

Existence results for some classes of differential systems with ”maxima”

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Dedicated to the memory of Prof. Antonio Avantaggiati

Abstract. Local existence of solutions of initial value problems associated with a new type of systems of differential equations with ”maxima” are investigated.

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1. Introduction

In this short note we consider a class of functional differential equations (and systems) that can be used to describe complex evolutionary phenomena in which the future behaviour depends not only on the present state but also on the past history. The model problem is an initial value problem (IVP) associated with a modified logistic equation which contains the maximum of the square of the unknown function over a past interval:

$$\begin{cases} \dot{x}(t) = x(t) - \max_{[0,t]} x^2(s) & t \geq 0; \\ x(0) = x_0 \end{cases} \quad (1.1)$$

where $x_0 \in \mathbb{R}$.

As it is emphasized in the book [3], the application of the classical logistic equations in the setting of experimental sciences entails two order of difficulties: on one hand the necessity of experimentally setting some of the parameters appearing in the equation, and on the other hand the fact that the derivative changes sign exactly when a certain value of the function is reached. To tackle with the second problem,

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often an apriori set delay τ is considered in the equation. It is evident that there are situations in which neither the delay nor the parameters can be determined on an experimental base. The problem (1.1) seems to be more appropriate to deal with those cases.

Analyzing (1.1), it is obvious that, if $x_0 = 0$ (resp. $x_0 = 1$), then the constant function $x \equiv 0$ (resp. $x \equiv 1$) is a solution. Moreover, if $x \in C^1([0, T])$ is a solution of (1.1), we observe that:

- if $x_0 < 0$ or $x_0 > 1$, then $\dot{x}(0) < 0$. Therefore, in a neighbourhood of 0, $\dot{x}(t) < 0$ and the equation reduces to $\dot{x}(t) = x(t) - x_0^2$.
- if $0 < x_0 < 1$, then $\dot{x}(0) > 0$. Therefore, in a neighbourhood of 0, $\dot{x}(t) > 0$ and the equation reduces to the well know equation $\dot{x}(t) = x(t) - x^2(t)$.

These easy considerations show that the problem (1.1) somehow "contains" two different types of problems, on the basis of the initial value.

Moreover the IVP (1.1) features also the following strange behaviour. Let $t_0 > 0$ and assume that $0 < x_1 < 1$: then a solution of the following IVP

$$\dot{x}(t) = x(t) - \max_{[0,t]} x^2(s) \quad t_0 \leq t; \quad x(t_0) = x_1 \tag{1.2}$$

could be an extension of a solution either of the IVP

$$\dot{x}(t) = x(t) - \max_{[0,t]} x^2(s) \quad 0 \leq t; \quad x(0) = y_0 \tag{1.3}$$

or of the IVP

$$\dot{x}(t) = x(t) - \max_{[0,t]} x^2(s) \quad 0 \leq t; \quad x(0) = z_0 \tag{1.4}$$

for suitable $0 < y_0 < 1, \quad 1 < z_0$. This "uncertainty" situation for a solution $x = x(t)$ could appear at all time $t > 0$ for which $0 < x(t) < 1$.

More generally, we are going to consider the system

$$\begin{cases} \dot{x}(t) = f\left(t, x(t), \max_{s \in [0,t]} g_1(x_1(s)), \dots, \max_{s \in [0,t]} g_m(x_m(s))\right), & t \geq 0 \\ x(0) = x_0 \end{cases} \tag{1.5}$$

where $x_0 \in \mathbb{R}^m, f \in C([0, +\infty[\times \mathbb{R}^{2m}, \mathbb{R}^m)$ and is locally Lipschitz with respect to the second variable and the functions $g_i \in C(\mathbb{R})$ are locally Lipschitz on \mathbb{R} , for every $i = 1, \dots, m$.

This type of systems belongs to the class of systems of differential equations with "maxima". Much attention has been paid to this type of equations and systems in the last years. Without any pretensions to being exhaustive, we recall only the recent papers [1, 4, 6, 5, 7], while we refer to the monograph [2] for a survey of motivations and techniques on the subject. In particular, Section 3.3 of [2] is devoted to the study of IVP associated with scalar differential equations of the type

$$\begin{cases} \dot{x}(t) = f(t, x(t), \max_{s \in [0,t]} x(s)) & t \geq 0 \\ x(0) = x_0 \end{cases} \tag{1.6}$$

Clearly, even in the scalar case, the class of problems (1.5) is wider than (1.6).

