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On generalized close-to-convexity related with strongly Janowski functions

Khalida Inayat Noor and Shujaat Ali Shah

Abstract. Strongly Janowski functions are used to define certain classes of analytic functions which generalize the concepts of close-to-convexity and bounded boundary rotation. Coefficient results, a necessary condition, distortion bounds, Hankel determinant problem and several other interesting properties of these classes are studied. Some significant well known results are derived as special cases.

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1. Introduction

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. If the functions f and g are analytic in E, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwartz function w in $\mathbb{D}$ such that $f(z) = g(w(z))$. Furthermore, if the function g is univalent in $\mathbb{D}$, then we have the following equivalence

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{D}) \subset g(\mathbb{D}).$$

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Let f be given by (1.1) and $g \in \mathcal{A}$ is of the form $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. Then the convolution (Hadamard product) of f and g is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{D}.$$

Let $\mathcal{S} \subset \mathcal{A}$ be the class of univalent functions in $\mathbb{D}$ and let $\mathcal{C}$, $\mathcal{S}^*$ and $\mathcal{K}$ be the subclasses of $\mathcal{S}$ consisting of convex, starlike and close-to-convex functions, respectively. Also, let p be analytic in $\mathbb{D}$ with $p(0) = 1$. Then the function p is known a strongly Janowski type functions of order α if

$$p(z) \prec \left(\frac{1+Az}{1+Bz}\right)^{\alpha}, \quad \alpha \in (0,1], \ -1 \leq B < A \leq 1 \text{ and } z \in \mathbb{D}.$$

We note that, when $\alpha = 1$, $A = 1$ and $B = -1$, then p is a Carathéodory function of positive real part.

Definition 1.1. Let p be analytic in $\mathbb{D}$ with $p(0) = 1$ and let ϕ be convex univalent in $\mathbb{D}$. Then $p \in \mathcal{P}_m(\phi)$, $m \geq 2$, if and only if there exists functions p_i with $p_i(0) = 1$, $i = 1, 2$ such that

$$p(z) = \left(\frac{m}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right) p_2(z), \tag{1.2}$$

where $p_i \prec \phi$.

Special cases:

Let $\phi(z) = \left(\frac{1+Az}{1+Bz}\right)^{\alpha}$, $\alpha \in (0,1]$, $-1 \leq B < A \leq 1$. Then the series representation of $\phi(z)$ is given by

$$\phi(z) = 1 + \alpha (A - B) z + \left[-\alpha (A - B) B + \frac{1}{2}\alpha (\alpha - 1) (A - B)^2\right] z^2 + \dots .$$

On differentiating we get

$$\phi'(z) = \left(\frac{1+Az}{1+Bz}\right)^{\alpha} \frac{\alpha (A - B)}{(1 + Az)(1 + Bz)}.$$

Now, for $-1 \leq B < A \leq 1$ and $z \in \mathbb{D}$, we have

$$\Re (\phi'(z)) \geq \left\{ \alpha |A - B| \frac{(1 - |A|)^{\alpha - 1}}{(1 - |B|)^{\alpha + 1}} \right\} > 0,$$

and by simple calculations we can easily prove that

$$\Re \left\{ \frac{(z\phi'(z))'}{\phi'(z)} \right\} \geq 0.$$

This implies that $\phi(z)$ is convex univalent function in $\mathbb{D}$.
Thus we have

$$\mathcal{P}_m\left(\left(\frac{1+Az}{1+Bz}\right)^{\alpha}\right) = \mathcal{P}_{m,\alpha}[A, B] \subset \mathcal{P}_m(\rho),$$

where $\rho = \left(\frac{1-A}{1-B}\right)^{\alpha}$. Also, we note that $\mathcal{P}_{m,1}[1,-1] = \mathcal{P}_m$, see [19]. Moreover, $\mathcal{P}_{2,1}[1,-1] = \mathcal{P}$ is the well-known class of Carathéodory functions of positive real part. When $m=2$, then $p \in \mathcal{P}_{2,\alpha}[1,-1]$ implies $|\arg p(z)| \leq \frac{\alpha\pi}{2}$. When $m=2$, $\alpha=1$, $A = 1-2\beta$ and $B=-1$, we obtain the class $\mathcal{P}(\beta)$, $\beta \in (0,1]$, of functions with real part greater than β.

For the class $\mathcal{P}_m(\rho)$, we refer to [18]. It is worth noting that $\mathcal{P}_{2,\alpha}[1,-1] = \mathcal{P}_\alpha$ and the class $\mathcal{P}_{\frac{1}{2}}[1,0] = \pounds\mathcal{P}$ is associated with the right-half of the Lemniscate of Bernoulli $\partial\pounds$ (see [11]) enclosing the region

$$\pounds = \left\{w \in \mathbb{C} : \Re(w) > 0,\ \left|w^2-1\right| < 1\right\},$$

where $\pounds \subset \left\{w \in \mathbb{C} : |\arg w| < \frac{\pi}{4}\right\}$.

The well-known hypergeometric function $G(a,b,c;z)$ is of the form

$$\begin{aligned} G(a,b,c;z) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+b)\Gamma(b+n)}{\Gamma(c+n)}\cdot\frac{z^n}{n!} \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 u^{a-1}(1-u)^{c-a-1}(1-zu)^{-b}\,du, \end{aligned}$$

where $\Re(a) > 0$, $\Re(c-a) > 0$ and Γ represents notation for Gamma function.

Definition 1.2. Let $f \in \mathcal{A}$. Then $f \in \mathcal{R}_{m,\alpha}[A,B]$ if and only if

$$\frac{zf'(z)}{f'(z)} \in \mathcal{P}_{m,\alpha}[A,B].$$

Definition 1.3. Let $f \in \mathcal{A}$. Then $f \in \mathcal{V}_{m,\alpha}[A,B]$ if and only if

$$\frac{(zf'(z))'}{f'(z)} \in \mathcal{P}_{m,\alpha}[A,B].$$

We note the following special cases.

(i) $\mathcal{R}_{2,1}[A,B] = \mathcal{S}^*[A,B]$ and $\mathcal{V}_{2,1}[A,B] = \mathcal{C}[A,B]$, see [9].

(ii) $\mathcal{R}_{m,1}[1,-1] = \mathcal{R}_m$ and $\mathcal{V}_{m,1}[1,-1] = \mathcal{V}_m$, the class of functions with bounded radius and bounded boundary rotations, respectively; see [2, 19].

