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Global nonexistence of solutions to a logarithmic nonlinear wave equation with infinite memory and delay term

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Abstract. As a continuity to the study by M. Kafini [24], we consider a logarithmic nonlinear wave condition with delay term. We obtain a blow-up result of solutions under suitable conditions.

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1. Introduction

In this paper, we are concerned with the blow-up in finite-time of solutions for the initial boundary value problem:

$$\begin{gathered} u_{tt} - \Delta u + \int_0^\infty g(t-s)\Delta u(x,s)ds + \mu_1 u_t(x,t) \\ +\mu_2 u_t(x,t-\tau) = u|u|^{p-2}\ln|u|^k, \quad \text{in } \Omega\times(0,\infty), \\ u(x,t)=0, \quad x\in\partial\Omega, \end{gathered} \tag{1.1}$$

and the initial conditions

$$\begin{gathered} u_t(x,t-\tau) = f_0(x,t-\tau), \quad \text{in } (0,\tau), \\ u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \quad x\in\Omega. \end{gathered}$$

where $u = u(x,t)$, $t \geq 0$, $x \in \Omega$, Δ means the Laplacian administrator regarding the x variable, Ω is an ordinary and limited area of $\mathbb{R}^n$, $n \geq 1$, $p \geq 2$, k, μ_1, are positive constants, μ_2 is a genuine number, $\tau > 0$ speak to the time delay. The capacity $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded C^1 function, the unwinding capacity exposed to

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conditions to be determined and u_0, u_1, f_0 are given capacities having a place with reasonable spaces.

Presenting the defer term $\mu_2 u_t(x, t-\tau)$ makes the issue unique in relation to those considered in the writing.

In [24] the nonappearance of the viscoelastic term $(g = 0)$, the issue has been widely examined and numerous outcomes concerning neighborhood presence result has been set up utilizing the semigroup hypothesis. Likewise, for negative introductory energy, a limited time explode result is demonstrated. For example, for the condition

$$u_{tt} - \Delta u + \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau) = u|u|^{p-2} \ln |u|^k, \quad \text{in } \Omega \times (0, \infty). \tag{1.2}$$

In [22], Han studied the global existence of weak solutions for the initial boundary value problem

$$\begin{gathered} u_{tt} - \Delta u + u - u \ln |u|^2 + u_t + u|u|^2 = 0, \quad in\ \Omega \times (0,T), \\ u(x,t) = 0, \quad x \in \partial\Omega, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega. \end{gathered} \tag{1.3}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$. The model (1.1) is closely related to the following equation with logarithmic nonlinearity

$$\begin{gathered} u_{tt} - u_{xx} + u - \varepsilon u \ln |u|^2 + u_t = 0, \text{ in } O \times (0,T), \\ u(x,t) = 0, \quad x \in \partial O, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in O. \end{gathered} \tag{1.4}$$

where $O = [a, b]$, the parameter $\varepsilon \in [0, 1]$ [22].

The remainder of our paper is coordinated as follows. In section 2, we review the documentation, speculations, and some fundamental primers. In section 3, we demonstrate the globale nonexistence result utilizing the semigroup hypothesis [24]. In section 4, we present the statement and the proof of our main blow-up result.

2. Preliminaries and assumptions

In this section, we give notations, hypotheses, $(.,.)$ and $\|.\|_p$ denote the inner prodution in the space $L(\Omega)$ and the norm of the space $L^p(\Omega)$, respectively. For breviy, we denote $\|.\|_2$ by $\|.\|$.
For the relaxation function g we assume the following.
(G) : We assume that the function $g : \mathbb{R}^+ \to \mathbb{R}$ is of class C^1 satisfying:

$$1 - \int_0^\infty g(s)ds = l > 0, \quad g(t) \geq 0, \quad g'(t) \leq 0.$$

and under the assumption

$$\mu_1 \geq |\mu_2|.$$

By using the direct calculations, we have

$$\begin{aligned} \int_0^\infty g(t-s) \left(\nabla u_t(t), \nabla u(s)\right) ds = &-\frac{1}{2} g(t) \|u(t)\|_2^2 + \frac{1}{2}(g' \circ \nabla u)(t) \\ &- \frac{1}{2}\frac{d}{dt}\left[(g \circ \nabla u)(t) - \left(\int_0^\infty g(s)ds\right) \|\nabla u(t)\|_2^2\right] \end{aligned}$$

where

$$(g \circ u)(t) = \int_0^\infty g(t-s) \|u(t) - u(s)\|_2^2 \, ds.$$

3. Local existence

We introduce the variable

$$z(x,\rho,t) = u_t(x, t-\tau\rho), \ (x,\rho,t) \in \ \Omega \times (0,1) \times (0,\infty).$$

Consequently, we have

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \ (x,\rho,t) \in \ \Omega \times (0,1) \times (0,\infty).$$

Therefore, problem (1.1) is equivalent to:

$$\begin{gathered} u_{tt}(x,t) - \Delta u(x,t) + \int_0^\infty g(t-s)\Delta u(x,s)ds + \mu_1 u_t(x,t) \\ +\mu_2 z(x,1,t) = u(x,t)|u(x,t)|^{p-2} ln|u(x,t)|^k, \ \text{ in } \Omega \times (0,\infty), \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad x \in \Omega, \quad \rho \in (0,1), \ t > 0, \end{gathered} \tag{3.1}$$

and the initial conditions

$$\begin{gathered} z(x,1,t) = f_0(x, t-\tau), \ \text{ in } \Omega \times (0,1), \\ u(x,t) = 0, \quad x \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x \in \Omega. \end{gathered}$$

Let $\upsilon = u_t$ and denote by

$$\Phi = (u, \upsilon, z)^T, \quad \Phi(0) = \Phi_0 = (u_0, u_1, f_0(.,-\rho\tau))^T,$$

Then Φ satisfies the problem

$$\begin{gathered} \partial_t \Phi + A\Phi = J(\Phi) \\ \Phi(0) = \Phi_0, \end{gathered} \tag{3.2}$$

where the operator $A : D(A) \longrightarrow \mathcal{H}$ is defined by

$$A\Phi = \begin{pmatrix} -\upsilon \\ -\Delta u + \mu_1 \upsilon + \mu_2 z(1,.) + \int_0^\infty g(t-s)\Delta u(x,s)ds \\ \frac{1}{\tau} z_\rho \end{pmatrix}$$

and

$$J(\Phi) = \left(0, \ u|u|^{p-2} ln|u|^k, \ 0\right)^T.$$

We introduce the following Hilbert space:

$$\mathcal{H} = \left(H_0^1(\Omega) \cap L_g^2(\mathbb{R}^+, H_0^1(\Omega))\right) \times L^2(\Omega) \times L^2(\Omega \times (0,1)),$$

where $L_g^2(\mathbb{R}^+, H_0^1(\Omega))$ denotes the Hilbert space of H_0^1-valued functions on $\mathbb{R}^+$, endowed with the inner product

$$\langle \phi, \vartheta \rangle_{L_g^2(\mathbb{R}^+, H_0^1(\Omega))} = \int_\Omega \int_0^\infty g\,(t-s)\, \nabla\phi(x,s) \nabla\vartheta(x,s) ds dx.$$

We define the inner product in the energy space $\mathcal{H}$,

$$\left\langle \Phi, \widetilde{\Phi} \right\rangle_{\mathcal{H}} = \int_{\Omega} (\nabla u \nabla \widetilde{u} + \upsilon \widetilde{\upsilon}) dx + \tau |\mu_2| \int_0^1 \int_{\Omega} z \widetilde{z} dx d\rho + \int_{\Omega} \int_0^{\infty} g(t-s) \nabla u \nabla \widetilde{u} ds dx,$$

for all $\Phi = (u, \upsilon, z)^T$ and $\widetilde{\Phi} = (\widetilde{u}, \widetilde{\upsilon}, \widetilde{z})^T$ in $\mathcal{H}$. The domain of A is

$$D(A) = \left\{ \begin{array}{c} \Phi \in H : \ u \in H^2(\Omega) \cap L_g^2(\mathbb{R}^+, H_0^1(\Omega)), \ \upsilon \in H_0^1(\Omega), \ z(1,.) \in L^2(\Omega), \\ z, \ z_\rho \in L^2 (\Omega \times (0,1)), \ z(0,.) = \upsilon. \end{array} \right\}.$$

Lemma 3.1. [24] *For every ε, there exists $A > 0$, such that the real function*

$$j(s) = |s|^{p-2} ln|s|, \quad p > 2,$$

satisfies

$$|j(s)| \leq A + |s|^{p-2+\varepsilon}.$$

We have the following existence and uniqueness result:

Theorem 3.2. *Assume that $\mu_1 \geq |\mu_2|$ and p be such that*

$$\left\{ \begin{array}{lll} 2 < p < \infty & & if \ \ n = 1, 2, \\ 2 < p < \frac{2(n-1)}{n-2} & & if \ \ n \geq 3. \end{array} \right. \tag{3.3}$$

Then for any $\Phi_0 \in \mathcal{H}$, problem (3.2) has a unique weak solution $\Phi \in C([0,T]; \mathcal{H})$.