Our aim is to provide first, via fixed point theory, a local existence and uniqueness result for the general system (1.5). Afterwards, in particular situations as (1.1) and for systems of the type

$$\begin{cases} \dot{x}(t) = x(t) - \max_{s \in [0,t]} y(s) \\ \dot{y}(t) = y(t) - \max_{s \in [0,t]} x(s) \end{cases} \quad \text{or} \quad \begin{cases} \dot{x}(t) = x(t) - \max_{s \in [0,t]} y^2(s) \\ \dot{y}(t) = y(t) - \max_{s \in [0,t]} x^2(s) \end{cases},$$

we will provide more precise existence results by the use of Peano-Picard's approximation.

2. Local existence results via contraction theorem

We start with two remarks that will help along the proofs of our results.

Remark 2.1. Let $g, h \in C([a, b])$. Then

$$|\max_{[a,b]} g - \max_{[a,b]} h| \leq \max_{[a,b]} |g - h|.$$

Indeed, assume that $\max_{[a,b]} g \geq \max_{[a,b]} h$ and let $x_0 \in [a, b]$ such that

$$\max_{[a,b]} g = g(x_0).$$

Then,

$$|\max_{[a,b]} g - \max_{[a,b]} h| = g(x_0) - \max_{[a,b]} h \leq g(x_0) - h(x_0) = |g(x_0) - h(x_0)| \leq \max_{[a,b]} |h - g|.$$

Remark 2.2. Let $g \in C([a, b])$. Then the function

$$h(s) = \max_{\tau \in [0,s]} g(\tau), \quad s \in [a, b]$$

is continuous. Indeed let $s_0 \in [a, b]$. Fix $\varepsilon > 0$ and consider $\delta > 0$ such that

$$|g(\tau) - g(s)| < \varepsilon \text{ if } |\tau - s| < \delta.$$

For any $s_0 < s < s_0 + \delta$, it can happen that $h(s) = h(s_0)$ or that $h(s) = g(\bar{\tau})$ for some $\bar{\tau} \in [s_0, s]$. In the first case obviously $h(s) - h(s_0) < \varepsilon$, while in the second case

$$|h(s) - h(s_0)| = h(s) - h(s_0) \leq g(\bar{\tau}) - g(s_0) < \varepsilon.$$

Therefore $\lim_{s \rightarrow s_0^+} h(s) = h(s_0)$.

If $s_0 - \delta < s < s_0$, then $h(s_0) = h(s)$ or $h(s_0) = g(\bar{\tau})$ for some $\bar{\tau} \in [s, s_0]$. In the last case,

$$|h(s) - h(s_0)| = h(s_0) - h(s) \leq g(\bar{\tau}) - g(s) < \varepsilon.$$

So we get that $\lim_{s \rightarrow s_0^-} h(s) = h(s_0)$.

Theorem 2.3. Let $x_0 \in \mathbb{R}^m$, $f \in C([0, +\infty[\times \mathbb{R}^{2m}, \mathbb{R}^m)$ and locally Lipschitz with respect to the second variable and $g_i \in C(\mathbb{R})$ locally Lipschitz on \mathbb{R} , for every $i = 1, \dots, m$.

Given $\alpha > 0$ and $T > 0$, set

$$M_{\alpha,T} := \max \left\{ \|f(t, u, v)\| : t \in [0, T], u \in [x_0 - \alpha, x_0 + \alpha]^m, v \in \prod_{i=1}^m g_i([x_0 - \alpha, x_0 + \alpha]) \right\}$$

and assume $M_{\alpha,T} > 0$. Let $L_{\alpha,T} > 0$ and $L_\alpha > 0$ be such that for every $t \in [0, T]$, $u_1, u_2 \in [x_0 - \alpha, x_0 + \alpha]^m$, $v_1, v_2 \in \prod_{i=1}^m g_i([x_0 - \alpha, x_0 + \alpha])$ and for every $x, y \in [x_0 - \alpha, x_0 + \alpha]$

$$\begin{aligned} \|f(t, u_1, v_1) - f(t, u_2, v_2)\| &\leq L_{\alpha,T} (\|u_1 - u_2\| + \|v_1 - v_2\|) \\ |g_i(x) - g_i(y)| &\leq L_\alpha |x - y|. \end{aligned}$$

