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Global solution for a diffusive epidemic model (HIV/AIDS) with an exponential behavior of source

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Abstract. We consider the question of global existence and uniform boundedness of nonnegative solutions of a system of reaction-diffusion equations with exponential nonlinearity, without any restriction on initial data, using maximum principle and Lyapunov function techniques.

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1. Introduction

In this paper we consider the following reaction-diffusion system

$$\frac{\partial u}{\partial t} - a\Delta u = \Pi - f(u,v) - \alpha u \quad (x,t) \in \Omega \times \mathbb{R}_+ \tag{1.1}$$

$$\frac{\partial v}{\partial t} - b\Delta v = f(u,v) - \sigma\kappa(v) \quad (x,t) \in \Omega \times \mathbb{R}_+ \tag{1.2}$$

with the boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0 \quad \text{on } \partial\Omega \times \mathbb{R}_+, \tag{1.3}$$

and the initial data

$$u(0,x) = u_0(x) \geq 0; \quad v(0,x) = v_0(x) \geq 0 \quad \text{in } \Omega, \tag{1.4}$$

where Ω is a smooth open bounded domain in $\mathbb{R}^n$, with boundary $\partial\Omega$ of class C^1 and

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η is the outer normal to $\partial\Omega$. The constants of diffusion a, b are positive and such that $a \neq b$ and Π, α, σ are positive constants, κ and f are nonnegative functions of class $C^1(\mathbb{R}_+)$ and $C^1(\mathbb{R}_+ \times \mathbb{R}_+)$ respectively.

The reaction-diffusion system $(1.1)-(1.4)$ arises in the study of physical, chemical, and various biological processes including population dynamics (especially AIDS, see C. Castillo-Chavez et al. [4], for further details see [6] [11] [17] [21] [22]).

The case $\Pi = 0$, $\alpha = 0, \sigma = 0$ and $f(u,v) = h(u)Q(v)$, with $h(u) = u$ (for simplicity), has been studied by many authors. Alikakos [1] established the existence of global solutions when $Q(v) \leq C(1 + |v|^{(n+2)/n})$. Then Massuda [18] obtained a positive result for the case $Q(v) \leq C(1 + |v|^{\alpha})$ with arbitrary $\alpha > 0$. The question when $Q(v) = e^{\alpha v^{\beta}}$, $0 < \beta < 1$, $\alpha > 0$ was positively answered by Haraux and Youkana [13], using Lyapunov function techniques, see also Barabanova [2] for $\beta = 1$, with some conditions and later on by Kanel [16], using useful properties inherent to the Green function. For $Q(v) = e^{\alpha v^{\beta}}$, $\beta > 1$, Rebiai [3] proved the global existence. The idea behind the Lyapunov functional stems from Zelenyak's article [23], which has also been used by Crandall et al. [5] for other purposes.
The case $\Pi > 0$, $\alpha > 0, \sigma > 0$ L. Melkemi et al. [19] established the existence of global solutions, when $f(\xi, \tau) \leq \psi(\xi)\varphi(\tau)$ such that

$$\lim_{\tau \to +\infty} \frac{\ln(1 + \varphi(\tau))}{\tau} = 0.$$

For $f(v) = e^{\alpha v^{\beta}}$, $\beta > 1$, Djebara et al [9] showed the global existence.
The goal of this work is to generalize the existing result in [7], where it is proved the existence of global solutions with following exponential nonlinearity

$$0 \leq f(\xi, \tau) \leq \varphi(\xi)(\tau + 1)^{\lambda} e^{r\tau}, \tag{1.5}$$

with restriction on initial data

$$\max\left(\| u_0 \|_{\infty}, \frac{\Pi}{\alpha} \right) < \frac{\theta^2}{2 - \theta} \quad \frac{8ab}{rn(a-b)^2}. \tag{1.6}$$

Hence, the main purpose of this paper is to give a positive answer, concerning the global existence and the uniform boundedness in time, of solutions of system $(1.1) - (1.4)$, with out any restriction on inital data u_0 and v_0 and same exponential nonlinearity, i.e,

(S1) $\forall \tau \geq 0$, $f(0, \tau) = 0$,
(S2) $\forall \xi \geq 0$, $\forall \tau \geq 0$, $0 \leq f(\xi, \tau) \leq \varphi(\xi)(\tau + 1)^{\lambda} e^{r\tau}$,
(S3) $\kappa(\tau) = \tau^{\mu}$, $\mu \geq 1$,

where r, λ are positive constants, such that $\lambda \geq 1$, φ is a nonnegative function of class $C(\mathbb{R}^+)$.
For this end we use maximum principle and Lyapunov function techniques, and an idea inspired from [8].