Proof. First, for all $\Phi \in D(A)$, we have

$$\begin{aligned} \langle A\Phi, \Phi \rangle_{\mathcal{H}} &= \left\langle \left(\begin{array}{c} -\upsilon \\ -\Delta u + \mu_1 \upsilon + \mu_2 z(1,.) + \int_0^{\infty} g(t-s) \Delta u(x,s) ds \\ \frac{1}{\tau} z_\rho \end{array} \right), \left(\begin{array}{c} u \\ \upsilon \\ z \end{array} \right) \right\rangle \\ &= -\int_{\Omega} \nabla u \nabla \upsilon dx + \int_{\Omega} \upsilon \left[-\Delta u + \mu_1 \upsilon + \mu_2 z(1,.) + \int_0^{\infty} g(t-s) \Delta u(x,s) ds \right] dx \\ &+ |\mu_2| \int_0^1 \int_{\Omega} z z_\rho dx d\rho + \int_{\Omega} \int_0^{\infty} g(t-s) \nabla u \nabla \upsilon ds dx \\ &= \mu_1 \int_{\Omega} |\upsilon|^2 dx + \mu_2 \int_{\Omega} \upsilon z(1,.) dx + \frac{|\mu_2|}{2} \int_{\Omega} |z(1,.)|^2 dx - \frac{|\mu_2|}{2} \int_{\Omega} |\upsilon|^2 dx \\ &+ \int_{\Omega} \upsilon(x,t) \left(\int_0^{\infty} g(t-s) \Delta u(x,s) ds \right) dx \\ &+ \int_{\Omega} \int_0^{\infty} g(t-s) \nabla u(x,s) \nabla \upsilon(x,s) ds dx. \end{aligned} \tag{3.4}$$

Looking now at the last term on the right-hand side of (3.4), we have

$$\begin{aligned} |\mu_2| \int_0^1 \int_{\Omega} z_\rho(x,\rho) z(x,\rho) dx d\rho &= |\mu_2| \int_{\Omega} \int_0^1 \frac{1}{2} \frac{\partial}{\partial \rho} z^2(x,\rho) d\rho dx \\ &= \frac{|\mu_2|}{2} \int_{\Omega} \left(z^2(x,1) - z^2(x,0) \right) dx. \\ &= \frac{|\mu_2|}{2} \int_{\Omega} \left(z^2(x,1) - \upsilon^2 \right) dx. \end{aligned}$$

Using Young's inequality, estimate (3.4) becomes

$$-\mu_2 vz \leq \frac{|\mu_2|}{2}|v|^2 + \frac{|\mu_2|}{2}|z|^2.$$

By combining all the estimates,

$$\begin{aligned}\langle A\Phi, \Phi\rangle_{\mathcal{H}} &\geq (\mu_1 - |\mu_2|)\int_\Omega |v|^2 dx + \frac{1}{2}\int_\Omega g(t)\left(\nabla u(x,t)\right)^2 dx \\ &\quad -\frac{1}{2}\int_\Omega\int_0^\infty g'(t-s)\left(\nabla u(x,t) - \nabla u(x,s)\right)^2 dxds \\ &\geq 0.\end{aligned}$$

Therefore, A is a monotone operator.
Next, we prove the operator A is maximal. It is sufficient to show that the operator $(I+A)$ is subjective. Indeed, for any $F = (f_1, f_2, f_3)^T \in \mathcal{H}$, we prove that there exists a unique $V = (u, v, z)^T \in D(A)$ such that

$$(I+A)V = F.$$

Or, equivalently

$$\begin{aligned} u - v &= f_1 \\ v - \Delta u + \mu_1 v + \mu_2 z(1,.) + \int_0^\infty g(t-s)\Delta u(x,s)ds &= f_2 \\ \tau z + z_\rho &= \tau f_3. \end{aligned} \tag{3.5}$$

Noting that $v = u - f_1$, we deduce, from $(3.5)_3$, that

$$z(\rho,.) = (u - f_1)e^{-\rho\tau} + \tau e^{-\rho\tau}\int_0^t f_3(\tau,.)e^{\gamma\tau}d\gamma. \tag{3.6}$$

Substituting (3.6) in $(3.5)_2$, we obtain

$$\sigma u - \Delta u + \int_0^\infty g(t-s)\Delta u(x,s)ds = f_2, \tag{3.7}$$

where,

$$\sigma = 1 + \mu_1 + \mu_2 e^{-\tau} > 0,\ G = f_2 + \sigma f_1 - \tau\mu_2 e^{-\tau}\int_0^1 f_3(\tau,.)e^{\gamma\tau}d\gamma \in L^2(\Omega). \tag{3.8}$$

Now we define, over $H_0^1(\Omega)$, the bilinear and linear forms

$$B(u,w) = \sigma\int_\Omega uw + \left(1 - \int_0^\infty g(s)ds\right)\int_\Omega \nabla u.\nabla w, \quad L(w) = \int_\Omega f_2 w.$$

Thus, for some $\alpha > 0$

$$B(u,u) \geq \alpha\|u\|^2_{H_0^1(\Omega)}$$

Thus B is coercive and L is continuous on $H_0^1(\Omega)$. According to Lax-Milgram Theorem, we can easily obtain unique

$$u \in H_0^1(\Omega),$$

satisfying

$$B(u,w) = L(w), \quad \forall w \in H_0^1(\Omega). \tag{3.9}$$

Consequently, $\upsilon = u - f_1 \in H_0^1(\Omega)$, $\upsilon = u - f_1 \in H_0^1(\Omega)$ and, $z_\rho \in L^2(\Omega \times (0,1))$. Thus, $V \in H$. Using (3.9), we get

$$\sigma \int_\Omega uwdx + \left(1 - \int_0^\infty g(s)ds\right) \int_\Omega \nabla u . \nabla w dx = \int_\Omega Gwdx, \quad w \in H_0^1(\Omega).$$

The standard elliptic regularity theory, gives $u \in H^2(\Omega)$. And using Green's formula and $(3.5)_2$, we obtain

$$\int_\Omega \left[(1+\mu_1)\, \upsilon - \Delta u + \int_0^\infty g(t-s)\Delta u(x,s)ds + \mu_2 z(1,.) - f_2\right] wdx = 0, \ \forall w \in H_0^1(\Omega).$$

Hence,

$$(1+\mu_1)\, \upsilon - \Delta u + \int_0^\infty g(t-s)\Delta u(x,s)ds + \mu_2 z(1,.) = f_2 \in L^2(\Omega)$$

Therefore,

$$V = (u, \upsilon, z)^T \in D(A).$$

Consequently, $I + A$ is surjective and then A is maximal.
We prove that $J : H \to H$ is locally Lipschitz. So, if we set

$$F(s) = |s|^{p-2} s ln|s|^k \text{ then } F'(s) = k\left[1 + (p-1)ln|s|\right]|s|^{p-2}.$$

