Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 4, 895–905
DOI: 10.24193/subbmath.2023.4.16

Decay rate of solutions to the Cauchy problem for a coupled system of viscoelastic wave equations with a strong delay in $\mathbb{R}^n$

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Abstract. Using weighted spaces, we establish a general decay rate properties of solutions as $T \to \infty$ for a coupled system of viscoelastic wave equations in $\mathbb{R}^n$ under some conditions on g_1, g_2, ϕ. We exploit a density function to introduce weighted spaces for solutions and using an appropriate Lyapunov function.

Mathematics Subject Classification (2010): 35L05, 35L15, 35L70, 35B40.

Keywords: Lyapunov function, relaxation function, density, decay rate, weighted spaces.

1. Introduction and statement

Let us consider the following problem

$$\begin{cases} u_1'' + \alpha u_2 + \Delta u_1'(x, t-\tau) = \phi(x)\Delta_x \left(u_1 + \int_0^t g_1(s)u_1(t-s, x)ds\right), x \in \mathbb{R}^n \times \mathbb{R}^+ \\ u_2'' + \alpha u_1 + \Delta u_2'(x, t-\tau) = \phi(x)\Delta_x \left(u_2 + \int_0^t g_2(s)u_2(t-s, x)ds\right), x \in \mathbb{R}^n \times \mathbb{R}^+ \\ u_1'(x, t-\tau) = f_1(x, t-\tau), \quad u_2'(x, t-\tau) = f_2(x, t-\tau) \quad t \in (0, \tau) \\ (u_1(0, x), u_2(0, x)) = (u_{10}(x), u_{20}(x)) \in (\mathcal{H}(\mathbb{R}^n))^2, \\ (u_1'(0, x), u_2'(0, x)) = (u_{11}(x), u_{21}(x)) \in (L^2_\rho(\mathbb{R}^n))^2, \end{cases} \tag{1.1}$$

where the space $\mathcal{H}(\mathbb{R}^n)$ defined in (1.11) and $l, n \geq 2$, $\phi(x) > 0$, $\forall x \in \mathbb{R}^n$, $(\phi(x))^{-1} = \rho(x)$ defined in (**A2**).

Received 19 October 2020; Accepted 10 November 2020.

In this paper we are going to consider the solutions in spaces weighted by the density function $\rho(x)$ in order to compensate for the lack of Poincare's inequality which is useful in the proof.

In this framework, (see [5], [9]), it is well known that, for any initial data $(u_{10}, u_{20}) \in (\mathcal{H}(\mathbb{R}^n))^2, (u_{11}, u_{21}) \in (L^l_\rho(\mathbb{R}^n))^2$, then problem (P) has a global solution $(u_1, u_2) \in (C([0,T), \mathcal{H}(\mathbb{R}^n)))^2, (u_1', u_2') \in (C([0,T), L^l_\rho(\mathbb{R}^n))^2$ for T small enough, under hypothesis (**A1**)-(**A2**).

The energy of (u_1, u_2) at time t is defined by

$$E(t) = \frac{1}{2}\sum_{i=1}^{2} \|u_i'\|^2_{L^2_\rho(\mathbb{R}^n)} + \frac{1}{2}\sum_{i=1}^{2}(1 - \int_0^t g_i(s)ds)\|\nabla_x u_i\|_2^2 + \frac{1}{2}\sum_{i=1}^{2}(g_i \circ \nabla_x u_i) + \alpha \int_{\mathbb{R}^n} \rho u_1 u_2 dx. \tag{1.2}$$

When α is sufficiently small, we deduce that:

$$E(t) \geq \frac{1}{2}(1 - |\alpha|\|\rho\|^{-1}_{L^s})\left[\sum_{i=1}^{2} \|u_i'\|^2_{L^l_\rho} + \sum_{i=1}^{2}(1 - \int_0^t g_i(s)ds)\|\nabla_x u_i\|_2^2 + \sum_{i=1}^{2}(g_i \circ \nabla_x u_i)\right]$$

and the following energy functional law holds, which means that, our energy is uniformly bounded and decreasing along the trajectories.