2. Existence of local solutions

The usual norms in spaces $L^p(\Omega)$, $L^\infty(\Omega)$ and $C(\overline{\Omega})$ are respectively denoted by

$$\| u \|_p^p = \frac{1}{|\Omega|}\int_\Omega | u(x) |^p \, dx, \quad \| u \|_\infty = \max_{x\in\Omega} | u(x) | .$$

Concerning a local existence, we can conclude directly from the theory of abstract semilinear equations (see A. Friedman [10], D. Henry [14], A. Pazy [20]), that for nonnegative functions u_0 and v_0 in $L^\infty(\Omega)$, there exists a unique local nonnegative solution (u, v) of system (1.1) – (1.4) in $C(\overline{\Omega})$ on $]0, T^*[$, where T^* is the eventual blowing-up time.

3. Existence of global solutions

Using the comparison principle, one obtains

$$0 \leq u(t, x) \leq \max\left(\| u_0 \|_\infty, \frac{\Pi}{\alpha}\right) = M, \tag{3.1}$$

from which it remains to establish the uniform boundedness of v.

According to the results of [12], it is enough to show that

$$\| f(u, v) - \sigma\kappa(v) \|_p \leq C \tag{3.2}$$

(where C is a nonnegative constant independent of t) for some $p > \frac{n}{2}$. To reach this goal, let us start with this preliminaries results.

We consider the following reaction-diffusion system:

$$\frac{\partial u_1}{\partial t} - a_1\Delta u_1 = 1 - h(u_1, u_2) - u_1 \quad (x, t) \in \Omega_1 \times \mathbb{R}_+ \tag{3.3}$$

$$\frac{\partial u_2}{\partial t} - (2-\sqrt{3})a_1\Delta u_2 = h(u_1, u_2) - \delta u_2 \quad (x, t) \in \Omega_1 \times \mathbb{R}_+ \tag{3.4}$$

$$\frac{\partial u_1}{\partial \eta} = \frac{\partial u_2}{\partial \eta} = 0 \quad \text{on } \partial\Omega_1 \times \mathbb{R}_+, \tag{3.5}$$

$$u_1(0, x) = u_{1,0}(x) \geq 0; \quad u_2(0, x) = u_{2,0}(x) \geq 0 \quad \text{in } \Omega_1, \tag{3.6}$$

where Ω_1 is a smooth open bounded domain in $\mathbb{R}^2$, with boundary $\partial\Omega_1$ of class C^1 and η is the outer normal to $\partial\Omega_1$ and $a_1 > 0$ is the diffusion constant, δ is a positive constant and $\|u_{1,0}\|_\infty = \frac{1}{2}$, h is differentiable nonnegative function such that:

(A1) $\forall\tau \geq 0, \quad h(0, \tau) = 0,$

(A2) $\forall\xi \geq 0, \forall\tau \geq 0, \quad 0 \leq h(\xi, \tau) = \xi\varphi(\tau) \leq \xi(\tau + \alpha_1)e^{\frac{1}{16}\tau},$

where φ is differentiable nonnegative function and

$$\alpha_1 = \max\left(\frac{48}{5}, \left(\frac{3}{2}\frac{M}{|\Omega_1|}\right)^{\frac{1}{4}}\right). \tag{3.7}$$

Using the maximum principle, we obtain

$$0 \leq u_1(t, x) \leq 1. \tag{3.8}$$

To establish the boundness of u_2, we use the results of [14, 15], where it is enough to show that

$$\| h(u_1, u_2) - \delta u_2 \|_4 \leq C, \tag{3.9}$$

where C is a nonnegative constant independent of t. For this end we need the following

Lemma 3.1. *Let ϕ be a nonnegative function of class $C(\mathbb{R}^+)$, such that*

$$\lim_{\tau \to +\infty} \frac{\phi(\tau)}{\tau} = 0$$

and let A be positive constant. Then there exists $\Pi_2 > 0$, such that

$$\left[\frac{\phi(\tau)}{\tau} - A \right] \tau h_1(\tau) \leq \Pi_2, \tag{3.10}$$

for all $\tau > 0$; h_1 is a nonnegative function of class $C(\mathbb{R}^+)$.