Therefore,

$$\begin{aligned} \left\| J(\Phi) - J(\widetilde{\Phi}) \right\|_H^2 &= \left\| (0, u|u|^{p-2} ln|u|^k - \widetilde{u}|\widetilde{u}|^{p-2} ln|\widetilde{u}|^k, 0) \right\|_{\mathcal{H}}^2 \\ &= \left\| u|u|^{p-2} ln|u|^k - \widetilde{u}|\widetilde{u}|^{p-2} ln|\widetilde{u}|^k \right\|_{L^2(\Omega)}^2 \\ &= \| F(u) - F(\widetilde{u}) \|_{L^2(\Omega)}^2 . \end{aligned} \tag{3.10}$$

Consequently, using value theorem, we have, for $0 \leq \theta \leq 1$,

$$\begin{aligned} |F(u) - F(\widetilde{u})| &= |F'(\theta u + (1-\theta)\widetilde{u})(u - \widetilde{u})| \\ &\leq k\left[1 + (p-1) ln(\theta u + (1-\theta)\widetilde{u})\right] |\theta u + (1-\theta)\widetilde{u}|^{p-2} |u - \widetilde{u}| \\ &\leq k|\theta u + (1-\theta)\widetilde{u}|^{p-2} |u - \widetilde{u}| + k(p-1)\,|j(\theta u + (1-\theta)\widetilde{u})|\,|u - \widetilde{u}| \end{aligned}$$

By Lemma 3.1, we find

$$\begin{aligned} |F(u) - F(\widetilde{u})| &\leq k|\theta u + (1-\theta)\widetilde{u}|^{p-2} |u - \widetilde{u}| + k(p-1)A|u - \widetilde{u}| \\ &\quad + k(p-1)\,|\theta u + (1-\theta)\widetilde{u}|^{p-2+\varepsilon} |u - \widetilde{u}| \\ &\leq k(|u| + |\widetilde{u}|)^{p-2} |u - \widetilde{u}| + k(p-1)A|u - \widetilde{u}| \\ &\quad + k(p-1)(|u| + |\widetilde{u}|)^{p-2+\varepsilon} |u - \widetilde{u}| . \end{aligned} \tag{3.11}$$

As u, $\widetilde{u} \in H^1(\Omega)$, we then applying Hölder's inequality and the Sobolev embedding

$$H_0^1(\Omega) \hookrightarrow L^r(\Omega), \quad \forall\, 1 \leq r \leq \frac{2n}{n-2},$$

to get

$$\begin{aligned}
&\int_\Omega \left[(|u|+|\widetilde{u}|)^{p-2}|u-\widetilde{u}|\right]^2 dx \\
&\quad= \int_\Omega (|u|+|\widetilde{u}|)^{2(p-2)}|u-\widetilde{u}|^2 dx \\
&\quad\le C\left(\int_\Omega (|u|+|\widetilde{u}|)^{2(p-1)} dx\right)^{\frac{(p-2)}{(p-1)}} \times \left(\int_\Omega |u-\widetilde{u}|^{2(p-1)} dx\right)^{\frac{1}{(p-1)}} \\
&\quad\le C\left[\|u\|^{2(p-1)}_{L^{2(p-1)}(\Omega)} + \|\widetilde{u}\|^{2(p-1)}_{L^{2(p-1)}(\Omega)}\right]^{\frac{(p-2)}{(p-1)}} \times \|u-\widetilde{u}\|^2_{H^1_0(\Omega)}. \qquad (3.12)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_\Omega \left[(|u|+|\widetilde{u}|)^{p-2+\varepsilon}|u-\widetilde{u}|\right]^2 dx \\
&\quad= \int_\Omega (|u|+|\widetilde{u}|)^{2(p-2+\varepsilon)}|u-\widetilde{u}|^2 dx \\
&\quad\le \left(\int_\Omega (|u|+|\widetilde{u}|)^{\frac{2(p-2+\varepsilon)(p-1)}{(p-2)}} dx\right)^{\frac{(p-2)}{(p-1)}} \times \left(\int_\Omega |u-\widetilde{u}|^{2(p-1)} dx\right)^{\frac{1}{(p-1)}} \\
&\quad\le \left(\int_\Omega (|u|+|\widetilde{u}|)^{\frac{2(p-2+\varepsilon)(p-1)}{(p-2)}} dx\right)^{\frac{(p-2)}{(p-1)}} \times \|u-\widetilde{u}\|^2_{L^{2(p-1)}(\Omega)}. \qquad (3.13)
\end{aligned}$$

Since, $p < \frac{2(n-1)}{(n-2)}$, we can choose $\varepsilon > 0$ so small that

$$p^* = 2(p-1) + \frac{2\varepsilon(p-1)}{p-2} \le \frac{2n}{n-2}.$$

Therefore, we have

$$\begin{aligned}
&\int_\Omega (|u|+|\widetilde{u}|)^{2(p-2+\varepsilon)}|u-\widetilde{u}|^2 dx \\
&\qquad\le C\left[\|u\|^{p^*}_{L^{p^*}(\Omega)} + \|\widetilde{u}\|^{p^*}_{L^{p^*}(\Omega)}\right]^{\frac{(p-2)}{(p-1)}} \times \|u-\widetilde{u}\|^2_{L^{2(p-1)}(\Omega)} \\
&\qquad\le C\left[\|u\|^{p^*}_{H^1_0(\Omega)} + \|\widetilde{u}\|^{p^*}_{H^1_0(\Omega)}\right]^{\frac{(p-2)}{(p-1)}} \times \|u-\widetilde{u}\|^2_{H^1_0(\Omega)}. \qquad (3.14)
\end{aligned}$$

A combination with (3.9) – (3.14) gives

$$\begin{aligned}
&\left\|J(\Phi)-J(\widetilde{\Phi})\right\|^2_H \\
&\le \left[k^2(p-1)^2A^2\right]\|u-\widetilde{u}\|^2_{H^1_0(\Omega)} \\
&+C\left[\left(\|u\|^{2(p-1)}_{H^1_0(\Omega)} + \|\widetilde{u}\|^{2(p-1)}_{H^1_0(\Omega)}\right)^{\frac{(p-2)}{(p-1)}} + \left(\|u\|^{p^*}_{H^1_0(\Omega)} + \|\widetilde{u}\|^{p^*}_{H^1_0(\Omega)}\right)^{\frac{(p-2)}{(p-1)}}\right] \times \|u-\widetilde{u}\|^2_{H^1_0(\Omega)} \\
&\le C\left(\|u\|_{H^1_0(\Omega)} + \|\widetilde{u}\|_{H^1_0(\Omega)}\right)\|u-\widetilde{u}\|^2_{H^1_0(\Omega)},
\end{aligned}$$

since

$$\left\|J(\Phi)-J(\widetilde{\Phi})\right\|_H^2 \leq K\left\|u-\widetilde{u}\right\|_H^2 .$$

Hence, J is locally Lipschitz. See [25]. This completes the proof of Theorem 3.2. □

Remark 3.3. The weak solution is taken in the sense of [29]. That is, a function

$$\Phi=(u,u_t,z)\in C([0,T);H),$$

satisfying, for a.e $x\in\Omega$,

$$\begin{aligned}\frac{d}{dt}\int_\Omega u_t(x,t)w\,(x)&+\int_\Omega \nabla u(x,t).\nabla w\,(x)\,dx\\&-\int_\Omega\left[\left(\int_0^\infty g(t-s)\nabla u(x,s).\nabla w\,(x)\,ds\right)\right]dx\\&+\mu_1\frac{d}{dt}\int_\Omega u_t(x,t)w\,(x)\,dx+\mu_2\int_\Omega z(x,1,t)w\,(x)\,dx\\&=\int_\Omega u(x,t)|u(x,t)|^{p-2}\ln|u(x,t)|^k w\,(x)\,dx.\end{aligned}\tag{3.15}$$

for all $(w,\psi)\in H_0^1(\Omega)\times L^2(\Omega\times(0,1))$.

4. Main result

Our main blow-up result reads as follows.