(iii) $\mathcal{V}_{2,\alpha}[A,B] = \mathcal{C}_\alpha[A,B] \subset \mathcal{C}(\rho) \subset \mathcal{C}$, with $\rho = \left(\frac{1-A}{1-B}\right)^{\alpha}$, where $\mathcal{C}$ is the class of convex functions.

(iv) $\mathcal{R}_{m,\alpha}[A,B] \subset \mathcal{R}_m(\rho)$ and $\mathcal{V}_{m,\alpha}[A,B] \subset \mathcal{V}_m(\rho)$, see [18].

It is observed that

$$f \in \mathcal{V}_{m,\alpha}[A,B] \Leftrightarrow zf' \in \mathcal{R}_{m,\alpha}[A,B].$$

Definition 1.4. Let $f \in \mathcal{A}$. Then $f \in \mathcal{T}_{m,\alpha}[A,B]$ if and only if

$$\frac{f'(z)}{g'(z)} \in \mathcal{P}_\alpha[A,B],$$

for some $g \in \mathcal{V}_{m,\alpha}[1,-1]$.

The class $\mathcal{T}_{m,1}[1,-1] = \mathcal{T}_m$ has been introduced and studied in [17], and $\mathcal{T}_{2,1}[1,-1] = \mathcal{K}$, the class of close-to-convex functions, see [10].

In the present work, we derive coefficient inequalities and distortion results for certain subclasses of analytic functions. Further, necessary condition and radius problem are discussed. Also, the Hankel determinant problem is estimated. We need the following results in our investigations.

Lemma 1.5. [24] *If $f \in \mathcal{C}$, $g \in \mathcal{S}^*$, then for each h analytic in $\mathbb{D}$ with $h(0) = 1$,*

$$\frac{(f * hg)(\mathbb{D})}{(f * g)(\mathbb{D})} \subset \overline{CO}h(\mathbb{D}),$$

where $\overline{CO}h(\mathbb{D})$ denotes the closed convex hull of $h(\mathbb{D})$.

Using well-known distortion results for the class $\mathcal{P}$, we can easily prove:

Lemma 1.6. *Let $p(z)$ be analytic in $\mathbb{D}$ with $p(0) = 1$. Let*

$$p(z) \prec \left(\frac{1+Az}{1+Bz}\right)^{\alpha}, \ \alpha \in (0,1], \ -1 \leq B < A \leq 1.$$

Then

$$\left(\frac{1-Ar}{1-Br}\right)^{\alpha} \leq |p(z)| \leq \left(\frac{1+Ar}{1+Br}\right)^{\alpha}$$

and

$$\left|\frac{zp'(z)}{p(z)}\right| \leq \frac{\alpha(A-B)r}{(1-Ar)(1-Br)}.$$

Lemma 1.7. [21] *Let $\theta_1 < \theta_2 < \cdots < \theta_l < \theta_1 + 2n\pi$ and $\lambda \geq \lambda_j$ $(j = 1, 2, \cdots, l)$. If*

$$\begin{aligned} \Psi(z) &= \prod_{j=1}^{l}\left(1 - e^{-i\theta_j}z\right)^{-\lambda_j} \\ &= \sum_{n=1}^{\infty} b_n z^n, \end{aligned} \tag{1.3}$$

then

$$b_n = O(1).n^{\lambda-1}, \text{ as } n \to \infty.$$

Lemma 1.8. [8] *Let $p \in \mathcal{P}$ and $z = re^{i\theta}$. Then*

$$\int_0^{2\pi} \left|p(re^{i\theta})\right|^{\eta} d\theta < c(\eta)\frac{1}{(1-r)^{\eta-1}},$$

where $\eta > 1$ and $c(\eta)$ is a constant depending on η only.

2. Main results

This section presents our main investigations. In the following theorem we derive the coefficient inequalities. Here, we use terminology Schlicht disc d by the disc d contained in the image of $\mathbb{D}$ under univalent function f.

Theorem 2.1. *Let* $\frac{zf'}{g} \in \mathcal{P}_{m,\alpha}[A,B]$, $g \in \mathcal{V}_2$, *and let* f *be given by* (1.1). *Then*

$$|a_n| \leq \frac{\{m\alpha\,|A-B|\,(n-1)+4\}}{4n}.$$

Proof. Let $g \in \mathcal{V}_2$ be of the form

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

and since g is convex univalent in $\mathbb{D}$, so we have $|b_n| \leq 1$, for all n.

Let

$$\frac{zf'}{g} = p(z) \in \mathcal{P}_{m,\alpha}[A,B], \tag{2.1}$$

where $p(z)$ be of the form $p(z) = 1 + \sum\limits_{n=2}^{\infty} c_n z^n$. We write $p(z)$ as given in (1.2) with $p_i(z) = 1 + \sum\limits_{n=2}^{\infty} c_{n,i} z^n$, $i = 1, 2$. Then $|c_{n,i}| \leq \alpha\,|A-B|$ by using a result due to Rogosinski [22]. From this, it easily follows that

$$|c_n| \leq \frac{m\alpha\,|A-B|}{2}, \quad (n \geq 1). \tag{2.2}$$

Now, using the expansions of $f(z)$, $g(z)$ and $p(z)$ in (2.1) to get

$$z + \sum_{n=2}^{\infty} n a_n z^n = \left(z + \sum_{n=2}^{\infty} b_n z^n\right)\left(1 + \sum_{n=2}^{\infty} c_n z^n\right).$$