Proof. Since

$$\lim_{\tau \to +\infty} \frac{\phi(\tau)}{\tau} = 0,$$

there exists $\tau_0 > 0$, such that for all $\tau > \tau_0$, we have

$$\left[\frac{\phi(\tau)}{\tau} - A \right] \tau h_1(\tau) \leq 0.$$

Now if τ is in the compact interval $[0, \tau_0]$, then the continuous function

$$[\phi(\tau) - A\tau] h_1(\tau)$$

is bounded. □

Lemma 3.2. *Assume that (A1) and (A2) hold and let (u_1, u_2) be a solution of (3.3)-(3.6) on $]0, T^*[$, with arbitrary $u_{2,0}$. Let*

$$G_1(t) = \int_{\Omega_1} \left(\frac{1}{\frac{3}{2} - u_1}\right)(u_2 + \alpha_1)^4 e^{\frac{1}{4}u_2} dx. \tag{3.11}$$

Then there exist a positive constant Π_1 such that

$$\frac{dG_1}{dt}(t) \leq -\sigma_1 G_1(t) + \Pi_1, \tag{3.12}$$

where σ_1 is a positive constant.

Proof. We put $q(u_1) = \left(\frac{1}{\frac{3}{2} - u_1}\right)$, so that

$$G_1(t) = \int_{\Omega_1} q(u_1)(u_2 + \alpha_1)^4 e^{\frac{1}{4}u_2} dx.$$

Differentiating G_1 with respect to t and a simple use of Green's formula gives

$$G_1'(t) = I_1 + J_1,$$

where

$$\begin{aligned} I_1 &= -a_1 \int_{\Omega_1} q''(u_1)(u_2+\alpha_1)^4 e^{\frac{1}{4}u_2} |\nabla u_1|^2 dx \\ &- (3-\sqrt{3})a_1 \int_{\Omega_1} q'(u)[4+\frac{1}{4}(u_2+\alpha_1)](u_2+\alpha_1)^3 e^{\frac{1}{4}u_2} \nabla u_1 \nabla u_2 dx \\ &- (2-\sqrt{3})a_1 \int_{\Omega_1} q(u)[12+2(u_2+\alpha_1)+\frac{1}{16}(u_2+\alpha_1)^2](u_2+\alpha_1)^2 e^{\frac{1}{4}u_2}|\nabla u_2|^2 dx, \\ J_1 &= \int_{\Omega_1} q'(u_1)(u_2+\alpha_1)^4 e^{\frac{1}{4}u_2} dx - \int_{\Omega_1} q'(u_1)u_1(u_2+\alpha_1)^4 e^{\frac{1}{4}u_2} dx \\ &+ \int_{\Omega_1} \Big(q(u_1)[4+\frac{1}{4}(u_2+\alpha_1)] - q'(u_1)(u_2+\alpha_1) \Big)(u_2+\alpha_1)^3 h(u_1,u_2) e^{\frac{1}{4}u_2} dx \\ &- \int_{\Omega_1} \delta[4+\frac{1}{4}(u_2+\alpha_1)]u_2(u_2+\alpha_1)^3 e^{\frac{1}{4}u_2} dx. \end{aligned}$$

I_1 involves a quadratic form with respect to ∇u_1 and ∇u_2, which is nonnegative if

$$(3-\sqrt{3})^2[4+\frac{1}{4}(u_2+\alpha_1)]^2 - 8(2-\sqrt{3})[12+2(u_2+\alpha_1)+\frac{1}{16}(u_2+\alpha_1)^2]$$

$$= \big[-2[4+\frac{1}{4}(u_2+\alpha_1)]^2+32\big](2-\sqrt{3}) = \big[1-[1+\frac{1}{16}(u_2+\alpha_1)]^2\big]32(2-\sqrt{3}) \le 0.$$