Lemma 4.1. *Now, we introduce the energy functional defined by*

$$\begin{aligned}E(t):=\frac{1}{2}\left\|u_t\right\|_2^2&+\frac{1}{2}\left(1-\int_0^\infty g(s)ds\right)\left\|\nabla u\right\|_2^2+\frac{1}{2}(g\circ\nabla u)\,(t)+\frac{k}{p}\left\|\nabla u\right\|_p^p\\&+\frac{\xi}{2}\int_\Omega\int_0^1|z(x,\rho,t)|^2\,dxd\rho-\frac{1}{p}\int_\Omega|u|^p\ln|u|^k dx,\end{aligned}$$

where

$$\tau|\mu_2|<\xi<\tau(2\mu_1-|\mu_2|),\quad \mu_1>|\mu_2|,\tag{4.1}$$

satisfies the estimate

$$\begin{aligned}E'(t)\leq -C_0&\left[\int_\Omega\left(|u_t|^2+|z(x,1,t)|^2\right)dx\right]\\&-\frac{1}{2}g(t)\int_\Omega|\nabla u\,(t,x)|^2\,dx+\frac{1}{2}(g'\circ\nabla u)(t)\leq 0.\end{aligned}\tag{4.2}$$

Proof. We approximate the initial data $(u_0,u_1,f_0(.,-\rho\tau))$ by a sequence

$$(u_0^v,u_1^v,f_0^v)\in C_0^\infty(\Omega)\times C_0^\infty(\Omega)\times C_0^\infty(\Omega\times(0,1)).$$

Then problem (3.1) has a unique classical solution $(u^{\upsilon}, u_t^{\upsilon}, z^{\upsilon})$ such that (3.15) takes the form

$$\begin{aligned}\frac{d}{dt}\int_{\Omega} u_t^{\upsilon}(x,t)w\,(x) \quad &+ \quad \int_{\Omega} \nabla u^{\upsilon}(x,t).\nabla w\,(x)\,dx \\ &- \quad \int_{\Omega}\left[\left(\int_0^{\infty} g(t-s)\nabla u^{\upsilon}(x,s).\nabla w\,(x)\,ds\right)\right]dx \\ &+ \quad \mu_1 \frac{d}{dt}\int_{\Omega} u^{\upsilon}(x,t)w\,(x)\,dx \\ &+ \quad \mu_2 \int_{\Omega} z^{\upsilon}(x,1,t)w\,(x)\,dx \\ &= \quad \int_{\Omega} u^{\upsilon}(x,t)|u^{\upsilon}(x,t)|^{p-2}\ln|u^{\upsilon}(x,t)|^{k} w\,(x)\,dx, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned}\frac{d}{dt}\int_{\Omega}\int_0^1 \tau z^{\upsilon}(x,\rho,t)\psi\,(x,\rho)\,dxd\rho \quad &+ \quad \int_{\Omega} z^{\upsilon}(x,\rho,t)\psi\,(x,\rho)\,dx \\ &= \quad \int_{\Omega} u_t^{\upsilon}(x,t)\psi\,(x,\rho)\,dx. \end{aligned} \tag{4.4}$$

By replacing w by u_t^{υ} and ψ by z^{υ} and integrating over $(0,\infty)$, we obtain

$$\begin{aligned} &\frac{1}{2}\int_{\Omega}\left(|u_t^{\upsilon}(x,t)|^2 + |\nabla u^{\upsilon}(x,t)|^2\right)dx \\ &\qquad +\frac{1}{2}\left[(g\circ\nabla u^{\upsilon})(t) - \left(\int_0^{\infty} g(s)ds\right)\|u^{\upsilon}\,(t)\|_2^2\right] + \mu_1\int_{\Omega}\int_0^1 |u_t^{\upsilon}(x,s)|^2\,ds \\ &= \quad \frac{1}{2}\int_{\Omega}\left(|u_1^{\upsilon}(x)|^2 + |\nabla u_0^{\upsilon}(x)|^2\right)dx - \frac{1}{2}\int_0^{\infty} g(s)\,\|u^{\upsilon}\,(s)\|_2^2\,ds + \frac{1}{2}(g'\circ\nabla u^{\upsilon})(t) \\ &= \quad \frac{1}{2}\int_{\Omega}\left(|u_1^{\upsilon}(x)|^2 + |\nabla u_0^{\upsilon}(x)|^2\right)dx \\ &\qquad -\frac{1}{2}\int_0^{\infty} g(s)\,\|u^{\upsilon}\,(s)\|_2^2\,ds + \frac{1}{2}(g'\circ\nabla u^{\upsilon})(t) \\ &\qquad -\mu_2\int_{\Omega}\int_0^1 z^{\upsilon}(x,1,s)u_t^{\upsilon}(x,s)dxds \\ &\qquad +\frac{1}{p}\int_{\Omega}\left[(u^{\upsilon}(x,t))^p\ln|u^{\upsilon}(x,t)|^k - k|u^{\upsilon}(x,t)|^p\right]dx \\ &\qquad -\frac{1}{p}\int_{\Omega}\left[(u_0^{\upsilon}(x))^p\ln|u_0^{\upsilon}(x)|^k - k|u_0^{\upsilon}(x)|^p\right]dx \end{aligned} \tag{4.5}$$

and

$$\tau z_t(x,\rho,t)z(x,\rho,t) + z_{\rho}(x,\rho,t)z(x,\rho,t) = 0$$

integrating over $(0,\infty)$ and $\rho \in (0,1)$, then

$$\begin{aligned}\frac{\xi}{2}\int_0^1\int_\Omega |z^v(x,\rho,t)|^2\,dxd\rho &= \frac{\xi}{2}\int_0^1\int_\Omega |f_0(x,-\rho\tau)|^2\,dxd\rho \\ &+\frac{\xi}{2\tau}\int_0^\infty\int_\Omega |u_t^v(x,t)|^2\,dxd\rho \\ &-\frac{\xi}{2\tau}\int_0^\infty\int_\Omega |z^v(x,1,s)|^2\,dxds \end{aligned}\tag{4.6}$$

where $\xi > 0$ is defined in (4.1). Also integration by parts, we get

$$\begin{aligned}&\int_\Omega \left[u_t(x,t)\left(\int_0^\infty g(t-s)\Delta u(x,s)ds\right)\right]dx \\ &\qquad = -\int_0^\infty \left[g(t-s)\left(\int_\Omega \nabla u_t(x,t).\nabla u(x,s)dx\right)\right]ds,\end{aligned}\tag{4.7}$$

and using

$$\begin{aligned}-\nabla u_t(x,t).\nabla u(x,s) &= \frac{1}{2}\frac{d}{dt}\left\{|\nabla u(x,s)-\nabla u(x,t)|^2\right\} \\ &-\frac{1}{2}\frac{d}{dt}\left\{|\nabla u(x,t)|^2\right\},\end{aligned}\tag{4.8}$$

then

$$\begin{aligned}-&\int_0^\infty \left[g(t-s)\left(\int_\Omega \nabla u_t(x,t).\nabla u(x,s)dx\right)\right]ds \\ &=\frac{1}{2}\int_0^\infty \left[g(t-s)\left(\int_\Omega \frac{d}{dt}\left\{|\nabla u(x,s)-\nabla u(x,t)|^2\right\}dx\right)\right]ds \\ &-\frac{1}{2}\int_0^\infty \left[g(t-s)\left(\frac{d}{dt}\left\{|\nabla u(x,t)|^2\right\}dx\right)\right]ds.\end{aligned}\tag{4.9}$$