$$E'(t) = \frac{1}{2}\sum_{i=1}^{2}(g_i' \circ \nabla_x u_i)(t) - \frac{1}{2}\sum_{i=1}^{2} g_i(t)\|\nabla_x u_i(t)\|_2^2, \forall t \geq 0. \tag{1.3}$$

The following notation will be used throughout this paper

$$(\Phi^s \circ \Psi)(t) = \int_0^t \Phi^s(t-\tau)\,\|\Psi(t) - \Psi(\tau)\|_2^2\,d\tau \tag{1.4}$$

For the literature, in $\mathbb{R}^n$ we quote essentially the results of [1], [5], [6], [7], [9], [11]. In [6], authors showed for one equation that, for compactly supported initial data and for an exponentially decaying relaxation function, the decay of the energy of solution of a linear Cauchy problem (1.1) with $l = 2, \rho(x) = 1$is polynomial. The finite-speed propagation is used to compensate for the lack of Poincars inequality. In the case $l = 2$, in [5], author looked into a linear Cauchy viscoelastic equation with density. His study included the exponential and polynomial rates, where he used the spaces weighted by density to compensate for the lack of Poincar's inequality. The same problem traited in [5], was considred in [7], where they consider a Cauchy problem for a viscoelastic wave equation. Under suitable conditions on the initial data and the relaxation function, they prove a polynomial decay result of solutions. Conditions used, on the relaxation function g and its derivative g' are different from the usual ones.

The problem (1.1) for the case $l = 2, \rho(x) = 1$, in a bounded domain $\Omega \subset \mathbb{R}^n, (n \geq 1)$ with a smooth boundary $\partial\Omega$ and g is a positive nonincreasing function was considred as equation in [11], where they established an explicit and general decay rate result for relaxation functions satisfying:

$$g'(t) \le -H(g(t)), t \ge 0, H(0) = 0 \tag{1.5}$$

for a positive function $H \in C^1(\mathbb{R}^+)$ and H is linear or strictly increasing and strictly convex C^2 function on $(0, r], 1 > r$. Wich improve the conditions considred recently by Alabau-Boussouira and Cannarsa [1] on the relaxation functions

$$g'(t) \le -\chi(g(t)), \chi(0) = \chi'(0) = 0 \tag{1.6}$$

where χ is a non-negative function, strictly increasing and strictly convex on $(0, k_0], k_0 > 0$. They required that

$$\int_0^{k_0} \frac{dx}{\chi(x)} = +\infty, \int_0^{k_0} \frac{xdx}{\chi(x)} < 1, \lim \inf_{s \to 0^+} \frac{\chi(s)/s}{\chi'(s)} > \frac{1}{2} \tag{1.7}$$

and proved a decay result for the energy of equation (1.1) with $\alpha = 0, l = 2, \rho(x) = 1$ in a bounded domain. In addition to these assumptions, if

$$\lim \sup_{s \to 0^+} \frac{\chi(s)/s}{\chi'(s)} < 1 \tag{1.8}$$

then, in this case, an explicit rate of decay is given.

We omit the space variable x of $u(x, t), u'(x, t)$ and for simplicity reason denote $u(x, t) = u$ and $u'(x, t) = u'$, when no confusion arises. We denote by

$$|\nabla_x u|^2 = \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i} \right)^2, \quad \Delta_x u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}.$$

The constants c used throughout this paper are positive generic constants which may be different in various occurrences also the functions considered are all real valued, here $u' = du(t)/dt$ and $u'' = d^2u(t)/dt^2$.

The main purpose of this work is to allow a wider class of relaxation functions and improve earlier results in the literature. The basic mechanism behind the decay rates is the relation between the damping and the energy. In section 2, we prove decay estimates of the solution of our problem (1.1) when g_1 and g_2 are of general decay rate. Our approach involves a perturbed energy method and leverages properties of convex functions.

