17. *Let $\chi \in [a_1, c_1]^\gamma$, $\varrho \in \mathbb{R}^\gamma$, $y \in [c_1, b_1]^l$ and $q \in \mathbb{R}^l$ be such that*

$$\int_{a_1}^{c_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma) \tilde{T}_{n-2}^{[a_1,c_1]}\, (\varsigma, u)\, d\varsigma \geq 0, \qquad \forall\ u \in [a_1, c_1], \tag{3.32}$$

$$\int_{c_1}^{b_1} \sum_{i=1}^{\gamma} q_i G_l(y_i, \varsigma) \tilde{T}_{n-2}^{[c_1,b_1]}\, (\varsigma, u)\, d\varsigma \geq 0, \qquad \forall\ u \in [c_1, b_1], \tag{3.33}$$

$$\int_{a_1}^{c_1} \left(\int_{a_1}^{c_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma) \tilde{T}_{w,n-2}^{[a_1,c_1]}(\varsigma, u) d\varsigma \right) du$$
$$= \int_{c_1}^{b_1} \left(\int_{c_1}^{b_1} \sum_{i=1}^{\ell} q_i G_l(y_i, \varsigma) \tilde{T}_{w,n-2}^{[c_1,b_1]}(\varsigma, u) d\varsigma \right) du, \tag{3.34}$$

where $\tilde{T}_{w,n-2}^{[a_1,c_1]}$, $\tilde{T}_{w,n-2}^{[c_1,b_1]}$, Λ_1 and Ξ_1 are given by (3.20), (3.21), (3.22) *and* (3.23) *respectively. If $\zeta : [a_1, b_1] \to \mathbb{R}$ is $(n+1)-$convex at point c_1, then*

$$\Lambda_1(\zeta) \leq \Xi_1(\zeta). \tag{3.35}$$

If inequalities in (3.32) *and* (3.33) *are reversed, then* (3.35) *is valid with reversed sign of inequality.*

Remark 3.18. From proof of Theorem 3.17 we have

$$\Lambda_1(\zeta) \leq \frac{K}{n!}\Lambda_1(e_n) = \frac{K}{n!}\Xi_1(e_n) \leq \Xi_1(\zeta).$$

In fact, inequality (3.35) still is valid if we replace assumption (3.34) with weaker assumption that

$$K\left(\Xi_1(e_n) - \Lambda_1(e_n)\right) \geq 0.$$

Remark 3.19. If we put $w(u) = \frac{1}{b_1-a_1}$ in above identity, we capture Theorem 2.11 of [7].

Here we have another similar result.

Theorem 3.20. *Let* $\chi \in [a_1, c_1]^\gamma$*,* $\varrho \in \mathbb{R}^\gamma$*,* $y \in [c_1, b_1]^l$ *and* $q \in \mathbb{R}^l$ *be such that*

$$\int_{a_1}^{c_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma) T_{n-2}^{[a_1,c_1]}(\varsigma, u)\, d\varsigma \geq 0, \qquad \forall\ u \in [a_1, c_1], \tag{3.36}$$

$$\int_{c_1}^{b_1} \sum_{i=1}^{\gamma} q_i G_l(y_i, \varsigma) T_{n-2}^{[c_1,b_1]}(\varsigma, u)\, d\varsigma \geq 0, \qquad \forall\ u \in [c_1, b_1], \tag{3.37}$$

$$\begin{aligned} &\int_{a_1}^{c_1}\left(\int_{a_1}^{c_1}\sum_{i=1}^{\gamma}\varrho_i G_l(\chi_i,\varsigma)T_{w,n-2}^{[a_1,c_1]}(\varsigma,u)d\varsigma\right)du \\ =&\int_{c_1}^{b_1}\left(\int_{c_1}^{b_1}\sum_{i=1}^{\ell}q_i G_l(y_i,\varsigma)T_{w,n-2}^{[c_1,b_1]}(\varsigma,u)d\varsigma\right)du, \end{aligned} \tag{3.38}$$

where $T_{w,n-2}^{[a_1,c_1]}$*,* $T_{w,n-2}^{[c_1,b_1]}$*,* Λ_2 *and* Ξ_2 *are given by* (3.18), (3.19), (3.26) *and* (3.27) *respectively. If* $\zeta : [a_1, b_1] \to \mathbb{R}$ *is* $(n+1)-$*convex at point* c_1*, then*

$$\Lambda_2(\zeta) \leq \Xi_2(\zeta). \tag{3.39}$$

If inequalities in (3.36) *and* (3.37) *are reversed, then* (3.39) *is valid with reversed sign of inequality.*

Remark 3.21. If we put $w(u) = \frac{1}{b_1-a_1}$ in above identity, we capture Theorem 2.13 of [7].

Remark 3.22. Similar results can also be stated for integral versions as well by using functionals $\Lambda_3(\zeta)$, $\Xi_3(\zeta)$, $\Lambda_4(\zeta)$ and $\Xi_4(\zeta)$ as defined in (3.28), (3.29) (3.30) and (3.31) respectively.