Using the direct account and (G), we find

$$\begin{aligned}&\frac{1}{2}\int_0^\infty \left[g(t-s)\left(\int_\Omega \frac{d}{dt}\left\{|\nabla u(x,s)-\nabla u(x,t)|^2\right\}dx\right)\right]ds \\ &=\frac{1}{2}\int_0^\infty \left[\left(\frac{d}{dt}\left(g(t-s)\left(\int_\Omega |\nabla u(x,s)-\nabla u(x,t)|^2\,dx\right)\right)\right)\right]ds \\ &-\frac{1}{2}\int_0^\infty g'(t-s)\int_\Omega\left(\int_\Omega |\nabla u(x,s)-\nabla u(x,t)|^2\,dx\right)ds \\ &=\frac{1}{2}\frac{d}{dt}\left[\int_0^\infty g(t-s)\int_\Omega |\nabla u(x,s)-\nabla u(x,t)|^2\,dxds\right] \\ &-\frac{1}{2}\int_0^\infty g'(t-s)\left(\int_\Omega |\nabla u(x,s)-\nabla u(x,t)|^2\,dx\right)ds \\ &=\frac{1}{2}\frac{d}{dt}\left\{(g\circ\nabla u)(t)\right\}-\frac{1}{2}(g'\circ\nabla u)(t);\end{aligned}\tag{4.10}$$

$$\begin{aligned}
&-\frac{1}{2}\int_0^\infty \left[g(t-s)\left(\frac{d}{dt}\int_\Omega \left\{|\nabla u(x,t)|^2\right\}dx\right)\right]ds\\
&=-\frac{1}{2}\left(\int_0^\infty g(t-s)ds\right)\left(\frac{d}{dt}\int_\Omega \left\{|\nabla u(x,t)|^2\right\}dx\right)\\
&=-\frac{1}{2}\left(\int_0^\infty g(s)ds\right)\left(\frac{d}{dt}\int_\Omega \left\{|\nabla u(x,t)|^2\right\}dx\right)\\
&=-\frac{1}{2}\frac{d}{dt}\left[\left(\int_0^\infty g(s)ds\right)\left(\int_\Omega \left\{|\nabla u(x,t)|^2\right\}dx\right)\right]\\
&+\frac{1}{2}g(t)\int_\Omega \left\{|\nabla u(x,t)|^2\right\}dx. \qquad (4.11)
\end{aligned}$$

By replacement of (4.6) – (4.10), we get

$$\begin{aligned}
&\int_\Omega \left[u_t(x,t)\left(\int_0^\infty g(t-s)\Delta u(x,s)ds\right)\right]dx\\
&=\frac{1}{2}\frac{d}{dt}\left\{(g\circ\nabla u)(t)\right\}-\frac{1}{2}(g'\circ\nabla u)(t)\\
&-\frac{1}{2}\frac{d}{dt}\left[\left(\int_0^\infty g(s)ds\right)\left(\int_\Omega \left\{|\nabla u(x,t)|^2\right\}dx\right)\right]\\
&+\frac{1}{2}g(t)\int_\Omega \left\{|\nabla u(x,t)|^2\right\}dx\\
&=\frac{d}{dt}\left\{\frac{1}{2}(g\circ\nabla u)(t)-\frac{1}{2}\left(\int_0^\infty g(s)ds\right)\left(\int_\Omega \left\{|\nabla u(x,t)|^2\right\}dx\right)\right\}\\
&-\frac{1}{2}(g'\circ\nabla u)(t)+\frac{1}{2}g(t)\int_\Omega \left\{|\nabla u(x,t)|^2\right\}dx. \qquad (4.12)
\end{aligned}$$

Combining (4.6) and (4.10) we get

$$\begin{aligned}
&\int_\Omega\left(\frac{1}{2}\left|u_t^\upsilon(x,t)\right|^2+\frac{1}{2}\left|\nabla u^\upsilon(x,t)\right|^2-\int_\Omega \frac{1}{p}\left(u^\upsilon(x,t)\right)^p\ln|u^\upsilon(x,t)|^k dx+\frac{k}{p^2}|u^\upsilon(x,t)|^p\right)dx\\
&+\frac{1}{2}\left[(g\circ\nabla u^\upsilon)(t)-\left(\int_0^\infty g(s)ds\right)\left\|u^\upsilon\left(t\right)\right\|_2^2\right]\\
&+\frac{\xi}{2}\int_\Omega\int_0^1\left|z^\upsilon(x,\rho,t)\right|^2d\rho dx\\
&=-\mu_1\int_0^\infty\int_\Omega\left|u_t^\upsilon(x,s)\right|^2dxds-\mu_2\int_0^\infty\int_\Omega z^\upsilon(x,1,s)u_t^\upsilon(x,s)dxds\\
&+\frac{1}{2}\int_\Omega\left(\left|u_1^\upsilon(x)\right|^2+\left|\nabla u_0^\upsilon(x)\right|^2\right)dx\\
&-\frac{1}{2}\int_0^\infty g(s)\left\|u^\upsilon\left(s\right)\right\|_2^2ds+\frac{1}{2}\int_0^\infty(g'\circ\nabla u^\upsilon)(s)ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\xi}{2}\int_0^1\int_\Omega |f_0(x,-\rho\tau)|^2\,dxd\rho \\
&-\frac{1}{p}\int_\Omega\left[(u_0^\upsilon(x))^p\ln|u_0^\upsilon(x)|^k-\frac{k}{p}|u_0^\upsilon(x)|^p\right]dx \\
&+\frac{\xi}{2\tau}\int_0^t\int_\Omega |u_t^\upsilon(x,t)|^2\,dxd\rho-\frac{\xi}{2\tau}\int_0^\infty\int_\Omega |z^\upsilon(x,1,s)|^2\,dxds. \qquad (4.13)
\end{aligned}
$$

Repeating the steps (3.6) – (3.11) of [2], we conlude that for any $\upsilon\in\mathbb{N}$,

$$
\begin{aligned}
&(u^\upsilon) \text{ is uniformly bounded in } L^\infty((0,T);H_0^1(\Omega))\\
&(u_t^\upsilon) \text{ is uniformly bounded in } L^\infty((0,T);H_0^1(\Omega))\\
&(z^\upsilon) \text{ is uniformly bounded in } L^\infty((0,T);L^2(\Omega\times(0,1))).
\end{aligned}
$$

thus, we get

$$
\begin{aligned}
u^\upsilon &\rightharpoonup u \text{ weakly star in } L^\infty((0,T);H_0^1(\Omega))\\
u_t^\upsilon &\rightharpoonup u_t \text{ weakly star in } L^\infty((0,T);L^2(\Omega))\\
z^\upsilon &\rightharpoonup z \text{ weakly star in } L^\infty((0,T);L^2(\Omega\times(0,1)))
\end{aligned}
$$

and by using Loins-Aubin theorem,

$$
u^\upsilon \rightharpoonup u \text{ in } L^2(\Omega\times(0,T)) \text{ and for a.e } (x,t) \text{ in } \Omega\times(0,T).
$$

By integrating (4.3) over $(0,\infty)$, we arrive at

$$
\begin{aligned}
\int_\Omega u_t^\upsilon(x,t)w(x)\,dx \quad + \quad & \mu_1\int_\Omega u^\upsilon(x,t)w(x)\,dx \\
= \quad & -\int_0^\infty\int_\Omega \nabla u^\upsilon(x,s).\nabla w(x)\,dxds \\
& +\int_\Omega\left[\int_0^\infty\left(\int_0^\xi g(\xi-s)\nabla u^\upsilon(x,s).\nabla w(x)\,ds\right)d\xi\right]dx \\
& -\mu_2\int_0^\infty\int_\Omega z^\upsilon(x,1,s)u_t^\upsilon(x,s)dxds \\
& +\int_\Omega\int_0^\infty u^\upsilon(x,t)|u^\upsilon(x,t)|^{p-2}\ln|u^\upsilon(x,t)|^k w(x)\,dsdx \\
& -\int_\Omega u_1^\upsilon(x,t)w(x)\,dx-\mu_1\int_\Omega u_0^\upsilon(x,t)w(x)\,dx, \qquad (4.14)
\end{aligned}
$$

and

$$
\begin{aligned}
\tau\int_\Omega z^\upsilon(x,\rho,t)\psi(x,\rho)\,dx = \quad & -\quad \int_0^\infty\int_\Omega z_\rho^\upsilon(x,\rho,t)\psi(x,\rho)\,dxd\rho \\
& +\quad \tau\int_\Omega f_0(x,-\rho\tau)\psi(x,\rho)\,dx. \qquad (4.15)
\end{aligned}
$$