On simplification and equating the coefficients of z^n $(n \geq 2)$, we have

$$n\,|a_n| \leq \sum_{k=1}^{n-1} |b_k|\,|c_{n-k}| + |b_n|,$$

using $|b_n| \leq 1$ together with (2.2), we obtain

$$\begin{aligned}
n\,|a_n| &\leq \frac{m\alpha\,|A-B|}{2}\sum_{k=1}^{n-1} k + 1 \\
&= \frac{m\alpha\,|A-B|}{2}\left[\frac{n\,(n-1)}{2}\right] + 1 \\
&= \frac{m\alpha\,|A-B|\,n\,(n-1)}{4} + 1,
\end{aligned}$$

and this implies

$$|a_n| \leq \frac{m\alpha\,|A-B|\,(n-1)}{4} + \frac{1}{n}.$$

This proves our required result.
In particular, we have

$$|a_2| \leq \frac{m\alpha |A-B|}{4} + \frac{1}{2}$$

and

$$|a_3| \leq \frac{m\alpha |A-B|}{2} + \frac{1}{3}.$$

□

Corollary 2.2. *Let* $\frac{zf'}{g} \in \mathcal{P}_{2,\alpha}[A,B]$, $g \in \mathcal{V}_2$, *and let* f *be given by* (1.1). *Then* $f(\mathbb{D})$ *contains the disc* d *such that*

$$d = \left\{ w : w < \frac{2}{5+\alpha |A-B|} \right\}.$$

Proof. Let w_0 ($w_0 \neq 0$) be any complex number such that $f(z_0) \neq w_0$ for $z \in E$. Then, the function

$$F(z) = \frac{w_0 f(z)}{w_0 - f(z)} = z + \left(a_2 + \frac{1}{w_0}\right) z^2 + ...$$

is analytic and univalent in E, see [7]. Now, using the well known Bieberbach theorem for the best bound of second coefficient of univalent functions, we have

$$\begin{aligned} \frac{1}{|w_0|} &\leq 2 + |a_2| \leq \frac{\alpha |A-B| + 1}{2} + 2 \\ &= \frac{5 + \alpha |A-B|}{2}. \end{aligned}$$

this implies

$$|w_0| \geq \frac{2}{5+\alpha |A-B|}.$$

Thus, $f(\mathbb{D})$ contains the disc d such that

$$d = \left\{ w : w < \frac{2}{5+\alpha |A-B|} \right\}.$$

□

Theorem 2.3. *Let* $\frac{f'}{g'} \in \mathcal{P}_\alpha$ *for* $g \in \mathcal{V}_{m,\alpha}[A,B]$. *Then*

$$\begin{aligned} \frac{2^{(2\alpha-1)} r_1^{\xi}}{\xi} \left[G(a,b,c,-1) - G(a,b,c,-r_1) \right] &\leq |f(z)| \\ &\leq \frac{2^{(2\alpha-1)} r_1^{-\xi}}{\xi} \left[G(a,b,c,-1) - G\left(a,b,c,-r_1^{-1}\right) \right], \end{aligned}$$

where $\xi = \left[(1-\varrho)\left(\frac{m}{2}-1\right) + \alpha + 1\right]$ *with* $\varrho = ((1-A)/(1-B))^{\alpha}$, $r_1 = \frac{1-r}{1+r}$, G *is hypergeometric function and* a, b, c *are given in* (2.9).

Proof. If $\frac{f'}{g'} \in \mathcal{P}_\alpha$ for $g \in \mathcal{V}_{m,\alpha}[A,B]$, then we can write

$$f'(z) = g'(z)\, p(z), \quad p \in \mathcal{P}_\alpha. \tag{2.3}$$

Since $g \in \mathcal{V}_{m,\alpha}[A,B] \subset \mathcal{V}_m(\varrho)$, with $\varrho = ((1-A)/(1-B))^\alpha$ implies

$$g'(z) = (g_1'(z))^{1-\varrho}, \text{ for } g_1 \in \mathcal{V}_m, \text{ (see [18])}.$$

Therefore, by using distortion results of $\mathcal{V}_m$ [2, 19], we have

$$\left[\frac{(1-r)^{\frac{m}{2}-1}}{(1+r)^{\frac{m}{2}+1}}\right]^{(1-\varrho)} \leq |g'(z)| \leq \left[\frac{(1+r)^{\frac{m}{2}-1}}{(1-r)^{\frac{m}{2}+1}}\right]^{(1-\varrho)}. \tag{2.4}$$

Also, for $p \in \mathcal{P}_\alpha$, we have

$$\left(\frac{1-r}{1+r}\right)^\alpha \leq |p(z)| \leq \left(\frac{1+r}{1-r}\right)^\alpha. \tag{2.5}$$

Therefore, from (2.3) to (2.5), it follows that

$$\frac{(1-r)^{\left\{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha\right\}}}{(1+r)^{\left\{(1-\varrho)\left(\frac{m}{2}+1\right)+\alpha\right\}}} \leq |f'(z)| \leq \frac{(1+r)^{\left\{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha\right\}}}{(1-r)^{\left\{(1-\varrho)\left(\frac{m}{2}+1\right)+\alpha\right\}}}. \tag{2.6}$$

Let $d_r = |f(z)|$ denote the radius of the largest Schlicht disc centered at the origin and contained in the image of $|z| < r$ under $f(z)$. Then there is a point z_0, $|z_0| = r$ such that $|f(z_0)| = d_r$.
Thus, we have

$$\begin{aligned} d_r &= |f(z_0)| = \int_C |f'(z)|\,|dz| \\ &\geq \int_C \frac{(1-|z|)^{\left\{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha\right\}}}{(1+|z|)^{\left\{(1-\varrho)\left(\frac{m}{2}+1\right)+\alpha\right\}}}\,|dz| \\ &\geq \int_0^{|z|} \frac{(1-s)^{\left\{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha\right\}}}{(1+s)^{\left\{(1-\varrho)\left(\frac{m}{2}+1\right)+\alpha\right\}}}\,ds \\ &= \int_0^{|z|} \left(\frac{1-s}{1+s}\right)^{\left\{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha\right\}} \frac{ds}{(1+s)^{2(1-\varrho)}}. \end{aligned} \tag{2.7}$$

Let $\frac{1-s}{1+s} = t$. Then $\frac{-2}{(1+s)^2}ds = dt$ and we can write (2.7) as

$$|f(z_0)| \geq 2^{2\varrho-1} \int_{\frac{1-|z|}{1+|z|}}^1 t^{\left\{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha\right\}} (1+t)^{-2\varrho}\,dt. \tag{2.8}$$