3.2. Bounds for identities related to the Popoviciu-type inequalities

Under the assumptions of Theorems 3.1 and 3.9, we denote the following functions Ω_j, $j \in \{1,2,3,4\}$, define as

$$\Omega_1(\tau) = \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G(\chi_i,\varsigma)\tilde{T}_{w,n-2}(\varsigma,u)d\varsigma, \quad u \in [a_1,b_1];$$

$$\Omega_2(\tau) = \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G(\chi_i,\varsigma)T_{w,n-2}(\varsigma,u)d\varsigma \geq 0, \quad u \in [a_1,b_1];$$

$$\Omega_3(\tau) = \int_{a_1}^{b_1} \int_{\alpha_1}^{\beta_1} \varrho\,(\chi)\,G(g(\chi),\varsigma)\,\tilde{T}_{w,n-2}(\varsigma,u)\,d\chi\,d\varsigma, \quad u \in [a_1,b_1];$$

$$\Omega_4(\tau) = \int_{a_1}^{b_1} \int_{\alpha_1}^{\beta_1} \varrho\,(\chi)\,G(g(\chi),\varsigma)\,T_{w,n-2}(\varsigma,u)d\chi\,d\varsigma, \quad u \in [a_1,b_1].$$

Theorem 3.23. *Let $n \in \mathbb{N}$, $n \geq 3$, $\zeta : [a_1,b_1] \to \mathbb{R}$ be such that $\zeta^{(n)}$ is an absolutely continuous function with $(\cdot - a_1)(b_1 - \cdot)[\zeta^{(n+1)}]^2 \in L[a_1,b_1]$ and let $\chi \in [a_1,b_1]^\gamma$ and $\varrho \in \mathbb{R}^\gamma$ satisfy*

$$\sum_{i=1}^{\gamma} \varrho_i = 0 \text{ and } \sum_{i=1}^{\gamma} \varrho_i\chi_i = 0.$$

Then

$$\begin{aligned}\sum_{i=1}^{\gamma} \varrho_i\zeta(\chi_i) &= \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{\kappa!} \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i,\varsigma) \\ &\times \left[w(\varsigma)(\varsigma - a_1)^\kappa + \kappa \int_{a_1}^{\varsigma} w(u)(\varsigma - a_1)^{\kappa-1}du \right] d\varsigma \\ &+ \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{\kappa!} \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i,\varsigma) \\ &\times \left[-w(\varsigma)(\varsigma - b_1)^\kappa + \kappa \int_{\varsigma}^{b_1} w(u)(\varsigma - b_1)^{\kappa-1}du \right] d\varsigma \\ &+ \frac{\zeta^{(n-1)}(b_1) - \zeta^{(n-1)}(a_1)}{(n-3)!(b_1 - a_1)} \int_{a_1}^{b_1} \Omega_1(\varsigma)d\varsigma + R_n^1(\zeta; a_1,b_1), \end{aligned} \tag{3.40}$$

and

$$\begin{aligned}\sum_{i=1}^{\gamma} \varrho_i\zeta(\chi_i) &= \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i,\varsigma) \left(\int_{a_1}^{b_1} w(\tau)\zeta''(\tau)d\tau \right) d\varsigma \\ &+ \sum_{\kappa=3}^{n-1} \frac{1}{(\kappa-2)!} \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i,\varsigma) \end{aligned}$$

$$\times \left[\zeta^{(\kappa)}(a_1)\int_{a_1}^{\varsigma} w(u)\left((\varsigma-a_1)^{\kappa-2}-(u-a_1)^{\kappa-2}\right)du\right.$$
$$\left.+\zeta^{(\kappa)}(b_1)\int_{\varsigma}^{b_1} w(u)\left((\varsigma-b_1)^{\kappa-2}-(u-b_1)^{\kappa-2}\right)du\right]d\varsigma$$
$$+\frac{\zeta^{(n-1)}(b_1)-\zeta^{(n-1)}(a_1)}{(n-3)!(b_1-a_1)}\int_{a_1}^{b_1}\Omega_2(\varsigma)d\varsigma+R_n^2(\zeta;a_1,b_1), \tag{3.41}$$

where the remainders $R_n^j(\zeta;a_1,b_1)$, $j=1,2$, satisfy the bounds

$$|R_n^j(\zeta;a_1,b_1)| \le \frac{1}{(n-3)!}$$
$$\times\left(\frac{(b_1-a_1)}{2}\left|T(\Omega_j,\Omega_j)\int_{a_1}^{b_1}(\varsigma-a_1)(b_1-\varsigma)[\zeta^{(n+1)}(\varsigma)]^2 d\varsigma\right|\right)^{1/2}. \tag{3.42}$$

Remark 3.24. If we put $w(u)=\frac{1}{b_1-a_1}$, $u\in[a_1,b_1]$ above identity reduces to Theorem 3.3 of [9].

By using Proposition 2.24, we obtain the following Grüss type inequality.