By passing to the limit, we get

$$\begin{aligned}
\int_\Omega u_t(x,t)w(x)\,dx \quad &+ \quad \mu_1 \int_\Omega u(x,t)w(x)\,dx \\
&= \quad -\int_0^\infty \int_\Omega \nabla u(x,s).\nabla w(x)\,dxds \\
&\quad + \int_0^\infty \int_\Omega \left[\left(\int_0^\xi g(\xi - s)\nabla u(x,s).\nabla w(x)\,ds\right)\right] dxd\xi \\
&\quad -\mu_2 \int_0^\infty \int_\Omega z(x,1,s)u_t^v(x,s)dxds \\
&\quad + \int_0^\infty \int_\Omega u(x,t)|u(x,s)|^{p-2} \ln |u(x,s)|^k w(x)\,dxds \\
&\quad - \int_\Omega u_1(x,t)w(x)\,dx - \mu_1 \int_\Omega u_0(x,t)w(x)\,dx, \qquad (4.16)
\end{aligned}$$

and

$$\begin{aligned}
\tau \int_\Omega z(x,\rho,t)\psi(x,\rho)\,dx = \quad & - \quad \int_0^\infty \int_\Omega z_\rho(x,\rho,t)\psi(x,\rho)\,dxd\rho \\
& + \quad \tau \int_\Omega f_0(x,-\rho\tau)\psi(x,\rho)\,dx, \qquad (4.17)
\end{aligned}$$

for all $(w,\psi) \in H_0^1(\Omega) \times L^2(\Omega \times (0,1)))$.

Notice that the right hand sides of (4.16) and (4.17) are absolutely continuous. So, by differentiating, we obtain, for a.e $x \in \Omega$,

$$\begin{aligned}
&\int_\Omega u_{tt}(x,t)w(x)\,dx + \int_\Omega \nabla u(x,t).\nabla w(x)\,dx \\
&\quad - \quad \int_\Omega \left[\left(\int_0^\infty g(t-s)\nabla u(x,s).\nabla w(x)\,ds\right)\right] dx \\
&\quad + \quad \mu_1 \int_\Omega u_t(x,t)w(x)\,dx + \mu_2 \int_\Omega z_\rho(x,1,t)\psi(x,\rho)\,dx \\
&\quad = \quad \int_\Omega u(x,t)|u(x,t)|^{p-2} \ln |u(x,t)|^k w(x)\,dx \qquad (4.18)
\end{aligned}$$

$$\tau \int_\Omega z_t(x,\rho,t)\psi(x,\rho)\,dx + \int_\Omega z_\rho(x,\rho,t)\psi(x,\rho)\,dx = 0 \qquad (4.19)$$

for all $(w,\psi) \in H_0^1(\Omega) \times L^2(\Omega \times (0,1))$.

We use the density of $H_0^1(\Omega)$ in $L^2(\Omega)$ to replace (w,ψ) by (u_t,z) in (4.18) and (4.19). Then we integrate (4.18) over $(0,t)$ and (4.19) over $(0,t)\times(0,1)$, we obtain

$$\begin{aligned}E(t) =& -\left(\mu_1-\frac{\xi}{2\tau}\right)\int_0^\infty\int_\Omega |u_t(x,s)|^2\,dxds\\ &+\frac{1}{2}\left(1-\int_0^\infty g(s)ds\right)\|\nabla u\|_2^2+\frac{1}{2}(g\circ\nabla u)(t)\\ &-\frac{\xi}{2\tau}\int_0^t\int_\Omega |z^\upsilon(x,1,s)|^2\,dxds-\mu_2\int_0^\infty\int_\Omega z(x,1,t)u_t(x,t)dxds+E(0).\end{aligned}$$

Therefore,

$$\begin{aligned}E'(t) =& -\left(\mu_1-\frac{\xi}{2\tau}\right)\int_\Omega |u_t(x,s)|^2\,dx+\frac{1}{2}(g'\circ\nabla u)(t)-\frac{1}{2}g(t)\int_\Omega |\nabla u\,(t,x)|^2\,dx\\ &-\frac{\xi}{2\tau}\int_\Omega |z(x,1,t)|^2\,dx-\mu_2\int_\Omega z(x,1,t)u_t(x,t)dx. \end{aligned}\tag{4.20}$$

for a.e $t\in(0,T)$.
Using Young's inequality, we estimate

$$-\mu_2\int_\Omega z(x,1,t)u_t(x,t)dx \le \frac{|\mu_2|}{2}\int_\Omega\left(|u_t(x,s)|^2+|z(x,1,t)|^2\right)dx.$$

Hence, from (4.20), we obtain

$$\begin{aligned}E'(t) \le& -\left(\mu_1-\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right)\int_\Omega |u_t(x,s)|^2\,dx-\left(\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right)\int_\Omega |z(x,1,t)|^2\,dx\\ &+(g'\circ\nabla u)(t)-g(t)\int_\Omega |\nabla u\,(t,x)|^2\,dx.\end{aligned}\tag{4.21}$$

Using (4.1), we have, for some $C_0>0$,

$$\begin{aligned}E'(t) \le & - C_0\left[\int_\Omega\left(|u_t|^2+|z(x,1,t)|^2\right)dx\right]\\ & - \frac{1}{2}g(t)\int_\Omega |\nabla u\,(t,x)|^2\,dx+\frac{1}{2}(g'\circ\nabla u)(t)\le 0.\end{aligned}\tag{4.22}$$

where $C_0=\min\left\{\mu_1-\frac{\xi}{2\tau}-\frac{|\mu_2|}{2},\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right\}$, which is positive by (4.1). □

Lemma 4.2. *There exists a positive constant $C>0$ such that*

$$\left(\int_\Omega |u|^p\ln|u|^k dx\right)^{\frac{s}{p}}\le C\left[\int_\Omega |u|^p\ln|u|^k dx+\|\nabla u\|_2^2\right].$$

For any $u\in L^{p+1}(\Omega)$ and $2\le s\le p$, provided that $\int_\Omega |u|^p\ln|u|^k dx\ge 0$.

Proof. If $\int_\Omega |u|^p\ln|u|^k dx>1$ then

$$\left(\int_\Omega |u|^p\ln|u|^k dx\right)^{\frac{s}{p}}\le\int_\Omega |u|^p\ln|u|^k dx.\tag{4.23}$$

If $\int_\Omega |u|^p \ln|u|^k dx \le 1$ then we set

$$\Omega_1 = \{x \in \Omega \ \mid |u| > 1\}$$

and, for any $\beta \le 2$, we have

$$\begin{aligned}\left(\int_\Omega |u|^p \ln|u|^k dx\right)^{\frac{s}{p}} &\le \left(\int_\Omega |u|^p \ln|u|^k dx\right)^{\frac{\beta}{p}} \le \left(\int_{\Omega_1} |u|^p \ln|u|^k dx\right)^{\frac{\beta}{p}} \\ &\le \left(\int_{\Omega_1} |u|^{p+1} dx\right)^{\frac{\beta}{p}} \le \left(\int_\Omega |u|^{p+1} dx\right)^{\frac{\beta}{p}} = \|u\|_{P+1}^{\frac{\beta(p+1)}{P}}.\end{aligned}$$

We choose $\beta = \frac{2p}{p+1} < 2$ to get

$$\left(\int_\Omega |u|^p \ln|u|^k dx\right)^{\frac{s}{p}} \le \|u\|_{P+1}^2 \le C\,\|\nabla u\|_2^2. \tag{4.24}$$

Combining (4.23) and (4.24), we get the desired result. □

Lemma 4.3. *There exists a positive constant $C > 0$ such that for any $u \in L^p(\Omega)$ we have*

$$\|u\|_P^p \le C\left[\int_\Omega |u|^p \ln|u|^k dx + \|\nabla u\|_2^2\right]. \tag{4.25}$$

provided that $\int_\Omega |u|^p \ln|u|^k dx \ge 0$.