First we recall and make use the following assumptions on the functions ρ and g for $i = 1, 2$ as:

A1: To guarantee the hyperbolicity of the system, we assume that the function $g_i : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ (for $i = 1, 2$) is of class C^1 satisfying:

$$1 - \int_0^{\infty} g_i(t)dt \ge k_i > 0, g_i(0) = g_{i0} > 0 \tag{1.9}$$

and there exist nonincreasing continuous functions ξ_1,ξ_2: $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying

$$g_i'(t) \le -\xi_i g_i(t). \tag{1.10}$$

A2: The function $\rho : \mathbb{R}^n \to \mathbb{R}_+^*, \rho(x) \in C^{0,\gamma}(\mathbb{R}^n)$ with $\gamma \in (0, 1)$ and $\rho \in L^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, where $s = \frac{2n}{2n - qn + 2q}$.

Definition 1.1 ([5], [12]). We define the function spaces of our problem and its norm as follows:

$$\mathcal{H}(\mathbb{R}^n) = \left\{ f \in L^{2n/(n-2)}(\mathbb{R}^n) : \nabla_x f \in L^2(\mathbb{R}^n) \right\} \tag{1.11}$$

and the spaces $L^2_\rho(\mathbb{R}^n)$ to be the closure of $C_0^\infty(\mathbb{R}^n)$ functions with respect to the inner product

$$(f,h)_{L^2_\rho(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \rho f h dx.$$

For $1 < p < \infty$, if f is a measurable function on $\mathbb{R}^n$, we define

$$\|f\|_{L^q_\rho(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \rho |f|^q dx \right)^{1/q}. \tag{1.12}$$

Corollary 1.2. *The separable Hilbert space $L^2_\rho(\mathbb{R}^n)$ with*

$$(f,f)_{L^2_\rho(\mathbb{R}^n)} = \|f\|^2_{L^2_\rho(\mathbb{R}^n)}.$$

consist of all f for which $\|f\|_{L^q_\rho(\mathbb{R}^n)} < \infty, 1 < q < +\infty$.

The following technical lemma will be pivotal in the next section.

Lemma 1.3. [4] (Lemma 1.1) *For any two functions g, $v \in C^1(\mathbb{R})$ and $\theta \in [0,1]$ we have*

$$\begin{aligned} \int_{\mathbb{R}^n} v'(t) \int_0^t g(t-s) v(s) ds dx &= -\frac{1}{2}\frac{d}{dt}(g \circ v)(t) + \frac{1}{2}\frac{d}{dt}\left(\int_0^t g(s) ds \right) \|v(t)\|_2^2 \\ &+ \frac{1}{2}(g' \circ v)(t) - \frac{1}{2} g(t) \|v(t)\|_2^2. \end{aligned} \tag{1.13}$$

and

$$\int_{\mathbb{R}^n} \left(\int_0^t g(t-s) |v(s) - v(t)| ds \right)^2 dx \le \left(\int_0^t g^{2(1-\theta)}(s) ds \right) (g^{2\theta} \circ v) \tag{1.14}$$

We are now ready to state and prove our main results

2. Results and proofs

Lemma 2.1. [8] *Let ρ satisfies (**A2**), then for any $u \in \mathcal{H}(\mathbb{R}^n)$*

$$\|u\|_{L^q_\rho(\mathbb{R}^n)} \le \|\rho\|_{L^s(\mathbb{R}^n)} \|\nabla_x u\|_{L^2(\mathbb{R}^n)}, \quad \text{with } s = \frac{2n}{2n - qn + 2q}, 2 \le q \le \frac{2n}{n-2}.$$

Corollary 2.2. *If $q = 2$, then Lemma 2.1. yields*

$$\|u\|_{L^2_\rho(\mathbb{R}^n)} \le \|\rho\|_{L^{n/2}(\mathbb{R}^n)} \|\nabla_x u\|_{L^2(\mathbb{R}^n)},$$

where we can assume $\|\rho\|_{L^{n/2}(\mathbb{R}^n)} = C_0 > 0$ to get

$$\|u\|_{L^2_\rho(\mathbb{R}^n)} \le C_0 \|\nabla_x u\|_{L^2(\mathbb{R}^n)}. \tag{2.1}$$

Using Cauchy-Schwarz, Poincare's inequalities, the proof of the following Lemma is immediate.