Now, let $\frac{1-r}{1+r} = r_1$ and $t = r_1 u$. Then, from (2.8), we get

$$\begin{aligned} |f(z_0)| &\geq 2^{2\varrho-1} \int_{r_1}^1 (r_1 u)^{\left\{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha\right\}} (1+r_1 u)^{-2\varrho} (r_1 du) \\ &= 2^{2\varrho-1} r_1^{\left\{(1-\varrho)\frac{m}{2}+2\alpha+1\right\}} [I_1 - I_2], \end{aligned}$$

with

$$\begin{aligned} I_2 &= \int_0^{r_1} u^{\left\{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha\right\}} (1+r_1 u)^{-2\varrho} (du) \\ &= \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} G(a,b,c,-r_1), \end{aligned} \tag{2.9}$$

where $G(a,b,c,z)$ represents hypergeometric function and

$$a = (1-\varrho)\left(\frac{m}{2}-1\right)+\alpha+1,\ b = 2\varrho,\ c = a+1.$$

Therefore,

$$I_2 = \frac{1}{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha+2}\left[G(a,b,c,-r_1)\right].$$

Also,

$$\begin{aligned} I_1 &= \int_0^{1} u^{\left\{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha\right\}} (1+r_1 u)^{-2\varrho} (du) \\ &= \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} G(a,b,c,-1). \end{aligned} \tag{2.10}$$

Thus

$$|f(z_0)| \geq 2^{2\varrho-1} r_1^{\left\{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha+1\right\}} \frac{1}{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha+2} \times \left[G(a,b,c,-1) - G(a,b,c,-r_1)\right].$$

For the upper bound, we use (2.6) with similar method and routine computations and have

$$\begin{aligned} &2^{2\varrho-1} r_1^{\left\{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha+1\right\}} \frac{1}{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha+2} \times \left[G(a,b,c,-1) - G(a,b,c,-r_1)\right] \\ &\leq |f(z)| \leq 2^{2\varrho-1} r_1^{-\left\{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha+1\right\}} \frac{1}{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha+2} \times \left[G(a,b,c,-1) - G\left(a,b,c,-r_1^{-1}\right)\right]. \end{aligned}$$

□

Corollary 2.4. (*Covering result*) *Let $r \to 1$ and f satisfy the condition of Theorem 2.3. Then $f(\mathbb{D})$ contains the Schlicht disc $|z| < \frac{2^{2\varrho-1}}{\xi}$, $\xi = \left\{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha+1\right\}$.*

As special cases, we note that the radius of this disc is
(i) $\frac{1}{m+2}$, when $A = 1$, $B = -1$ and $\alpha = 1$, (see [15]).
(ii) $\frac{2^{2\varrho-1}}{2(1+\alpha)-\varrho}$, when $m = 2$ and for $\varrho = \alpha = \frac{1}{2}$, it is $\frac{2}{5}$.
(ii) $m = 4$ gives $\frac{2^{2\alpha-1}}{2(\alpha-\varrho)+3}$ and for $\varrho = \alpha = \frac{1}{2}$, it is $\frac{1}{3}$.

Theorem 2.5. *Let* $\frac{f'}{g'} \in \mathcal{P}_\alpha$ *for* $g \in \mathcal{V}_{m,\alpha}[A,B]$ *and let* $f(z)$ *be given by* (1.1). *Then, for* $m > 2 + \frac{2-\alpha}{1-\rho}$, . *Thus, by taking* $r = \left(1 - \frac{1}{n}\right)$, $n \to \infty$, *it follows that*

$$a_n = O(1)n^\beta \quad \text{with} \quad \beta = \left\{(1-\rho)\left(\frac{m}{2} - 1\right) + \alpha\right\},$$

where $O(1)$ *is a constant depending only on* α, m, A, B *and* $\rho = \left(\frac{1-A}{1-B}\right)^\alpha$.

Proof. We can write

$$f'(z) = g'(z)p(z), \ \ g \in \mathcal{V}_{m,\alpha}[A,B] \subset \mathcal{V}_m(\rho),$$

where $\rho = \left(\frac{1-A}{1-B}\right)^\alpha$ and $p \in \mathcal{P}_\alpha$ implies, for $z \in \mathbb{D}$,

$$p(z) = (p_1(z))^\alpha, \quad p_1 \in \mathcal{P}. \tag{2.11}$$

For $g \in \mathcal{V}_m(\rho)$, it is well known that there exists $g_1 \in \mathcal{V}_m$ such that

$$g'(z) = (g_1'(z))^{1-\rho}, \quad z \in \mathbb{D}. \tag{2.12}$$

Also, it is known [3] that, for $g_1 \in \mathcal{V}_m$,

$$g_1'(z) = s(z)h^{\frac{m}{2}-1}(z),\ m > 2,\ s \in \mathcal{S}^*,\ h \in \mathcal{P}. \tag{2.13}$$

From (2.11), (2.12), (2.13) and Cauchy theorem, we have

$$\begin{aligned}
n|a_n| &\leq \frac{1}{2\pi r^n}\int_0^{2\pi} |s(z)|^{1-\rho}|h(z)|^{\left(\frac{m}{2}-1\right)(1-\rho)}|p(z)|^\alpha d\theta \\
&\leq \frac{1}{r^n}\left(\frac{r}{(1-r)^2}\right)^{1-\rho}\left[\frac{1}{2\pi}\int_0^{2\pi}|h(z)|^{\left\{\left(\frac{m}{2}-1\right)(1-\rho)\right\}\frac{2}{2-\alpha}}d\theta\right]^{\frac{2-\alpha}{2}} \times \\
&\qquad \left[\frac{1}{2\pi}\int_0^{2\pi}|p(z)|^2 d\theta\right]^{\frac{\alpha}{2}} \\
&\qquad\qquad \leq C(\rho,m,\alpha)\left\{\frac{1}{(1-r)}\right\}^{\left(\frac{m}{2}-1\right)(1-\rho)+\alpha+1},
\end{aligned}$$

where we have used distortion result for starlike functions, Holder's inequality and a result for the class $\mathcal{P}$, due to Hayman [8], with

$$m > 2 + \frac{2-\alpha}{1-\rho}, \qquad \rho = \left(\frac{1-A}{1-B}\right)^\alpha.$$

Thus, by taking $r = \left(1 - \frac{1}{n}\right)$, $n \to \infty$, it follows that

$$a_n = O(1).n^{\left\{(1-\rho)\left(\frac{m}{2}-1\right)+\alpha\right\}},\ (n \to \infty).$$

□

Special cases:

(i) We note that, for $m = 4$, we have

$$a_n = O(1).n^{(1-\rho+\alpha)}.$$

(ii) $A = 1$, $B = -1$ gives us $\rho = 0$ and with $\alpha = \frac{1}{2}$, $m = 5$, we get $\beta = 2$. Therefore, in this case

$$a_n = O(1).n^2, \ (n \to \infty).$$

(iii) Choosing ρ in such a way that $\rho = \alpha$ and $m = 4$, we have

$$a_n = O(1).n, \ (n \to \infty).$$

Theorem 2.6. *Let $\frac{f'}{g'} \in \mathcal{P}_\alpha$ for $g \in \mathcal{V}_{m,\alpha}[A, B]$. Then $f(z)$ is a convex function of order ρ for $|z| < r_*$, where*

$$r_* = \frac{2}{m_1 + \sqrt{m_1^2 - 4}}, \qquad \text{with} \quad m_1 = m + \frac{2\alpha}{1-\rho}.$$

Proof. We have

$$f'(z) = g'(z)p(z), \ \ p \in \mathcal{P}_\alpha. \tag{2.14}$$

Since $\mathcal{V}_{m,\alpha}[A, B] \subset \mathcal{V}_m(\rho)$ with $\rho = \left(\frac{1-A}{1-B}\right)^\alpha$, so

$$g'(z) = (g_1'(z))^{1-\rho}, \ \ g_1 \in \mathcal{V}_m.$$

Also, for $g_1 \in \mathcal{V}_m$, it is known [3] that there exists a starlike function s such that

$$g_1'(z) = \left(\frac{s(z)}{z}\right)(h(z))^{\left(\frac{m}{2}-1\right)}, \ \ m > 2, \ \ h \in \mathcal{P}. \tag{2.15}$$

From (2.14) and (2.15), we can write

$$f'(z) = \left(\frac{s(z)}{z}\right)^{1-\rho}(h(z))^{(1-\rho)\left(\frac{m}{2}-1\right)}(p_1(z))^\alpha, \ \ p_1 \in \mathcal{P}. \tag{2.16}$$

Logarithmic differentiation of ([19]) yields to us

$$\frac{zf''(z)}{f'(z)} = (1-\rho)\left(\frac{zs'(z)}{s(z)} - 1\right)(1-\rho)\left(\frac{m}{2} - 1\right)\frac{zh'(z)}{h(z)} + \alpha\frac{zp'(z)}{p(z)}.$$

Now, for h, p and h_1 in $\mathcal{P}$, we have

$$1 + \frac{zf''(z)}{f'(z)} = \rho + (1-\rho)\left\{h_1(z) + \left(\frac{m}{2} - 1\right)\frac{zh'(z)}{h(z)}\right\} + \alpha\frac{zp'(z)}{p(z)}.$$

That is,

$$\begin{aligned}
\Re\left[\frac{(zf'(z))'}{f'(z)} - \rho\right] &\geq (1-\rho)\left[\Re(h_1(z)) - \left(\frac{m}{2}-1\right)\left|\frac{zh'(z)}{h(z)}\right| - \alpha\left|\frac{zp'(z)}{p(z)}\right|\right] \\
&\geq (1-\rho)\left[\frac{1-r}{1+r} - \left(\frac{m}{2}-1\right)\frac{2r}{1-r^2}\right] - \frac{2\alpha r}{1-r^2} \\
&= (1-\rho)\left[\frac{1-2r+r^2-(m-2)r}{1-r^2}\right] - \frac{2\alpha r}{1-r^2},
\end{aligned}$$

where we have used Lemma 1.6 with $A = 1$ and $B = -1$. Therefore, we get

$$\frac{1}{(1-\rho)}\Re\left[\frac{(zf'(z))'}{f'(z)} - \rho\right] \geq \frac{1 - \left(m + \frac{2\alpha}{1-\rho}\right)r + r^2}{1-r^2} = \frac{T(r)}{1-r^2}.$$

We note $T(0)=1>0$ and $T(1)=1-m-\frac{2\alpha}{1-\rho}+1=2-\left(m+\frac{2\alpha}{1-\rho}\right)<0$. This shows $r_*\in(0,1)$. Solving $T(r)=0$ gives us the value of r_* which is

$$r_*=\frac{2}{\left(m+\frac{2\alpha}{1-\rho}\right)+\sqrt{\left(m+\frac{2\alpha}{1-\rho}\right)^2-4}}.$$ □

When $A=1$, $B=-1$, $\alpha=1$, then $\rho=0$ and $g\in\mathcal{V}_m$. This gives radius of convexity for $f\in\mathcal{T}_m$ for $|z|<r_*=\frac{2}{(m+2)+\sqrt{m^2+4m}}$. Furthermore, the case $m=2$ gives us $r_*=\frac{1}{2+\sqrt{3}}$ and this is the well-known radius of convexity for the class $\mathcal{K}$ of close-to-convex functions, see [7]. By assigning other permissible values to the parameters α, A, B and m, we obtain several new and known results.

Theorem 2.7. *Let $f\in\mathcal{T}_{2,\alpha}[A,B]$. Let, for $b>-1$,*

$$F(z)=\frac{b+1}{z^b}\int_0^z t^{b-1}f(t)dt. \tag{2.17}$$

Then $F\in\mathcal{T}_{2,\alpha}[A,B]$ in $\mathbb{D}$.

Proof. Since $f\in\mathcal{T}_{2,\alpha}[A,B]$, $\frac{f'}{g'}\prec\left(\frac{1+Az}{1+Bz}\right)^\alpha$, for some $g\in\mathcal{V}_{2,\alpha}[1,-1]$. We can write (2.17) as

$$F(z)=\phi_b(z)*f(z),$$

where $*$ represents convolution and $\phi_b(z)=\sum\limits_{n=1}^{\infty}\frac{b+1}{b+n}z^n$, see [23].
We define

$$G(z)=\frac{b+1}{z^b}\int_0^z t^{b-1}g(t)dt,\ \ g\in\mathcal{V}_{2,\alpha}[1,-1].$$

Then

$$\begin{aligned} G(z)&=\phi_b(z)*g(z)\\ zG'(z)&=\phi_b(z)*zg'(z)\\ z\left(zG'(z)\right)'&=\phi_b(z)*\frac{\left(zg'(z)\right)'}{g'(z)}.zg'(z). \end{aligned}$$