Theorem 3.25. *Let $n\in\mathbb{N}$, $n\ge 3$, $\zeta:[a_1,b_1]\to\mathbb{R}$ be such that $\zeta^{(n)}$ is an absolutely continuous function with $\zeta^{(n+1)}\ge 0$ and let $\chi\in[a_1,b_1]^{\gamma}$ and $\varrho\in\mathbb{R}^{\gamma}$ satisfy*

$$\sum_{i=1}^{\gamma}\varrho_i=0 \ \textit{and}\ \sum_{i=1}^{\gamma}\varrho_i\chi_i=0.$$

Then representations (3.40) *and* (3.41) *hold and the remainders $R_n^j(\zeta;a_1,b_1)$, $j=1,2$, satisfy the bounds*

$$|R_n^j(\zeta;a_1,b_1)| \le \frac{1}{(n-3)!}\|\Omega_j'\|_\infty\left\{\frac{b_1-a_1}{2}\left[\zeta^{(n-1)}(b_1)+\zeta^{(n-1)}(a_1)\right]\right.$$
$$\left.-\left[\zeta^{(n-2)}(b_1)-\zeta^{(n-2)}(a_1)\right]\right\}. \tag{3.43}$$

Remark 3.26. If we put $w(u)=\frac{1}{b_1-a_1}$, $u\in[a_1,b_1]$ above identity reduces to Theorem 3.4 of [9].

Remark 3.27. Similar results can also be stated for the integral version as well by using functional Ψ_j, where $j\in\{3,4\}$.

3.3. Ostrowski type inequalities via extension of Montgomery identity and Green functions

Here we present some Ostrowski-type inequalities related to the generalized linear inequalities. Throughout the section, we use the following functions Ω_j, $j\in\{1,2,3,4\}$ defined as in the previous subsection.

Theorem 3.28. *Let* $\zeta^{(n)} \in L_q[a_1, b_1]$, $1 \leq q, r \leq \infty$, $\frac{1}{q} + \frac{1}{r} = 1$, $n \geq 3$, $n \in \mathbb{N}$, $j \in \{1, 2, 3, 4\}$. *Then*

$$\begin{aligned}\Bigg|\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i) - \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{\kappa!} \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G(\chi_i, \varsigma) \\ \times \left[w(\varsigma)(\varsigma - a_1)^{\kappa} + \kappa \int_{a_1}^{\varsigma} w(u)(\varsigma - a_1)^{\kappa-1} du\right] d\varsigma \\ - \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{\kappa!} \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G(\chi_i, \varsigma) \\ \times \left[-w(\varsigma)(\varsigma - b_1)^{\kappa} + \kappa \int_{\varsigma}^{b_1} w(u)(\varsigma - b_1)^{\kappa-1} du\right] d\varsigma\Bigg| \\ \leq \frac{1}{(n-3)!} \|\zeta^{(n)}\|_q \|\Omega_j\|_r, \quad (3.44)\end{aligned}$$

and

$$\begin{aligned}\Bigg|\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i) - \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G(\chi_i, \varsigma) \left(\int_{a_1}^{b_1} w(\tau)\zeta''(\tau) d\tau\right) d\varsigma \\ - \sum_{\kappa=3}^{n-1} \frac{1}{(\kappa-2)!} \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G(\chi_i, \varsigma) \\ \times \left[\zeta^{(\kappa)}(a_1) \int_{a_1}^{\varsigma} w(\tau) \left((\varsigma - a_1)^{\kappa-2} - (\tau - a_1)^{\kappa-2}\right) dt\right. \\ \left. + \zeta^{(\kappa)}(b_1) \int_{\varsigma}^{b_1} w(\tau) \left((\varsigma - b_1)^{\kappa-2} - (\tau - b_1)^{\kappa-2}\right) dt\right] d\varsigma\Bigg| \\ \leq \frac{1}{(n-3)!} \|\zeta^{(n)}\|_q \|\Omega_2\|_r. \quad (3.45)\end{aligned}$$

The constant on the right hand sides of (3.44) *and* (3.45) *is the best possible for* $q = 1$ *and sharp for* $1 < q \leq \infty$.

Remark 3.29. If we put $w(\tau) = \frac{1}{b_1 - a_1}$ in Theorem 3.28, we capture Theorem 3.5 for $j \in \{1, 2\}$ and Theorem 3.8 for $j \in \{3, 4\}$ of [9].

4. Conclusion and remarks

In this article, we have given a generalization of the results stated in [8] and [9](see also [10]) by introducing weights which are probability density functions. If we put our weights equal to $\frac{1}{b_1 - a_1}$ in our proposed results, we will capture almost all the results of [8], [9] and [10] as our special cases. Due to the general nature of the article, in some places we have used the Leibnitz rule of integration due to the involvement of the variable of integration in the limit of the integral as well. In our subsections we stated results, related to $n-$convexity at a point for Popoviciu-type inequalities involving the weighted version of the extension of Montgomery's identity and similar results for Popoviciu-type inequalities involving the weighted version of the extended Montgomery's identity with Green functions. We have also discussed the bounds of remainders for our proposed results using *Čebyšev* functional and *Grüss* type inequalities. In the end of sections we obtained bounds of Ostrowski type. Such results are also valid in the context of Green functions.

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