Lemma 2.3. *There exist constants $c, c' > 0$ such that*

$$\int_{\mathbb{R}^n}\left(\int_0^t g_i(t-s)(u_i(t)-u_i(s))ds\right)^2 dx \le c(g_i\circ u_i)(t) \le c'(g_i'\circ \nabla u_i)(t) \qquad (2.2)$$

for any $u \in \mathcal{H}(\mathbb{R}^n)$.

To construct a Lyapunov functional L equivalent to E, we introduce the next functionals

$$\psi_1(t) = \sum_{i=1}^2 \int_{\mathbb{R}^n} \rho(x) u_i |u_i'|^{l-2}u_i' dx \qquad (2.3)$$

$$\psi_2(t) = -\sum_{i=1}^2 \int_{\mathbb{R}^n} \rho(x)|u_i'|^{l-2}u_i' \int_0^t g_i(t-s)(u_i(t)-u_i(s))dsdx \qquad (2.4)$$

Lemma 2.4. *Under the assumptions (* **A1**-**A2** *), the functional ψ_1 satisfies, along the solution of* (1.1)

$$\psi_1'(t) \le \sum_{i=1}^2 \|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} - (k+|\alpha|C_0-\delta-1)\sum_{i=1}^2 \|\nabla_x u_i\|_2^2 + \frac{(1-k)}{4\delta}\sum_{i=1}^2 (g_i\circ\nabla_x u_i) \quad (2.5)$$

Proof. From (2.3), integrate by parts over $\mathbb{R}^n$, we have

$$\begin{aligned}
\psi_1'(t) &= \int_{\mathbb{R}^n}\rho(x)u_1'^l dx + \int_{\mathbb{R}^n}\rho(x)u_1\left(|u_1'|^{l-2}u_1'\right)' dx\\
&+ \int_{\mathbb{R}^n}\rho(x)u_2'^l dx + \int_{\mathbb{R}^n}\rho(x)u_2\left(|u_2'|^{l-2}u_2'\right)' dx\\
&= \int_{\mathbb{R}^n}\left(\rho(x)u_1'^l + u_1\Delta_x u_1 - \alpha\rho(x)u_1u_2 - u_1\int_0^t g_1(t-s)\Delta_x u_1(s,x)ds\right)dx\\
&+ \int_{\mathbb{R}^n}\left(\rho(x)u_2'^l + u_2\Delta_x u_2 - \alpha\rho(x)u_1u_2 - u_2\int_0^t g_2(t-s)\Delta_x u_2(s,x)ds\right)dx\\
&\le \sum_{i=1}^2\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} - \sum_{i=1}^2 k_i\|\nabla_x u_i\|_2^2 - 2\alpha\int_{\mathbb{R}^n}\rho(x)u_1u_2dx\\
&+ \sum_{i=1}^2\int_{\mathbb{R}^n}\nabla_x u_i\int_0^t g_i(t-s)(\nabla_x u_i(s)-\nabla_x u_i(t))dsdx
\end{aligned}$$

Using Young's, Poincare's inequalities, Lemma (2.1) and Lemma (1.3), we obtain

$$\begin{aligned}
\psi_1'(t) &\le \sum_{i=1}^2\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} - \sum_{i=1}^2 k_i\|\nabla_x u_i\|_2^2 + (1-|\alpha|\|\rho\|^{-1}_{L^s(\mathbb{R}^n)})\sum_{i=1}^2\|\nabla_x u_i\|_2^2\\
&+ \delta\sum_{i=1}^2\|\nabla_x u_i\|_2^2 + \frac{1}{4\delta}\sum_{i=1}^2\int_{\mathbb{R}^n}\left(\int_0^t g_i(t-s)|\nabla_x u_i(s)-\nabla_x u_i(t)|ds\right)^2 dx\\
&\le \sum_{i=1}^2\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} - (k+|\alpha|C_0-\delta-1)\sum_{i=1}^2\|\nabla_x u_i\|_2^2 + \frac{(1-k)}{4\delta}\sum_{i=1}^2(g_i\circ\nabla_x u_i)
\end{aligned}$$