So

$$\frac{\left(zG'(z)\right)'}{G'(z)}=\frac{\phi_b(z)*\frac{\left(zg'(z)\right)'}{g'(z)}.zg'(z)}{\phi_b(z)*.zg'(z)}.$$

Since $g\in\mathcal{V}_{2,\alpha}[1,-1]$, this implies $zg'\in\mathcal{R}_{2,\alpha}[1,-1]\subset\mathcal{S}^*$, we use Lemma 1.5 and it follows that $G\in\mathcal{V}_{2,\alpha}[1,-1]$.
Now,

$$\frac{F'}{G'}=\frac{\phi_b(z)*\frac{f'(z)}{g'(z)}.zg'(z)}{\phi_b(z)*zg'(z)},$$

and this proves $F'(z)/G'(z)\prec\left((1+Az)/(1+Bz)\right)^\alpha$. Hence the class $\mathcal{T}_{2,\alpha}[A,B]$ is preserved under the integral operator given by (2.17). This operator is known as Bernardi operator, see [1]. □

Theorem 2.8. *Let $f \in \mathcal{T}_{m,\alpha}[0,-1]$. Then, for $z = re^{i\theta}$, $0 \leq \theta_1 < \theta_2 \leq 2\pi$*

$$\int_{\theta_1}^{\theta_2} \Re\left\{\frac{(zf'(z))'}{f'(z)}\right\} d\theta > \beta\pi,$$

where $\beta = (1-\rho_1)(m/2-1)+\alpha$, with $\rho_1 = (1/2)^{\alpha}$.

Proof. It can easily be seen that

$$\mathcal{V}_{m,\alpha}[0,-1] \subset \mathcal{V}_m(\rho_1), \quad \text{for } \rho_1 = (1/2)^{\alpha}.$$

So, for $g \in \mathcal{V}_m(\rho_1)$, there exists $g_1 \in \mathcal{V}_m$ such that

$$g'(z) = \left(g_1'(z)\right)^{1-\rho_1}. \tag{2.18}$$

Also, for $g_1 \in \mathcal{V}_m$, we have

$$\int_{\theta_1}^{\theta_2} \Re\left\{\frac{(zg_1'(z))'}{g_1'(z)}\right\} d\theta > -\left(\frac{m}{2}-1\right)\pi. \tag{2.19}$$

We have $h \in \mathcal{P}_\alpha$ which implies $h(z) \prec ((1+z)/(1-z))^{\alpha}$ and so $h(z) = (h_1(z))^{\alpha}$, $h_1 \in P$.
We observe, for $h_1 \in \mathcal{P}$

$$\begin{aligned} \frac{\partial}{\partial\theta}\arg h_1(re^{i\theta}) &= \frac{\partial}{\partial\theta}\Re\left\{-i\ln h_1(re^{i\theta})\right\} \\ &= \Re\left\{\frac{re^{i\theta}h_1'(re^{i\theta})}{h_1(re^{i\theta})}\right\}. \end{aligned}$$

Therefore

$$\int_{\theta_1}^{\theta_2} \Re\left\{\frac{re^{i\theta}h_1'(re^{i\theta})}{h_1(re^{i\theta})}\right\} d\theta = \arg h_1(re^{i\theta_2}) - \arg h_1(re^{i\theta_1}),$$

and

$$\max_{h_1\in P}\left|\int_{\theta_1}^{\theta_2} \Re\left\{\frac{re^{i\theta}h_1'(re^{i\theta})}{h_1(re^{i\theta})}\right\} d\theta\right| = \max_{h_1\in P}\left|\arg h_1(re^{i\theta_2}) - \arg h_1(re^{i\theta_1})\right|.$$

Since $h_1 \in \mathcal{P}$, it is known [25] that

$$\left|h_1(z) - \frac{1+r^2}{1-r^2}\right| \leq \frac{2r}{1-r^2},$$

and so

$$|\arg h_1(z)| \leq \sin^{-1}\left(\frac{2r}{1-r^2}\right).$$

This gives us

$$\begin{aligned} \max_{h_1\in P}\left|\int_{\theta_1}^{\theta_2} \Re\left\{\frac{re^{i\theta}h_1'(re^{i\theta})}{h_1(re^{i\theta})}\right\} d\theta\right| &\leq 2\sin^{-1}\left(\frac{2r}{1-r^2}\right) \\ &\leq \pi - 2\cos^{-1}\left(\frac{2r}{1-r^2}\right). \end{aligned} \tag{2.20}$$

For $f \in \mathcal{T}_{m,\alpha}[0,-1]$ we can write

$$f'(z) = \left(g_1'(z)\right)^{1-\rho_1} \left(h_1(z)\right)^{\alpha}, \quad \rho_1 = \left(\frac{1}{2}\right)^{\alpha}, \; g_1 \in \mathcal{V}_m, \, h_1 \in \mathcal{P}. \tag{2.21}$$

Hence, from (2.18), (2.19), (2.20) and (2.21) together with some computations, it follows that

$$\max_{h_1 \in P} \left| \int_{\theta_1}^{\theta_2} \Re \left\{ \frac{z f'(z)}{f'(z)} \right\} d\theta \right| > -\left\{ (1-\rho_1)\left(\frac{m}{2}-1\right) + \alpha \right\} \pi, \;\; z = re^{i\theta}, \; (r \to 1). \tag{2.22}$$

□

Remark 2.9. It has been proved in [10] by Kaplan that f satisfying (2.22) is close-to-convex in $\mathbb{D}$ if and only if $\beta = \left\{(1-\rho_1)\left(\frac{m}{2}-1\right)+\alpha\right\} \leq 1$. Thus $f \in \mathcal{T}_{m,\alpha}[0,-1]$ is univalent in $\mathbb{D}$ for $2 \leq m \leq 2 + \frac{2(1-\alpha)}{(1-\rho_1)}$, with $\rho_1 = \left(\frac{1}{2}\right)^{\alpha}$.

We shall now discuss the rate of growth of qth Hankel determinant $L_q(n)$ of $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n \in \mathcal{T}_{m,\alpha}[0,B]$, $B \in [-1,0)$, $\alpha \in (0,1]$, and $L_q(n)$, $q \geq 1$, $n \geq 1$ is defined as

$$L_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}, \tag{2.23}$$

Hankel determinant problem has been studied by several prominent researchers in the past, see [4, 5, 12, 13, 14, 16, 20, 21].