Concerning the second term J_1, we can observe that

$$\begin{aligned} J_1 &\le \int_{\Omega_1} (2-\frac{1}{4}\delta u_2)\frac{1}{\frac{3}{2}-u_1}(u_2+\alpha_1)^4 e^{\frac{1}{4}u_2} dx \\ &+ \int_{\Omega_1} \Big([4+\frac{1}{4}(u_2+\alpha_1)] - \frac{1}{\frac{3}{2}-u_1}(u_2+\alpha_1)\Big)\frac{1}{\frac{3}{2}-u_1}(u_2+\alpha_1)^3 h(u_1,u_2) e^{\frac{1}{4}u_2} dx. \end{aligned}$$

Now we introduce a positive constant σ_1, such that

$$\begin{aligned} J_1 &\le \int_{\Omega_1} -\sigma_1 \frac{1}{\frac{3}{2}-u_1}(u_2+\alpha_1)^4 e^{\frac{1}{4}u_2} + \Big(\frac{2+\sigma_1}{u_2} - \frac{1}{4}\delta\Big)\frac{1}{\frac{3}{2}-u_1} u_2(u_2+\alpha_1)^4 e^{\frac{1}{4}u_2} dx \\ &+ \int_{\Omega_1} (4-\frac{5}{12}\alpha_1)\frac{1}{\frac{3}{2}-u_1}(u_2+\alpha_1)^3 h(u_1,u_2) e^{\frac{1}{4}u_2} dx. \end{aligned}$$

using the Lemma 3.1 and the choice in the formula 3.7, let us get

$$J_1 \le -\sigma_1 G_1(t) + \Pi_2|\Omega_1|.$$

It follows that

$$\frac{dG_1(t)}{dt} \le -\sigma_1 G_1(t) + \Pi_1,$$

where $\Pi_1 = \Pi_2|\Omega_1|$. □

Theorem 3.3. *Under the assumptions* $(A1)$ *and* $(A2)$*, the solutions of* $(3.3)-(3.6)$ *are global and uniformly bounded on* $[0,+\infty[$.

Proof. Multiplying (3.12) by $e^{\sigma_1 t}$ and integrating the inequality on $(0,t)$, it implies the existence of a positive constant $C_3 > 0$ independent of t such that

$$G_1(t) \le C_3. \tag{3.13}$$

Then we have

$$\int_{\Omega_1} h^4(u_1,u_2)dx \le \frac{3}{2}G_1(t) \le \frac{3}{2}C_3. \tag{3.14}$$

□

Remark 3.4. From the choice (3.7) we have for all $t \ge 0$

$$G_1(t) \ge \int_{\Omega_1} \frac{2}{3}\alpha_1^4 dx \ge M. \tag{3.15}$$

3.1. Main result

Now, we will state the main result

Theorem 3.5. *Under the assumptions* $(S1)-(S3)$*, the solutions of (1.1)-(1.4) are global and uniformly bounded on* $[0,+\infty[$*.*

The key result needed to prove the Theorem 3.5 is the following

Proposition 3.6. *Assume that* $(S1)-(S3)$ *hold and let* (u,v) *be a solution of (1.1)-(1.4) on* $]0,T^*[$*, with arbitrary* v_0 *and* u_0*. Let*

$$G(t) = \int_\Omega \left(\frac{M}{(2-\theta)M-u}\right)^\beta (v+\omega)^{\gamma p} e^{prv} dx + G_1(\psi(t)), \tag{3.16}$$

where ω, β, γ *and* θ *are positive constants such that* $\omega \ge 1$, $\theta < 1$ *and*

$$\beta = \theta\frac{4ab}{(a-b)^2}, \quad \gamma = \max\left(\lambda, \mu, \frac{(\beta+1)(2-\theta)Mr}{\beta\theta(1-\theta)}\right) \tag{3.17}$$

and

$$\psi(t) = \int_0^t \int_\Omega f(u,v)g(u)(v+\omega)^{\gamma p} e^{prv} dxds. \tag{3.18}$$