For α small enough and $k = \max\{k_1, k_2\}$. □

Lemma 2.5. *Under the assumptions (***A1-A2***), the functional ψ_2 satisfies, along the solution of (P), for any $\sigma \in (0,1)$*

$$\begin{aligned}\psi_2'(t) \quad &\leq \quad \sum_{i=1}^{2}\left(\delta - \int_0^t g_i(s)ds\right)\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} \\ &+ \quad \delta\sum_{i=1}^{2}\|\nabla_x u_i\|_2^2 + \frac{c}{\delta}\sum_{i=1}^{2}(g_i \circ \nabla_x u_i) - c_\delta C_0\sum_{i=1}^{2}(g_i' \circ \nabla_x u_i)^{l/2} \qquad (2.6)\end{aligned}$$

Proof. Exploiting Eq. in (1.1), to get

$$\begin{aligned}\psi_2'(t) = &-\sum_{i=1}^{2}\int_{\mathbb{R}^n}\rho(x)\left(|u_i'|^{l-2}u_i'\right)'\int_0^t g_i(t-s)(u_i(t)-u_i(s))dsdx \qquad (2.7)\\ &-\sum_{i=1}^{2}\int_{\mathbb{R}^n}\rho(x)|u_i'|^{l-2}u_i'\int_0^t g_i'(t-s)(u_i(t)-u_i(s))dsdx - \sum_{i=1}^{2}\int_0^t g_i(s)ds\|u_i'\|^l_{L^l_\rho}\end{aligned}$$

To simplify the first term in (2.7), we multiply (1.1) by $\int_0^t g_i(t-s)(u_i(t)-u_i(s))dsdx$ and integrate by parts over $\mathbb{R}^n$. So we obtain

$$\begin{aligned}&-\sum_{i=1}^{2}\int_{\mathbb{R}^n}\rho(x)\left(|u_i'|^{l-2}u_i'\right)'\int_0^t g_i(t-s)(u_i(t)-u_i(s))dsdx\\ &=\sum_{i=1}^{2}\int_{\mathbb{R}^n}\Delta u_i(x)\int_0^t g_i(t-s)(u_i(t)-u_i(s))dsdx\\ &-\sum_{i=1}^{2}\int_{\mathbb{R}^n}\left(\int_0^t g_i(t-s)(u_i(t)-u_i(s))\int_0^t g_i(t-s)\Delta u_i(s)\right)dx \qquad (2.8)\\ &-\alpha\int_{\mathbb{R}^n}\left[\rho u_2\int_0^t g_1(t-s)(u_1(t)-u_1(s))ds + \rho u_1\int_0^t g_2(t-s)(u_2(t)-u_2(s))ds\right]dx\end{aligned}$$

The first term in the right side of (2.8) is estimated as follows

$$\begin{aligned}&\int_{\mathbb{R}^n}\Delta u_i(x)\int_0^t g_i(t-s)(u_i(t)-u_i(s))dsdx\\ \leq\quad &-\int_{\mathbb{R}^n}\nabla_x u_i\int_0^t g_i(t-s)(\nabla_x u_i(t)-\nabla_x u_i(s))dsdx\\ \leq\quad &\int_{\mathbb{R}^n}\nabla_x u_i\int_0^t g_i(t-s)(\nabla_x u_i(s)-\nabla_x u_i(t))dsdx\\ \leq\quad &\delta\|\nabla_x u_i\|^2 + \frac{1}{4\delta}\left(\int_0^t g_i(s)\right)(g_i \circ \nabla u_i)(t)\\ \leq\quad &\delta\|\nabla_x u_i\|^2 + \frac{1-k}{4\delta}(g_i \circ \nabla u_i)(t).\end{aligned}$$