Then, for every

$$0 < \bar{T} < \min \left\{ \frac{\alpha}{M_{\alpha,T}}, \frac{1}{L_{\alpha,T}(1 + L_\alpha\sqrt{m})}, T \right\}$$

there exists $x \in C^1([0, \bar{T}]; \mathbb{R}^m)$ unique solution of the IVP (1.5).

Proof. We will apply the Banach Fixed Point Theorem.

Indeed, observe first that the existence of a C^1 solution of problem (1.5) is equivalent to the existence of a continuous solution of the integral problem

$$x(t) = x_0 + \int_0^t f\left(s, x(s), \max_{\tau \in [0,s]} g_1(x_1(\tau)), \dots, \max_{\tau \in [0,s]} g_m(x_m(\tau))\right) ds. \tag{2.1}$$

Fix

$$0 < \bar{T} < \min \left\{ \frac{\alpha}{M_{\alpha,T}}, \frac{1}{L_{\alpha,T}L_\alpha\sqrt{m}}, T \right\}$$

and consider the map $F : C([0, \bar{T}]; \mathbb{R}^m) \rightarrow C([0, \bar{T}]; \mathbb{R}^m)$ defined by

$$F(x)(t) = x_0 + \int_0^t f\left(s, x(s), \max_{\tau \in [0,s]} g_1(x_1(\tau)), \dots, \max_{\tau \in [0,s]} g_m(x_m(\tau))\right) ds$$

and the ball

$$X := \left\{ x \in C([0, \bar{T}]; \mathbb{R}^m) \mid \|x(t) - x_0\| \leq \alpha \quad \forall t \in [0, \bar{T}] \right\}.$$

Clearly X is a complete metric space, with respect the the distance induced by the norm of $C([0, \bar{T}]; \mathbb{R}^m)$:

$$\|x\|_\infty := \sup_{t \in [0, \bar{T}]} \|x(t)\|, \quad x \in C([0, \bar{T}]; \mathbb{R}^m).$$

If $x \in X$, then

$$\begin{aligned} &\|F(x) - x_0\|_\infty \\ &\leq \sup_{0 \leq t \leq \bar{T}} \int_0^t \left\| f\left(s, x(s), \max_{\tau \in [0,s]} g_1(x_1(\tau)), \dots, \max_{\tau \in [0,s]} g_m(x_m(\tau))\right) \right\| ds \\ &\leq \bar{T} M_{\alpha,T} \leq \alpha. \end{aligned}$$

Hence $F(X) \subseteq X$. On the other hand, for every $x, y \in X$, it holds

$$\begin{aligned} & \|F(x) - F(y)\|_\infty \\ & \leq \bar{T}L_{\alpha,T}(\|x - y\|_\infty \\ & + \max_{s \in [0, \bar{T}]} \left\| \left(\max_{\tau \in [0, s]} g_1(x_1(\tau)), \dots, \max_{\tau \in [0, s]} g_m(x_m(\tau)) \right) \right. \\ & \left. - \left(\max_{\tau \in [0, s]} g_1(y_1(\tau)), \dots, \max_{\tau \in [0, s]} g_m(y_m(\tau)) \right) \right\| \\ & \leq \bar{T}L_{\alpha,T} \left(\|x - y\|_\infty + \sqrt{m} \max_{s \in [0, \bar{T}]} \max_{i=1}^m \left| \max_{\tau \in [0, s]} g_i(x_i(\tau)) - \max_{\tau \in [0, s]} g_i(y_i(\tau)) \right| \right) \\ & \leq \bar{T}L_{\alpha,T} \left(\|x - y\|_\infty + \sqrt{m} \max_{i=1}^m \max_{\tau \in [0, \bar{T}]} |g_i(x_i(\tau)) - g_i(y_i(\tau))| \right) \\ & \leq \bar{T}L_{\alpha,T}(1 + \sqrt{m}L_\alpha)\|x - y\|_\infty. \end{aligned}$$