Then, there exist $p > n/2$ *and positive constant* Γ *such that*

$$\frac{dG}{dt} \le -sG + \Gamma, \tag{3.19}$$

where s *is a positive constant.*

It's very important to state this lemma, before proving this proposition,

Lemma 3.7. *For all* $\tau \ge 0$ *we have*

$$\left[\frac{\Pi\beta}{(1-\theta)M} - \sigma p\kappa(\tau)(\frac{\gamma}{\tau+\omega} + r)\right](\tau+\omega)^{\gamma p} e^{pr\tau} \le -s(\tau+\omega)^{\gamma p} e^{pr\tau} + B_1, \tag{3.20}$$

where B_1 *and* s *are positive constants.*

Proof. Let us put

$$\xi = \frac{\Pi\beta}{(1-\theta)M} + s$$

$$\begin{aligned}
&\frac{\Pi\beta}{(1-\theta)M}(\tau+\omega)^{p\gamma}e^{pr\tau} - \sigma p\kappa(\tau)[\gamma(\tau+\omega)^{\gamma p-1} + r(\tau+\omega)^{\gamma p}]e^{pr\tau} \\
&= \left(\frac{\Pi\beta}{(1-\theta)M} - \xi\right)(\tau+\omega)^{p\gamma}e^{pr\tau} + \left(\frac{\xi}{\kappa(\tau)} - \sigma rp\right)\kappa(\tau)(\tau+\omega)^{\gamma p}e^{pr\tau},
\end{aligned}$$

then, using Lemma 3.1 we can conclude the result. □

Proof. (of Proposition 3.2). Let

$$g(u) = \left(\frac{M}{(2-\theta)M-u}\right)^{\beta},$$

so that

$$G(t) = \int_\Omega g(u)(v+\omega)^{\gamma p}e^{prv}dx + G_1(\psi(t)).$$

Differentiating G with respect to t and a simple use of Green's formula gives

$$G'(t) = I + J,$$

where

$$\begin{aligned}
I = &-a\int_\Omega g''(u)(v+\omega)^{\gamma p}e^{prv}|\nabla u|^2dx \\
&-(a+b)\int_\Omega g'(u)[\gamma p(v+\omega)^{\gamma p-1} + pr(v+\omega)^{\gamma p}]e^{prv}\nabla u\nabla v dx \\
&-b\int_\Omega g(u)[\gamma p(\gamma p-1)(v+\omega)^{\gamma p-2} + 2\gamma p^2 r(v+\omega)^{\gamma p-1} + p^2r^2(v+\omega)^{\gamma p}]e^{prv}|\nabla v|^2dx, \\
J = &\int_\Omega \Pi g'(u)(v+\omega)^{\gamma p}e^{prv}dx - \int_\Omega \alpha g'(u)u(v+\omega)^{\gamma p}e^{prv}dx \\
&+\int_\Omega \left(g(u)\big[\gamma p(v+\omega)^{\gamma p-1} + rp(v+\omega)^{\gamma p}\big] - g'(u)(v+\omega)^{\gamma p}\right)f(u,v)e^{prv}dx \\
&-\int_\Omega \sigma[\gamma p(v+\omega)^{\gamma p-1} + rp(v+\omega)^{\gamma p}]\kappa(v)g(u)e^{prv}dx + \psi'(t)G_1'(\psi(t)).
\end{aligned}$$

We can see that I involves a quadratic form with respect to ∇u and ∇v, which is nonnegative if

$$\begin{aligned}
\delta = &\big(p(a+b)g'(u)[\gamma(v+\omega)^{\gamma p-1} + r(v+\omega)^{\gamma p}]\big)^2 \\
&-4ab\gamma p(\gamma p-1)g''(u)g(u)(v+\omega)^{2\gamma p-2} \\
&-4abg''(u)g(u)(v+\omega)^{\gamma p}[2\gamma p^2r(v+\omega)^{\gamma p-1} + p^2r^2(v+\omega)^{\gamma p}] \le 0.
\end{aligned}$$

Indeed

$$\begin{aligned}
\delta = &[(p\gamma)^2(a+b)^2\beta^2 - 4ab\beta(\beta+1)p\gamma(p\gamma-1)]\frac{g(u)^2(v+\omega)^{2p\gamma-2}}{((2-\theta)M-u)^2} \\
&+[(a+b)^2\beta^2 - 4ab\beta(\beta+1)]\frac{rp^2g(u)^2(v+\omega)^{2p\gamma-1}}{((2-\theta)M-u)^2}[2\gamma + r(v+\omega)],
\end{aligned}$$