Now, we prove

Theorem 2.10. *Let f given by $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ and let $\frac{f'}{g'} \in \mathcal{P}_\alpha[0,B]$, $B \in [-1,0)$ with $g \in \mathcal{V}_{m,\alpha}[0,B]$, $m > 2$. Then, for $k = 0,1,2,\cdots$, there are numbers γ_k and $c_{k\mu}$ $(\mu = 0,1,2,\cdots,k)$ that satisfy $|c_{k0}| = |c_{kk}| = 1$ and*

$$\sum_{l=0}^{\infty} \gamma_l \leq 3, \;\; 0 \leq \gamma_k \leq \frac{2}{k+1} \tag{2.24}$$

such that

$$\sum_{\mu=0}^{\infty} c_{k\mu} a_{n+\mu} = O(1) n^{\beta_1}, \;\; \beta_1 = \gamma_k + \left(\frac{m}{2}-1\right)(1-\rho_1) + \alpha - 2, \quad (n \to \infty).$$

The bounds (2.24) are the best possible.

Proof. We can write

$$f'(z) = g'(z)\, h(z),$$

where $g \in \mathcal{V}_m[0,B]$ with $B \in [-1,0)$ and $h(z) \prec \left(\frac{1}{1+Bz}\right)^{\alpha}$. Since $g \in \mathcal{V}_{m,\alpha}[0,B]$ implies $g \subset \mathcal{V}_m(\rho_1)$, where $\rho_1 = \left(\frac{1}{1-B}\right)^{\alpha}$.

Thus, we have

$$f'(z) = (g_1'(z))^{1-\rho_1}(h_1(z))^{\alpha}, \quad g_1 \in \mathcal{V}_m,\, h_1 \in \mathcal{P}. \tag{2.25}$$

It is shown [3] that, for all $m > 2$, there exists a starlike function s and $p \in \mathcal{P}$ such that

$$zg_1'(z) = s(z)(p(z))^{\left(\frac{m}{2}-1\right)}. \tag{2.26}$$

From (2.25) and (2.26), it follows that

$$f'(z) = \left[\frac{s(z)}{z}(p(z))^{\left(\frac{m}{2}-1\right)}\right]^{(1-\rho_1)}(h_1(z))^{\alpha}. \tag{2.27}$$

Now $s(z)$ can be represented by as

$$s(z) = z\exp\int_0^{2\pi}\log\frac{1}{1-ze^{-it}}dv(t),$$

where $v(t)$ is an increasing function and $v(2\pi) - v(0) = 2$. We here note the jumps of $v(t)$ as $\alpha_1 \geq \alpha_2 \geq \cdots$ at $t = t_1, t_2, \ldots$ and assume $t_1 = 0$. Then $\alpha_1 + \alpha_2 + \ldots \leq 2$ also $\alpha_1 + \alpha_2 + \ldots + \alpha_q = 2$, for some q, if and only if $s(z)$ is of the form

$$s(z) = z\prod_{j=1}^{q}\left(1 - e^{it_j}z\right)^{\frac{-2}{q}}. \tag{2.28}$$

Following the similar arguments given in [21], we define

$$\phi_k(z) = \prod_{\mu=1}^{k}\left(1 - e^{it_\mu}z\right)^{\frac{-2}{q}} = \sum_{\mu=0}^{k}C_{k\mu}z^{k-\mu},$$

and consider three cases. It is shown in [21] that the bounds (2.24) are the best possible.

We use Lemma 1.7 to complete the proof. We write

$$\phi_k.zf'(z) = \sum_{n=0}^{k}b_{kn}z^{n+k} + \sum_{n=1}^{\infty}(n+k)\,a_{kn}z^{n+k}, \tag{2.29}$$

where

$$b_{kn} = \sum_{v=0}^{n}(n+v)\,C_{k-v}a_{n-v},$$

$$a_{kn} = \sum_{\mu=0}C_{k\mu}a_{n+\mu}, \quad |C_{kn}| = |C_{kk}| = 1.$$

Let $s(z)$ in (2.27) be not of the form (2.28). Then $\alpha_1 + \alpha_2 + \ldots + \alpha_q < 2$ for $q \geq 1$ and in particular $\alpha_1 < 2$

$$0 \leq \gamma_k < \frac{2}{1+l},\ \eta_0 + \eta_1 + \ldots < 3.$$

It can easily be shown [21] that, in each of three cases considered in [21],

$$\underset{|z|=r}{Max}\,|\phi_k.s(z)| = O(1)\,(1-r)^{-\eta_k-\delta_k}, \tag{2.30}$$

where

$$\eta_k < \frac{2}{1+k}, \;\; \eta_1 + \eta_2 + \ldots < 3$$

and

$$\delta_k = \frac{1}{3} \min \left\{ \frac{2}{1+k} - \eta_k, \frac{1}{2^{1+k}} \left(3 - \sum_{j=0}^{k} \eta_j \right) \right\},$$

Thus, from (2.27), (2.29) and Cauchy integral formula, we proceed with $m > \left(2 + \frac{2-\alpha}{1-\rho_1}\right)$ for $\rho_1 = \left(\frac{1}{1-B}\right)$ and $B \in [-1, 0)$.

$$\begin{aligned}(k+n)\left|a_{kn}\right| &\leq \frac{1}{r^{n+k}}\left[\frac{1}{2\pi}\int_0^{2\pi}\left|\phi_k.\,(s(z))^{1-\rho_1}\right| |p(z)|^{\left(\frac{m}{2}-1\right)(1-\rho_1)}\, |h(z)|^{\alpha}\, d\theta\right] \\ &\leq \frac{4^{\rho_1}}{r^{n+k}} \max|\phi_k.s(z)| \left[\frac{1}{2\pi}\int_0^{2\pi} |p(z)|^{\left(\frac{(m-2)(1-\rho_1)}{2-\alpha}\right)}\right]^{\frac{2-\alpha}{2}} \times \\ &\quad \left(\frac{1}{2\pi}\int_0^{2\pi} |h_1(z)|^2\, d\theta\right)^{\frac{\alpha}{2}}. \qquad (2.31)\end{aligned}$$

Where we have used distortion result for starlike function $s(z)$ along with the Holder's inequality. Now using Lemma 1.8 and (2.30), we obtain from (2.31)

$$(l+n)\left|a_{kn}\right| \leq C(m,\alpha)\,(1-r)^{\left\{-\eta_k-\gamma_k-\left(\frac{m}{2}-1\right)(1-\rho_1)-1+\alpha\right\}}, \quad (r \to 1),$$

where $C(m,\alpha)$ is a constant $m > \left(2 + \frac{2-\alpha}{1-\rho_1}\right)$ with $\rho_1 = \left(\frac{1}{1-B}\right)^{\alpha}$.