Therefore F is a contraction on X and it has a unique fixed point. □

Remark 2.4. The previous result applies, for example, to the following types of problems

$$\begin{aligned} \dot{x}(t) &= \alpha(t)x(t) - \beta(t) \max_{s \in [0, t]} x^2(s) \quad 0 \leq t; \quad x(0) = x_0; \\ \dot{x}(t) &= \alpha(t)x(t) - \beta(t) \max_{s \in [0, t]} x(s) \quad 0 \leq t; \quad x(0) = x_0; \\ \dot{x}(t) &= \alpha(t)x(t) - \beta(t) \max_{s \in [0, t]} |x(s)| \quad 0 \leq t; \quad x(0) = x_0. \end{aligned}$$

under suitable conditions on the functions α, β .

3. Existence proofs with approximations

Theorem 3.1. Consider the following problem

$$\begin{cases} \dot{x}(t) = x(t) - \max_{s \in [0, t]} x^2(s) & t \geq 0; \\ x(0) = x_0 \end{cases} \tag{3.1}$$

with $x_0 \neq 1$. Let $\alpha > 1$ and

$$0 < T^* < \frac{\alpha - 1}{\alpha(1 + \alpha|x_0|)}.$$

Then there exists a unique solution $x \in C^1([0, T^*])$ of (3.1).

Proof. We prove the existence of a solution via Peano-Picard's approximations. Set $x_0(t) = x_0$ for every $t \geq 0$ and define

$$x_n(t) = x_0 + \int_0^t x_{n-1}(s)ds - \int_0^t \max_{\eta \in [0, s]} x_{n-1}^2(\eta)ds \quad t \in [0, T^*], \quad n \geq 1$$

It is immediate to prove that

$$x_n(t) = x_0 g_n(t) \quad \forall n \in \mathbb{N}, t \geq 0 \tag{3.2}$$

where $g_0 \equiv 1$ and

$$g_n(t) = [1 + \int_0^t g_{n-1}(s)ds - x_0 \int_0^t \max_{[0,s]} [g_{n-1}(\eta)]^2 ds].$$

By induction, using the choice of T^* , we easily get that

$$\forall n \in \mathbb{N}, t \in [0, T^*] \quad |g_n(t)| \leq \alpha,$$

and, as a consequence, that

$$|g_{n+1}(t) - g_n(t)| \leq \frac{|1 - x_0|}{1 + 2\alpha|x_0|} \frac{(1 + 2\alpha|x_0|)^{n+1}t^{n+1}}{(n + 1)!}.$$

Then the sequence $(g_n)_n$ is uniformly convergent on $[0, T^*]$ and therefore also the sequence $(x_n)_n$ is uniformly convergent on $[0, \overline{T}]$. It is immediate that its uniform limit is the solution of the problem (3.1). □

Remark 3.2. It is worth noticing that

$$T^* < \max_{\alpha \geq 1} \frac{\alpha - 1}{\alpha(1 + \alpha|x_0|)}.$$

We consider now the following system

$$\begin{cases} \dot{x}(t) = x(t) - \max_{s \in [0,t]} y(s) \\ \dot{y}(t) = y(t) - \max_{s \in [0,t]} x(s) \\ x(0) = x_0 \quad y(0) = y_0 \end{cases} \tag{3.3}$$

with $x_0, y_0 \in \mathbb{R}$. We remark that (3.3) is equivalent to the functional system

$$\begin{cases} x(t) = x_0 + \int_0^t x(s)ds - \int_0^t \max_{\tau \in [0,s]} y(\tau)ds, \\ y(t) = y_0 + \int_0^t y(s)ds - \int_0^t \max_{\tau \in [0,s]} x(\tau)ds, \end{cases} \tag{3.4}$$

The following theorem holds.