Proof. We set

$$\Omega_+ = \{x \in \Omega \ ||u| > e\} \text{ and } \ \Omega_- = \{x \in \Omega \ ||u| \le e\}.$$

Therefore,

$$\begin{aligned}\|u\|_P^p &= \int_{\Omega_+} |u|^p dx + \int_{\Omega_-} |u|^p dx \\ &\le \int_{\Omega_+} |u|^p \ln|u|^k dx + \int_{\Omega_-} e^p |\frac{u}{e}|^p dx \\ &\le \int_{\Omega_+} |u|^p \ln|u|^k dx + e^p \int_{\Omega_-} |\frac{u}{e}|^2 dx \\ &\le \int_{\Omega_+} |u|^p \ln|u|^k dx + e^{p-2} \int_{\Omega_-} |u|^2 dx \\ &\le C\left(\int_{\Omega_+} |u|^p \ln|u|^k dx + \|\nabla u\|_2^2\right).\end{aligned}$$

□

Corollary 4.4. *There exists a positive constant $C > 0$ such tha*

$$\|u\|_2^2 \le C\left[\left(\int_\Omega |u|^p \ln|u|^k dx\right)^{\frac{2}{p}} + \|\nabla u\|_2^{\frac{4}{p}}\right] \tag{4.26}$$

provided that $\int_\Omega |u|^p \ln|u|^k dx \ge 0$.

Lemma 4.5. *There exists a positive constant C such that for any $u \in L^p(\Omega)$ and $2 \leq s \leq p$, we have*

$$\|u\|_p^s \leq C\left[\|u\|_p^p + \|\nabla u\|_2^2\right]. \tag{4.27}$$

Proof. If $\|u\|_p \geq 1$ then

$$\|u\|_p^s \leq \|u\|_p^p.$$

If $\|u\|_p \leq 1$ then, $\|u\|_p^s \leq \|u\|_p^2$. Using Sobolev embedding theorems, we have

$$\|u\|_p^s \leq \|u\|_p^2 \leq C\|\nabla u\|_2^2.$$

Now we are ready to state and prove your main result. For this purpose, we define

$$\begin{aligned} H(t) = -E(t) &= \frac{1}{p}\int_\Omega |u(t)|^p \ln|u(t)|^k dx - \frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}\|u\|_2^2 - \frac{1}{2}\|\nabla u\|_2^2 \\ &- \frac{1}{2}\left(1 - \int_0^\infty g(s)ds\right)\|\nabla u\|_2^2 - \frac{1}{2}(g \circ \nabla u)(t) \\ &- \frac{k}{p}\|\nabla u\|_p^p - \frac{\xi}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx. \end{aligned}$$

□

Theorem 4.6. *Suppose that* (4.1) *and* (3.3) *hold. Assume further that*

$$\begin{aligned} E(0) &= \frac{1}{2}\|\nabla u_0\|_2^2 + \frac{1}{2}\|u_1\|_2^2 + \frac{k}{p}\|u_0\|_p^p \\ &+ \frac{\xi}{2}\int_\Omega\int_0^1 |f_0(x,-\rho\tau)|^2 d\rho dx - \frac{1}{p}\int_\Omega |u_0|^p \ln|u_0|^k dx < 0. \end{aligned} \tag{4.28}$$

Then the solution of (3.1) *blows up in finite time.*

Proof. As $E(t)$ is a nonincreasing function, we have

$$E(0) \geq E(t).$$

A differentiation of $H(t)$ gives

$$\begin{aligned} H'(t) &= -E'(t) \\ &\geq C_0\left[\int_\Omega \left(|u_t|^2 + |z(x,1,t)|^2\right)dx\right] + \frac{1}{2}g(t)\int_\Omega |\nabla u(t,x)|^2 dx \\ &-\frac{1}{2}\int_\Omega (g' \circ \nabla u)(t)dx \\ &\geq C_0\int_\Omega z^2(x,1,t)dx \geq 0. \end{aligned} \tag{4.29}$$

and

$$0 < H(0) \leq H(t) \leq \frac{1}{p}\int_\Omega u^p \ln|u_0|^k dx. \tag{4.30}$$

We set

$$L(t) = H^{1-\alpha}(t) + \varepsilon\int_\Omega u_t u dx + \varepsilon\frac{u_t}{2}\int_\Omega u^2 dx, \quad t \geq 0,$$

where $\varepsilon > 0$ to be specified later and

$$0 < \frac{2(p-2)}{p^2} < \alpha < \frac{p-2}{p^2} < 1. \tag{4.31}$$

Differentiating $L(t)$ we easily obtain

$$\begin{aligned} L'(t) &= (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon \|u_t\|_2^2 - \varepsilon \|\nabla u\|_2^2 \\ &\quad + \varepsilon \int_0^\infty g(t-s) \int_\Omega \nabla u(s,x) . \nabla u(t,x) ds \\ &\quad - \varepsilon \mu_2 \int_\Omega u z(x,1,t) dx + \frac{\varepsilon}{p} \int_\Omega u^p \ln |u|^k dx. \end{aligned} \tag{4.32}$$

Using Young's inequality, we estimate

$$\begin{aligned} & -\varepsilon \mu_2 \int_\Omega u z(x,1,t) dx \\ & \qquad \geq -\varepsilon |\mu_2| \left(\delta \int_\Omega u^2 dx + \frac{1}{4\delta} \int_\Omega z^2(x,1,t) dx \right), \quad \forall \delta > 0. \end{aligned} \tag{4.33}$$

and Cachy-Schwarz and Young inequalities, we have

$$\begin{aligned} & \int_0^\infty g(t-s) \int_\Omega \nabla u(s,x) . \nabla u(t,x) dx ds \\ &= \int_0^\infty g(t-s) \int_\Omega \nabla u(t,x) . (\nabla u(s,x) - \nabla u(t,x)) dx ds \\ &\quad + \int_0^\infty g(t-s) \|\nabla u\|_2^2 ds \\ &\geq \left(1 - \frac{1}{4\delta}\right) \left(\int_0^\infty g(s) ds \right) \|\nabla u\|_2^2 - \delta (g \circ \nabla u)(t), \quad \forall \delta > 0. \end{aligned}$$

We get, from (4.32),

$$\begin{aligned} L'(t) \geq & \left[(1-\alpha) H^{-\alpha}(t) - \frac{\varepsilon |\mu_2|}{4\delta C_0} \right] H'(t) \\ & + \varepsilon \left(1 - \frac{1}{4\delta}\right) \left(\int_0^\infty g(s) ds \right) \|\nabla u(t,x)\|_2^2 \\ & - \varepsilon \delta (g \circ \nabla u)(t) + \varepsilon \|u_t\|_2^2 - \varepsilon \|\nabla u\|_2^2 \\ & - \varepsilon \delta |\mu_2| \|u\|_2^2 + \varepsilon \int_\Omega |u|^p \ln |u|^k dx. \end{aligned} \tag{4.34}$$

Of course (4.34) remains valid even if δ is time dependent. Therefore by taking δ so that

$$\frac{|\mu_2|}{4\delta C_0} = \kappa H^{-\alpha}(t),$$

for large κ to be specified later, and substituting in (4.34) we arrive at

$$\begin{aligned} L'(t) &\geq \left[(1-\alpha)-\varepsilon\kappa\right] H^{-\alpha}(t)H'(t)+\varepsilon\left\|u_t\right\|_2^2-\varepsilon\left\|\nabla u\right\|_2^2 \\ &+\varepsilon\left(1-\frac{1}{4\delta}\right)\left(\int_0^\infty g(s)ds\right)\left\|\nabla u\right\|_2^2-\varepsilon\delta(g\circ\nabla u)(t) \\ &-\frac{\varepsilon|\mu_2|^2}{4\kappa C_0}H^\alpha(t)\left\|u\right\|_2^2+\varepsilon\int_\Omega |u|^p\ln|u|^k dx. \end{aligned}$$