while the second term becomes,

$$\begin{aligned}
&- \int_{\mathbb{R}^n}\left(\int_0^t g_i(t-s)(u_i(t)-u_i(s))\int_0^t g_i(t-s)\Delta u_i(s)\right)dx\\
&= \int_{\mathbb{R}^n}\left(\int_0^t g_i(t-s)(\nabla u_i(t)-\nabla u_i(s))\cdot\int_0^t g_i(t-s)\nabla u_i(s)\right)dx\\
&\leq \delta\int_{\mathbb{R}^n}\left(\int_0^t g_i(t-s)|\nabla u_i(s)-\nabla u_i(t))+\nabla u_i(t)|\right)^2\\
&+ \frac{1}{4\delta}\int_{\mathbb{R}^n}\left(\int_0^t g_i(t-s)(\nabla u_i(t)-\nabla u_i(s))\right)^2\\
&\leq 2\delta(1-k)^2\|\nabla u_i\|_2^2+\left(2\delta+\frac{1}{4\delta}\right)(1-k)(g_i\circ\nabla u_i)(t).
\end{aligned}$$

Now, using Young's and Poincare's inequalities we estimate

$$\begin{aligned}
&- \alpha\int_{\mathbb{R}^n}\rho u_2\int_0^t g_1(t-s)(u_1(t)-u_1(s))dsdx\\
&\leq -|\alpha|\delta\|u_2\|_{L^2_\rho}^2-\frac{|\alpha|C_0}{4\delta}(1-k)(g_1\circ\nabla u_1)(t)\\
&\leq -|\alpha|\delta C_0\|\nabla u_2\|_{L^2}^2-\frac{|\alpha|C_0}{4\delta}(1-k)(g_1\circ\nabla u_1)(t).
\end{aligned}$$

By Hölder's and Young's inegualities and Lemma (2.1) we estimate

$$\begin{aligned}
&- \int_{\mathbb{R}^n}\rho(x)|u_i'|^{l-2}u_i'\int_0^t g_i'(t-s)(u_i(t)-u_i(s))dsdx\\
&\leq \left(\int_{\mathbb{R}^n}\rho(x)|u_i'|^l dx\right)^{(l-1)/l}\times\left(\int_{\mathbb{R}^n}\rho(x)|\int_0^t -g_i'(t-s)(u_i(t)-u_i(s))ds|^l\right)^{1/l}\\
&\leq \delta\|u'\|_{L^l_\rho(\mathbb{R}^n)}^l+\frac{1}{4\delta}\|\rho\|_{L^s(\mathbb{R}^n)}^l\|\int_0^t -g'(t-s)(u(t)-u(s))ds\|_{L^l_\rho(\mathbb{R}^n)}^l\\
&\leq \delta\|u'\|_{L^l_\rho(\mathbb{R}^n)}^l-\frac{1}{4\delta}C_0(g'\circ\nabla_x u)^{l/2}(t).
\end{aligned}$$

Using Young's and Poincare's inequalities and Lemma (1.3), we obtain

$$\begin{aligned}
\psi_2'(t) &\leq \sum_{i=1}^2\left(\delta-\int_0^t g_i(s)ds\right)\|u_i'\|_{L^l_\rho(\mathbb{R}^n)}^l\\
&+ \delta\sum_{i=1}^2\|\nabla_x u_i\|_2^2+\frac{c}{\delta}\sum_{i=1}^2(g_i\circ\nabla_x u_i)-c_\delta C_0\sum_{i=1}^2(g_i'\circ\nabla_x u_i)^{l/2}.
\end{aligned}$$

□

Our main result reads as follows

Theorem 2.6. *Let* $(u_0, u_1) \in (\mathcal{H}(\mathbb{R}^n(\Omega)) \times L^l_\rho(\mathbb{R}^n)$ *and suppose that* $(\mathbf{A1}) - (\mathbf{A2})$ *hold. Then there exist positive constants* α_1, ω *such that the energy of solution given by* (1.1) *satisfies,*

$$E(t) \leq \alpha_1 E(t_0) \exp\left(-\omega \int_{t_0}^{t} \xi(s)ds\right), \forall t \geq t_0 \tag{2.9}$$

where $\xi(t) = \min\{\xi_1(t), \xi_2(t)\}, \quad \forall t \geq 0.$

In order to prove this theorem, let us define

$$L(t) = N_1 E(t) + \psi_1(t) + N_2 \psi_2(t) \tag{2.10}$$

for $N_1, N_2 > 1$. We require the following lemma, indicating an equivalence between the Lyapunov and energy functions