Theorem 3.3. *Assume that $x_0 > 0, y_0 > 0$ and $x_0 \neq y_0$. Then for all $T > 0$ there exists a unique solution $(x(t), y(t)) \in C^1([0, T])^2$ of the system (3.3).*

Proof. Assume $0 < y_0 < x_0$ and consider the sequences of functions (x_n) and (y_n) defined on $[0, +\infty[$ by

$$\begin{aligned} x_0(t) &= x_0 \quad y_0(t) = y_0 \\ x_{n+1}(t) &= x_0 + \int_0^t (x_n(s) - y_0)ds \\ y_{n+1}(t) &= y_0 + \int_0^t (y_n(s) - x_n(s))ds \end{aligned}$$

It holds that, for every $n \in \mathbb{N}$ and for every $t \geq 0, x_n(t) \geq y_0$ and $y_n(t) \leq x_n(t)$. Indeed, the assertion is obviously true if $n = 0$. Assuming that $x_n(t) \geq y_0$ and

$y_n(t) \leq x_n(t)$ for every $t \geq 0$, we get that

$$x_{n+1}(t) - y_0 = x_0 - y_0 + \int_0^t (x_n(s) - y_0) ds \geq 0,$$

$$x_{n+1}(t) - y_{n+1}(t) = x_0 - y_0 + \int_0^t (x_n(s) - y_n(s)) ds \leq 0.$$

As a consequence we get that, for every $n \in \mathbb{N}$, $\dot{x}_n \geq 0$ and $\dot{y}_n \leq 0$ and consequently

$$\max_{[0,s]} x(\tau) = x(s), \quad \max_{[0,s]} y_n(\tau) = y_n(0) = y_0.$$

Therefore, for the sequences (x_n) and (y_n) , it holds that

$$x_{n+1}(t) = x_0 + \int_0^t x_n(s) ds - \int_0^t \max_{[0,s]} y_n(\tau) ds,$$

$$y_{n+1}(t) = y_0 + \int_0^t y_n(s) ds - \int_0^t \max_{[0,s]} x_n(\tau) ds,$$

By induction, one can prove that for every $n \in \mathbb{N}$ and every $t \geq 0$

$$|x_{n+1}(t) - x_n(t)| \leq |x_0 - y_0| \frac{t^{n+1}}{(n+1)!}$$

$$|y_{n+1}(t) - y_n(t)| \leq |x_0 - y_0| T \frac{t^{n+1}}{n!}$$

Hence the sequences (x_n) and (y_n) are uniformly convergent on $[0, T]$ to continuous functions $x_\infty = x_\infty(t)$ and $y_\infty = y_\infty(t)$ and the couple (x_∞, y_∞) is the unique solution of the functional system (3.4). □

Remark 3.4. It is worth observing that the proof fails if $x_0 = y_0$. Moreover the proof highlights the difference with the system

$$\begin{cases} \dot{x}(t) = x(t) - y(t) \\ \dot{y}(t) = y(t) - x(t). \end{cases}$$

Remark 3.5. More interesting seems to be the study of the following general system

$$\begin{cases} \dot{x}(t) = a(t)x(t) - b(t) \max_{s \in [0,t]} y(s) \\ \dot{y}(t) = c(t)y(t) - d(t) \max_{s \in [0,t]} x(s) \\ x(0) = x_0 > 0, \quad y(0) = y_0 > 0 \end{cases}$$

where the functions a, b, c, d are continuous, non negative and defined on the interval $[0, T]$.

If the functions a, b, c, d are constant, one can prove the following partial results.

If $A = ax_0 - by_0 < 0$, $B = cy_0 - dx_0 < 0$ and $a > 0$, $c > 0$, then a solution is the following couple of functions

$$x(t) = x_0 + A \frac{1}{a} [e^{at} - 1] \quad y(t) = y_0 + B \frac{1}{c} [e^{ct} - 1].$$

and therefore more information follow. For example we have that

$$x(t) = 0 \Leftrightarrow t = \frac{1}{a} \log \frac{by_0}{|A|} \qquad y(t) = 0 \Leftrightarrow t = \frac{1}{c} \log \frac{dx_0}{|B|}.$$

For different situations, such as $A > 0, B < 0$, or $A < 0, B > 0$, or $A > 0, B > 0$ an explicit representation for the solution is not available.