For $0<a<1$, we have

$$\begin{aligned} L'(t) &\geq \left[(1-\alpha)-\varepsilon\kappa\right] H^{-\alpha}(t)H'(t)+\frac{\varepsilon a}{p}\int_\Omega |u|^p\ln|u|^k dx+\varepsilon\frac{p\left(1-a\right)+2}{p}\left\|u_t\right\|_2^2 \\ &+\varepsilon\left(\frac{p\left(1-a\right)}{2}-\left(\frac{p\left(1-a\right)-2}{2}+\frac{1}{4\delta}\right)\int_0^\infty g(s)ds\right)\left\|\nabla u\right\|_2^2 \\ &+\varepsilon\left(\frac{p\left(1-a\right)}{2}-\delta\right)(g\circ\nabla u)(t)+\varepsilon k\left(1-a\right)\left\|u\right\|_p^p-\frac{\varepsilon|\mu_2|^2}{4\kappa C_0}H^\alpha(t)\left\|u\right\|_2^2 \\ &+\varepsilon p\left(1-a\right)H(t)+\frac{\varepsilon\left(1-a\right)p\xi}{2}\int_\Omega\int_0^1 z^2(x,1,t)d\rho dx. \end{aligned} \tag{4.35}$$

Using (4.26), (4.30) and Young's inequality, we find

$$\begin{aligned} H^\alpha(t)\left\|u\right\|_2^2 &\leq \left(\int_\Omega |u|^p\ln|u|^k dx\right)^\alpha\left\|u\right\|_2^2 \\ &\leq C\left[\left(\int_\Omega |u|^p\ln|u|^k dx\right)^{\alpha+\frac{2}{p}}+\left(\int_\Omega |u|^p\ln|u|^k dx\right)^\alpha\left\|\nabla u\right\|_2^{\frac{4}{p}}\right] \\ &\leq C\left[\left(\int_\Omega |u|^p\ln|u|^k dx\right)^{\frac{p\alpha+2}{p}}+\left\|\nabla u\right\|_2^2+\left(\int_\Omega |u|^p\ln|u|^k dx\right)^{\frac{\alpha p}{p-2}}\right]. \end{aligned}$$

Exploiting (4.31), we have

$$2<\alpha p+2\leq p \text{ and } 2<\frac{\alpha p^2}{p-2}\leq p.$$

Thus, lemma 4.2 yields

$$H^\alpha(t)\left\|u\right\|_2^2\leq C\left[\int_\Omega |u|^p\ln|u|^k dx+\left\|\nabla u\right\|_2^2\right]. \tag{4.36}$$

Combining (4.35) and (4.36), we obtain

$$\begin{aligned}
L'(t) &\geq \left[(1-\alpha)-\varepsilon\kappa\right] H^{-\alpha}(t)H'(t)+\varepsilon\left(\frac{a}{p}-\frac{\varepsilon|\mu_2|^2}{4\kappa C_0}\right)\int_\Omega |u|^p \ln|u|^k dx \\
&+\varepsilon\left(\frac{p\,(1-a)}{2}-\frac{\varepsilon|\mu_2|^2}{4\kappa C_0}-\left(\frac{p\,(1-a)-2}{2}+\frac{1}{4\delta}\right)\int_0^\infty g(s)ds\right)\|\nabla u\|_2^2 \\
&\varepsilon\left(\frac{p\,(1-a)}{2}-\delta\right)(g\circ\nabla u)\,(t)+\varepsilon k\,(1-a)\,\|u\|_p^p+\varepsilon\frac{p\,(1-a)+2}{p}\,\|u_t\|_2^2 \\
&+\varepsilon p\,(1-a)\,H(t)+\frac{\varepsilon\,(1-a)\,p\xi}{2}\int_\Omega\int_0^1 z^2(x,1,t)d\rho dx. \qquad (4.37)
\end{aligned}$$

We choose $a>0$ so small that

$$\frac{p(1-a)-2}{2}>0,\ \frac{p\,(1-a)}{2}-\delta>0,$$

and κ so large that

$$\frac{p\,(1-a)}{2}-\frac{\varepsilon|\mu_2|^2}{4\kappa C_0}-\left(\frac{p\,(1-a)-2}{2}+\frac{1}{4\delta}\right)\int_0^\infty g(s)ds>0 \text{ and } \frac{a}{p}-\frac{\varepsilon|\mu_2|^2}{4\kappa C_0}>0.$$

We pick ε so small so that

$$(1-\alpha)-\varepsilon\kappa>0,\ H(0)+\varepsilon\int_\Omega u_0u_1dx>0.$$

Next, for some $\lambda>0$, estimate (4.37) becomes

$$\begin{aligned}
L'(t) &\geq \lambda\left[H(t)+\|u_t\|_2^2+\|\nabla u\|_2^2+\|u\|_p^p\right] \\
&+ \lambda\left[(g\circ\nabla u)\,(t)+\int_\Omega\int_0^1 z^2(x,1,t)d\rho dx+\int_\Omega |u|^p\ln|u|^k dx\right] \qquad (4.38)
\end{aligned}$$

and

$$L(t)\geq L(0)>0,\ t\geq 0. \qquad (4.39)$$

Using Hölder's inequality and the embedding $\|u\|_2\leq C\,\|u\|_p$, we have

$$\int_\Omega u_t u dx\leq \|u\|_2\,\|u_t\|_2\leq C\,\|u\|_p\,\|u_t\|_2\,,$$

and exploiting Young's inequality, we obtain

$$\left|\int_\Omega u_t u dx\right|^{\frac{1}{1-\alpha}}\leq C\left(\|u\|_p^{\frac{\mu}{1-\alpha}}+\|u_t\|_2^{\frac{\theta}{1-\alpha}}\right),\quad \text{for } \frac{1}{\mu}+\frac{1}{\theta}=1. \qquad (4.40)$$

To be able to use Lemma 4.2, we take $\theta=2(1-\alpha)$ which gives $\frac{\mu}{1-\alpha}=\frac{2}{1-2\alpha}\leq p$. Consequently, for $s=\frac{2}{1-2\alpha}$, estimate (4.40) gives

$$\left|\int_\Omega u_t u dx\right|^{\frac{1}{1-\alpha}}\leq C\left(\|u_t\|_2^2+\|u\|_p^s\right).$$

Hence, Lemma 4.2 gives

$$\left|\int_\Omega u_t u dx\right|^{\frac{1}{1-\alpha}} \leq C\left(\|\nabla u\|_2^2 + \|u_t\|_2^2 + \|u\|_p^p\right). \tag{4.41}$$

Consequently,

$$\begin{aligned}
L^{\frac{1}{1-\alpha}}(t) &= \left(H^{1-\alpha}(t) + \varepsilon\int_\Omega u_t u dx + \frac{u_t\varepsilon}{2}\int_\Omega u^2 dx\right)^{\frac{1}{1-\alpha}} \\
&\leq C\left[H(t) + (g\circ\nabla u)(t) + \left|\int_\Omega u_t u dx\right|^{\frac{1}{1-\alpha}} + \|u\|_2^{\frac{1}{1-\alpha}}\right] \\
&\leq C\left[H(t) + (g\circ\nabla u)(t) + \left|\int_\Omega u_t u dx\right|^{\frac{1}{1-\alpha}} + \|u\|_2^{\frac{2}{1-\alpha}}\right] \\
&\leq C\left[H(t) + (g\circ\nabla u)(t) + \|\nabla u\|_2^2 + \|u_t\|_2^2 + \|u\|_p^p\right]. \qquad (4.42)
\end{aligned}$$

Combining (4.38) and (4.42), we obtain

$$L'(t) \geq \Lambda L^{\frac{1}{1-\alpha}}(t), \text{ for } t \geq 0. \tag{4.43}$$

where Λ is a positive constant.
A direct integration over $(0,t)$ of (4.43) then yields

$$L^{\frac{1}{1-\alpha}}(t) \geq \frac{1}{L^{-\frac{\alpha}{1-\alpha}}(0) - \Lambda\frac{\alpha t}{1-\alpha}}, \text{ for } t \geq 0.$$

Therefore, $L(t)$ blows up in time

$$T \leq T^* = \frac{1-\alpha}{\Lambda\alpha L^{\frac{\alpha}{1-\alpha}}(0)}.$$

This completes the proof. □

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