Lemma 2.7. *For* $N_1, N_2 > 1$*, we have*

$$\beta_1 L(t) \leq E(t) \leq L(t)\beta_2, \tag{2.11}$$

holds for two positive constants β_1 *and* β_2*.*

Proof. By applying Young's inequality to (2.3) and using (2.4) and (2.10), we obtain

$$\begin{aligned} |L(t) - N_1 E(t)| &\leq |\psi_1(t)| + N_2|\psi_2(t)| \\ &\leq \sum_{i=1}^{2} \int_{\mathbb{R}^n} \left|\rho(x) u_i |u_i'|^{l-2} u_i'\right| dx \\ &+ N_2 \sum_{i=1}^{2} \int_{\mathbb{R}^n} \left|\rho(x) |u_i'|^{l-2} u_i' \int_0^t g_i(t-s)(u_i(t) - u_i(s))ds\right| dx \end{aligned}$$

Thanks to Hölder and Young's inequalities with exponents $\frac{l}{l-1}$, l, since $\frac{2n}{n+2} \geq l \geq 2$, we have by using Lemma 2.1

$$\begin{aligned} \int_{\mathbb{R}^n} \left|\rho(x) u_i |u_i'|^{l-2} u_i'\right| dx &\leq \left(\int_{\mathbb{R}^n} \rho(x)|u_i|^l dx\right)^{1/l} \left(\int_{\mathbb{R}^n} \rho(x)|u_i'|^l dx\right)^{(l-1)/l} \\ &\leq \frac{1}{l}\left(\int_{\mathbb{R}^n} \rho(x)|u_i|^l dx\right) + \frac{l-1}{l}\left(\int_{\mathbb{R}^n} \rho(x)|u_i'|^l dx\right) \\ &\leq c\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} + c\|\rho\|^l_{L^s(\mathbb{R}^n)}\|\nabla_x u_i\|_2^l. \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^n} \left|\left(\rho(x)^{\frac{l-1}{l}} |u_i'|^{l-2} u_i'\right)\left(\rho(x)^{\frac{1}{l}} \int_0^t g_i(t-s)(u_i(t) - u_i(s))ds\right)\right| dx \\ &\leq \left(\int_{\mathbb{R}^n} \rho(x)|u_i'|^l dx\right)^{(l-1)/l} \times \left(\int_{\mathbb{R}^n} \rho(x)\left|\int_0^t g_i(t-s)(u_i(t) - u_i(s))ds\right|^l\right)^{1/l} \\ &\leq \frac{l-1}{l}\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} + \frac{1}{l}\left\|\int_0^t g_i(t-s)(u_i(t) - u_i(s))ds\right\|^l_{L^l_\rho(\mathbb{R}^n)} \\ &\leq \frac{l-1}{l}\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} + \frac{1}{l}\|\rho\|^l_{L^s(\mathbb{R}^n)}(g_i \circ \nabla_x u_i)^{l/2}(t). \end{aligned}$$

then, since $l \geq 2$, we have

$$\begin{aligned}
|L(t) - N_1 E(t)| &\leq c\sum_{i=1}^{2}\Big(\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} + \|\nabla_x u_i\|_2^l + g_i \circ \nabla_x u_i)^{l/2}(t)\Big) \\
&\leq c(E(t) + E^{l/2}(t)) \\
&\leq c(E(t) + E(t)\cdot E^{(l/2)-1}(t)) \\
&\leq c(E(t) + E(t)\cdot E^{(l/2)-1}(0)) \\
&\leq cE(t).
\end{aligned}$$