Next we consider the following problem, for $t \geq 0$

$$\begin{cases} \dot{x}(t) = x(t) - \max_{s \in [0, t]} y^2(s) \\ \dot{y}(t) = y(t) - \max_{s \in [0, t]} x^2(s) \\ x(0) = x_0 > 0, \\ y(0) = y_0 > 0. \end{cases} \tag{3.5}$$

Theorem 3.6. *If $T, c_0 > 0$ satisfy*

$$\begin{aligned} |x_0| + |x_0 - y_0^2|T \leq c_0, \quad |y_0| + |y_0 - x_0^2|T \leq c_0; \\ |x_0| + c_0T + c_0^2T \leq c_0, \quad |y_0| + c_0T + c_0^2T \leq c_0. \end{aligned}$$

then there exists $(x, y) \in C^1([0, T]; \mathbb{R}^2)$ unique solution of the IVP (3.5)

Proof. The initial problem (3.5) is equivalent to the following functional system:

$$\begin{cases} x(t) = x_0 + \int_0^t x(s)ds - \int_0^t \max_{\eta \in [0, s]} y^2(\eta)ds; \\ y(t) = y_0 + \int_0^t y(s)ds - \int_0^t \max_{\eta \in [0, s]} x^2(\eta)ds. \end{cases}$$

As usual, we define the sequences of functions (x_n) and (y_n) on $[0, +\infty[$ by:

$$\begin{aligned} x_0(t) &= x_0, \quad y_0(t) = y_0 \\ x_{n+1} &= x_0 + \int_0^t x_n(s)ds - \int_0^t \max_{\eta \in [0, s]} y_n^2(\eta)ds \\ y_{n+1} &= y_0 + \int_0^t y_n(s)ds - \int_0^t \max_{\eta \in [0, s]} x_n^2(\eta)ds. \end{aligned}$$

Under the assumptions, it is immediate to prove by induction that

$$|x_n(t)| \leq c_0, \quad |y_n(t)| \leq c_0 \quad \forall n \in \mathbb{N}, \quad t \geq 0.$$

Consequently

$$\begin{aligned} |x_{n+1}(t) - x_n(t)| &\leq \frac{c_0}{T} (1 + 2c_0)^n \frac{t^{n+1}}{(n+1)!}; \\ |y_{n+1}(t) - y_n(t)| &\leq \frac{c_0}{T} (1 + 2c_0)^n \frac{t^{n+1}}{(n+1)!}. \end{aligned}$$

Indeed, the last assertion is immediately true if $n = 0$. Assuming it for n , we get that

$$\begin{aligned} & |x_{n+1}(t) - x_n(t)| \\ & \leq \int_0^t |x_n(t) - x_{n-1}(t)| dt + \int_0^t \left| \max_{\eta \in [0, s]} y_n^2(\eta) - \max_{\eta \in [0, s]} y_{n-1}^2(\eta) \right| ds \\ & \leq \frac{c_0}{T} (1 + 2c_0)^{n-1} \frac{t^{n+1}}{(n+1)!} + \int_0^t \max_{[0, s]} |y_n^2 - y_{n-1}^2| ds \\ & \leq \frac{c_0}{T} (1 + 2c_0)^{n-1} \frac{t^{n+1}}{(n+1)!} + 2c_0 \int_0^t \max_{[0, s]} |y_n - y_{n-1}| ds \\ & \leq \frac{c_0}{T} (1 + 2c_0)^n \frac{t^{n+1}}{(n+1)!}. \end{aligned}$$

Hence the sequences (x_n) and (y_n) are uniformly convergent to continuous functions x_∞, y_∞ defined in the interval $[0, T]$, that solve the functional system. \square

Remark 3.7. The methods we have considered could also be applied to investigate a version of Lotka-Volterra systems (see [8, 9] for the first steps in the study of the classical situation) with "maxima", namely

$$\dot{x}(t) = x(t) - \max_{s \in [0, t]} x(s)y(s); \quad \dot{y}(t) = y(t) + \max_{s \in [0, t]} x(s)y(s).$$

or other analogous equations and systems.

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