the choice of β and γ gives

$$\delta \leq [\beta + 1 - p\gamma(1-\theta)]\frac{4ab\beta p\gamma g(u)^2(v+\omega)^{2p\gamma-2}}{((2-\theta)M-u)^2} + 4ab(\theta-1)\frac{rp\beta g(u)^2(v+\omega)^{2p\gamma-1}}{((2-\theta)M-u)^2}[2+(rp)(v+\omega)] \leq 0,$$

it follows that

$$I \leq 0.$$

Concerning the second term J, we use (3.12), we can observe that

$$\begin{aligned} J \leq & \int_\Omega \Big(\frac{\Pi\beta}{(1-\theta)M} - \sigma p\kappa(v)[\frac{\gamma}{v+\omega} + r]\Big) g(u)(v+\omega)^{p\gamma} e^{prv} dx \\ & + \int_\Omega \Big(p[\frac{\gamma}{v+\omega} + r] - \frac{\beta}{(2-\theta)M-u}\Big) f(u,v) g(u)(v+\omega)^{\gamma p} e^{prv} dx \\ & + \psi'(t)\big(-\sigma_1 G_1(\psi(t)) + \Pi_1\big). \end{aligned}$$

Using Lemma 3.7 and by choosing $\sigma_1 = \frac{1}{M}(rp + \Pi_1)$, we get

$$\begin{aligned} J \leq & \int_\Omega [-s(v+\omega)^{p\gamma} e^{prv} + B_1] g(u) dx \\ & + \int_\Omega \Big(\frac{p\gamma}{v+\omega} - \frac{\theta}{2-\theta}\frac{4ab}{(a-b)^2 M}\Big) f(u,v) g(u)(v+\omega)^{\gamma p} e^{prv} dx. \end{aligned}$$

Since f is continuous function, applying the Lemma 3.1, it follows that there exist a positive constant N_1 such that

$$\begin{aligned} J \leq & \int_\Omega [-s(v+\omega)^{p\gamma} e^{prv} + B_1] g(u) dx \\ & + N_1 \int_\Omega g(u) dx. \end{aligned}$$

In addition

$$g(u) \leq \Big(\frac{1}{1-\theta}\Big)^\beta,$$

then

$$J \quad \leq \quad -sG(t) + (\mid \Omega \mid B_1 + N_1)\Big(\frac{1}{1-\theta}\Big)^\beta + sC_3,$$

it follows that

$$\frac{dG}{dt} \leq -sG + \Gamma,$$

where $\Gamma = (\mid \Omega \mid B_1 + N_1)\big(\frac{1}{1-\theta}\big)^\beta + sC_3$. □

Proof. (of Theorem 3.5)
Multiplying (3.19) by e^{st} and integrating the inequality, it implies the existence of a positive constant $C_1 > 0$ independent of t such that

$$G(t) \leq C_1.$$

Since

$$g(u) \geq \left(\frac{1}{2-\theta}\right)^{\beta},$$

$$\int_{\Omega} (v+\omega)^{\gamma p} e^{prv} dx \leq (2-\theta)^{\beta} G(t) \leq C_1 (2-\theta)^{\beta}.$$

Since $\omega \geq 1$ and (3.17) we have also,

$$\int_{\Omega} (v+1)^{\lambda p} e^{prv} dx \leq \int_{\Omega} (v+\omega)^{\gamma p} e^{prv} dx \leq C_1 (2-\theta)^{\beta},$$

$$\int_{\Omega} v^{\mu p} dx \leq \int_{\Omega} (v+\omega)^{\gamma p} dx \leq C_1 (2-\theta)^{\beta}.$$

We put

$$A = \max_{0 \leq \xi \leq M} \varphi(\xi),$$

according to $(S1) - (S3)$, we have

$$\int_{\Omega} f(u,v)^p dx \leq \int_{\Omega} A^p (v+1)^{\lambda p} e^{prv} dx \leq A^p C_1 (2-\theta)^{\beta} = A^p H^p,$$

we conclude

$$\|f(u,v) - \sigma\kappa(v)\|_p \leq \|f(u,v)\|_p + \|\sigma\kappa(v)\|_p \leq H(A+\sigma).$$

By the preliminary remarks (introduction of section 3), we conclude that the solution of (1.1)-(1.4) is global and uniformly bounded on $[0, +\infty[\times \Omega$. □

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