Consequently, (2.11) follows. □

Proof of Theorem 2.6. From (1.3), results of Lemmas (2.4) and (2.5), we have

$$\begin{aligned}
L'(t) &= N_1 E'(t) + \psi_1'(t) + N_2\psi_2'(t) \\
&\leq \left(\frac{1}{2}N_1 - c_\delta C_0 N_2\right)\sum_{i=1}^{2}(g_i' \circ \nabla_x u_i)^{l/2} + \left(\frac{4\xi_2 c + (1-l)}{4\delta}\right)\sum_{i=1}^{2}(g_i \circ \nabla_x u_i) \\
&- M_1\sum_{i=1}^{2}\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} - M_2\sum_{i=1}^{2}\|\nabla_x u_i\|_2^2
\end{aligned}$$

At this point, we choose ξ_2 large enough so that

$$M_1 := \left(N_2\left(\int_0^{t_1} g(s)ds - \delta\right) - 1\right) > 0,$$

We choose δ so small that $N_1 > 2c_\delta\|\rho\|^l_{L^s(\mathbb{R}^n)}N_2$. Given that δ is fixed, we can choose ξ_1, ξ_2 large enough so that

$$M_2 := \left(-N_2\sigma + \frac{1}{2}N_1 g(t_1) + (l-\sigma)\right) > 0,$$

and

$$\left(\frac{1}{2}N_1 - c_\delta C_0 N_2\right) > 0.$$

which yields

$$L'(t) \leq M_0\sum_{i=1}^{2}(g_i \circ \nabla_x u_i) - mE(t), \quad \forall t \geq t_1. \tag{2.13}$$

Multiplying (2.13) by $\xi(t)$ gives

$$\xi(t)L'(t) \leq -m\xi(t)E(t) + M_0\xi(t)\sum_{i=1}^{2}(g_i \circ \nabla_x u_i) \tag{2.14}$$

The last term can be estimated, using (**A1**), as follows

$$\begin{aligned} M_0\xi(t)\sum_{i=1}^{2}(g_i \circ \nabla_x u_i) &\leq M_0\sum_{i=1}^{2}\xi_i(t)\int_{\mathbb{R}^n}\int_0^t g_i(t-s)|u_i(t)-u_i(s)|^2 \\ &\leq M_0\sum_{i=1}^{2}\int_{\mathbb{R}^n}\int_0^t \xi_i(t-s)g_i(t-s)|u_i(t)-u_i(s)|^2 \\ &\leq -M_0\sum_{i=1}^{2}\int_{\mathbb{R}^n}\int_0^t g_i'(t-s)|u_i(t)-u_i(s)|^2 \\ &\leq -M_0\sum_{i=1}^{2} g_i' \circ \nabla u_i \leq -M_0E'(t). \end{aligned} \tag{2.15}$$

Thus, (2.13) becomes

$$\xi(t)L'(t)+M_0E'(t) \quad \leq \quad -m\xi(t)E(t) \quad \forall t\geq t_0. \tag{2.16}$$

Using the fact that ξ is a nonincreasing continuous function as ξ_1 and ξ_2 are nonincreasing, and so ξ is differentiable, with $\xi'(t)\leq 0$ for a.e t, then

$$(\xi(t)L(t)+M_0E(t))' \quad \leq \quad \xi(t)L'(t)+M_0E'(t)\leq -m\xi(t)E(t) \quad \forall t\geq t_0. \tag{2.17}$$

Since, using (2.11)

$$F=\xi L+M_0E\sim E, \tag{2.18}$$

we obtain, for some positive constant ω

$$F'(t)\leq -\omega\xi(t)F(t) \quad \forall t\geq t_0. \tag{2.19}$$

Integration over (t_0,t) leads to, for some constant $\omega>0$ such that

$$F(t)\leq \alpha_1F(t_0)\exp\left(-\omega\int_{t_0}^t\xi(s)ds\right), \forall t\geq t_0 \tag{2.20}$$

Recalling (2.18), estimate (2.20) yields the desired result (2.9). This completes the proof of Theorem 2.6.

Acknowledgement. This research work is supported by the General Direction of Scientific Research and Technological Development (DGRSDT), Algeria.

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