This implies, with $r = 1 - \frac{1}{n}$, $n \to \infty$

$$a_{kn} = O(1).n^{\left\{\gamma_k + \left(\frac{m}{2}-1\right)(1-\rho_1)+\alpha-2\right\}},$$

where $O(1)$ represents a constant.

The case when $s(z)$ is of the form (2.28) follows on similar lines. □

We can now easily prove the following.

Theorem 2.11. *Let the function f satisfy the conditions given in Theorem 2.10. Then, for $q \geq 1$, $n \geq 1$ and $m > 2 + \frac{2-\alpha}{1-\rho_1}$ with $\rho_1 = \left(\frac{1}{1-B}\right)^{\alpha}$.*

$$L_q(n) = O(1).n^{2+\left\{\left(\frac{m}{2}-1\right)(1-\rho_1)+\alpha-2\right\}q}.$$

We note some special cases:

(i) $B = -1$, $\rho_1 = \left(\frac{1}{1-B}\right)^{\alpha} = \left(\frac{1}{2}\right)^{\alpha}$, $\alpha = 1$. Then, for $m > 4$

$$L_q(n) = O(1).n^{2+\left\{\left(\frac{m}{4}-\frac{1}{2}\right)-1\right\}q}.$$

(ii) Also $L_1(n) = a_n$ and, from Theorem 2.5, we have

$$L_1(n) = O(1).n^{\left\{\left(\frac{m}{2}-1\right)(1-\rho_1)+\alpha\right\}},$$

for $m > \left(2 + \frac{2-\alpha}{1-\rho_1}\right)$ with $\rho_1 = \left(\frac{1}{1-B}\right)^{\alpha}$.

For the case $m = 2$, we solve this problem separately as follows.

Corollary 2.12. *Let* $\frac{f'}{g'} \in \mathcal{P}_\alpha$ *with* $g \in \mathcal{V}_{2,\alpha}[-1,0]$, $\alpha \in (\frac{1}{2}, 1]$. *Then, for* $q \geq 1$, $n \geq 1$ *and* $m = 2$,

$$L_q(n) = O(1).n^{2+(\alpha-2)q}.$$

Proof. Let $\frac{f'}{g'} \in \mathcal{P}_\alpha$ with $g \in \mathcal{V}_{2,\alpha}[-1,0]$, $\alpha \in (\frac{1}{2}, 1]$. Then

$$f'(z) = (g_1'(z))^{1-\rho_2} h^\alpha(z), \;\; g_1 \in \mathcal{V}_2, \;\; h \in \mathcal{P}.$$

We take $\frac{s(z)}{z} = g_1'(z)$, and $s(z)$ of the form (2.28) and in the case $\alpha_1 + \alpha_2 + \cdots = 2$, $\sum\limits_{l=0}^{\infty} \gamma_l \leq 3$, $0 \leq \gamma_k \leq \frac{2}{k+1}$. Also $\gamma_k = \frac{2}{k+1}$ implies that $k = q-1$, $\alpha_1 = \alpha_2 = \cdots = \alpha_q$. So using distortion result for $s(z)$ together with Cauchy's theorem, we can write

$$(k+n)\,|a_{kn}| \leq \frac{4^{\rho_2}}{2\pi r^{n+k}} \int_0^{2\pi} |\phi_k.s(z)|\,|h(z)|^\alpha \, d\theta,$$

by Holder's inequality, this implies

$$(k+n)\,|a_{kn}| \leq \frac{4^{\rho_2}}{r^{n+k}} \left[\frac{1}{2\pi}\int_0^{2\pi} |\phi_k.s(z)|^2 \, d\theta\right]^{\frac{1}{2}} \left(\frac{1}{2\pi}\int_0^{2\pi} |h(z)|^{2\alpha} \, d\theta\right)^{\frac{1}{2}}. \tag{2.32}$$

When we write $|\phi_k.s(z)|^2$ in the form (1.3) the exponent $(-\lambda_j)$ satisfy

$$\lambda_j \leq 2\gamma_k, \;\; (k = 1, 2, \ldots, q : k > 0).$$

Hence, using Lemma 1.7, we have

$$\int_0^{2\pi} |\phi_k.s(z)|^2 \, d\theta \leq C_1 n^{2\gamma_k - 1}, \;\; (n \to \infty). \tag{2.33}$$

Also, for $\alpha \in (\frac{1}{2}, 1]$, it follows from Lemma 1.8

$$\int_0^{2\pi} |h_1(z)|^{2\alpha} \, d\theta \leq C_2 n^{2\alpha - 1}, \;\; (n \to \infty). \tag{2.34}$$

Hence, from (2.32) to (2.34), we obtain

$$(n+k)\,|a_{kn}| \leq C_3 n^{\gamma_k + \alpha - 1}. \tag{2.35}$$

From (2.35), we have

$$a_{kn} = O(1).n^{\gamma_k + \alpha - 2}, \;\; (n \to \infty).$$

Thus, for $q \geq 1$, $n \geq 1$

$$L_q(n) = O(1).n^{2+(\alpha-2)q}. \qquad \square$$

Particularly, when $\alpha = 1$, $L_q(n) = O(1).n^{2-q}$, and the exponent $(2-q)$ is best possible, see [13]. C_i, $(i = 1, 2, 3)$, $O(1)$ represents constants, and f is close-to-convex in $\mathbb{D}$.

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Khalida Inayat Noor
COMSATS University Islamabad, Department of Mathematics,
Park Road, Islamabad, 44000, Pakistan
e-mail: khalidan@gmail.com

Shujaat Ali Shah
COMSATS University Islamabad, Department of Mathematics,
Park Road, 44000, Islamabad, Pakistan
and
Quaid-i-Awam University of Engineering Science and Technology,
Department of Mathematics and Satatistics,
Sakrand Road, Nawabshah, 67480, Pakistan
e-mail: shahglike@yahoo.com, shujaatali@quest.edu.pk