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Differential subordination implications for certain Carathéodory functions

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Abstract. In this article, we wish to establish some first order differential subordination relations for certain Carathéodory functions with nice geometrical properties. Moreover, several implications are determined so that the normalized analytic function belongs to various subclasses of starlike functions.

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1. Introduction

Denote the collection of all functions f which are analytic on the open unit disc by $\mathscr{H}$. Let $\mathcal{A} \subset \mathscr{H}$ be the subclass consisting of analytic functions given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and normalised by the conditions $f(0) = 0$ and $f'(0) - 1 = 0$. Further, let $\mathcal{S}^*$ and $\mathcal{C}$ denote the subclasses of univalent function consisting of starlike and convex functions, characterized by the quantities $zf'(z)/f(z)$ and $1+zf''(z)/f'(z)$ lying in the interior of the right half plane respectively. Let f and g be members of $\mathscr{H}$. We say f is subordinate to g (written as $f \prec g$) if there exists a function $w \in \mathscr{H}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. Equivalently, if g is univalent in $\mathbb{D}$, then the conditions $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$ together gives $f \prec g$. For more details, see [15]. The unified class of starlike functions $\mathcal{S}^*_{\varphi} := \{f \in \mathcal{A} : zf'(z)/f(z) \prec \varphi(z);\ \text{for all } z \in \mathbb{D}\}$ where φ is analytic, univalent, $\varphi(\mathbb{D})$ is starlike with respect to $\varphi(0)=1$ and $\operatorname{Re}(\varphi) > 0$, was introduced and studied by Ma and Minda [13]. Various subclasses of starlike functions have been studied by considering

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different choices of φ in recent years. For $\varphi(z) := (1+Az)/(1+Bz), (-1 \leq B < A \leq 1)$, the class $\mathcal{S}^*_\varphi$ reduces to the class $\mathcal{S}^*[A,B]$, introduced by Janowski [9]. A function $f \in \mathscr{H}$ is said to be a Carathéodory function if $f(0) = 1$ and $\mathrm{Re}(f(z)) > 0$. The class of such functions is denoted by $\mathcal{P}$. On taking some Carathéodory functions $\varphi(z) := e^z$, $\phi_q(z)$, $\phi_0(z), \phi_c(z)$, $\phi_{\lim}(z)$, $\mathcal{Q}(z)$, ϕ_{SG}, $\phi_s(z)$, the class $\mathcal{S}^*_\varphi$ reduce to subclasses $\mathcal{S}^*_e$ [14], $\mathcal{S}^*_q$ [19], $\mathcal{S}^*_R$ [11], $\mathcal{S}^*_c$ [20], $\mathcal{S}^*_{LC}$ [22], $\mathcal{S}^*_B$ [5], $\mathcal{S}^*_{SG}$ [8], $\mathcal{S}^*_s$ [4] respectively, where

$$\phi_q(z) := z + \sqrt{1+z^2},\ \phi_0(z) := 1 + \frac{z}{k}.\frac{k+z}{k-z}; \quad k = 1+\sqrt{2},$$

$$\phi_c(z) := 1 + \frac{4z}{3} + \frac{2z^2}{3},\ \phi_{\lim}(z) := 1 + \sqrt{2}z + \frac{z^2}{2},\ \phi_s(z) := 1 + \sin z.$$

Recently, Kumar *et al.*[5] introduced and studied differential subordination relations and radius estimates for the class $\mathcal{S}^*_B := \mathcal{S}^*(\mathcal{Q}(z)$, where

$$\mathcal{Q}(z) := e^{e^z - 1} \tag{1.1}$$

In 2020, Goel and Kumar [8] studied the subclass $\mathcal{S}^*_{SG} := \mathcal{S}^*(\phi_{SG})$, where

$$\phi_{SG}(z) = 2/(1+e^{-z}) \text{ for all z } \in \mathbb{D}. \tag{1.2}$$

These subclasses of starlike functions are well associated with the right half plane of the complex plane.

In 1989, for $p \in \mathcal{P}$, Nunokawa *et al.* [17] proved that the differential subordination $1 + zp'(z) \prec 1 + z$ implies $p(z) \prec 1 + z$. Further, authors [18] established sufficient conditions for starlike functions discussed by Silverman [21] to be strongly convex and strongly starlike in $\mathbb{D}$. In 2006, Kanas [10] determined the conditions for the functions to map $\mathbb{D}$ onto hyperbolic and parabolic regions using the concept of differential subordination. In 2007, Ali *et al.* [2] obtained conditions on $\beta \in \mathbb{R}$ for which $1 + \beta zp'(z)/p^j(z) \prec (1+Dz)/(1+Ez)$, $j = 0,1,2$ implies $p(z) \prec (1+Az)/(1+Bz)$, where $A, B, D, E \in [-1, 1]$. Later, Kumar and Ravichandran [12] determined sharp upper bounds on β such that $1 + \beta zp'(z)/p^j(z), j = 0,1,2$ is subordinate to some Carathéodory functions like $e^z, \phi_0(z)$ etc. implies $p(z) \prec e^z$ and $(1+Az)/(1+Bz)$. For more such results, we refer [1, 3, 7, 6].

In the present paper, we determine sharp estimate on β so that $p(z) \prec \phi_q(z)$, $\mathcal{Q}(z)$, $\phi_c(z)$, $\phi_0(z)$, $\phi_{\lim}(z)$, $\phi_s(z)$, $\phi_{SG}(z)$ whenever $1 + \beta zp'(z)/p^j(z) \prec \mathcal{Q}(z)$ and $\phi_{SG}(z)$; $(j = 0,1,2)$. Further the best possible bound on β is computed such that $p(z) \prec \mathcal{Q}(z)$ whenever $1 + \beta zp'(z)/p^j(z) \prec \phi_c(z)$; $(j = 0,1,2)$. At last, the upper bound on β is estimated so that the subordination $1+\beta zp'(z)/p^j(z) \prec \phi_0(z)$ and $\phi_c(z)$ implies $p(z) \prec \phi_{SG}(z)$. Moreover, sufficient conditions are obtained for an analytic function f to be a member of a certain subclass of starlike function.

2. Main results

First, we recall following lemma which plays a vital role in our proofs.

Lemma 2.1. [16, Theorem 3.4h, p.132] *Let $q : \mathbb{D} \to \mathbb{C}$ be analytic, and ψ and v be analytic in a domain $U \supseteq q(\mathbb{D})$ with $\psi(w) \neq 0$ whenever $w \in q(\mathbb{D})$. Set*

$$Q(z) := zq'(z)\psi(q(z)) \quad \textit{and} \quad h(z) := v(q(z)) + Q(z), z \in \mathbb{D}.$$

Suppose that

(i) *either* $h(z)$ *is convex, or* $Q(z)$ *is starlike univalent in* $\mathbb{D}$ *and*

(ii) $\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) > 0, z \in \mathbb{D}$.

If p *is analytic in* $\mathbb{D}$, *with* $p(0) = q(0)$, $p(\mathbb{D}) \subset U$ *and*

$$v(p(z)) + zp'(z)\psi(p(z)) \prec v(q(z)) + zq'(z)\psi(q(z))$$

then $p \prec q$, *and* q *is the best dominant.*

Throughout this paper, the following notations will be used:

$$\Psi_\beta(z, p(z)) = 1 + \beta zp'(z),\ \Lambda_\beta(z, p(z)) = 1 + \beta \frac{zp'(z)}{p(z)}, \text{ and}$$

$$\Theta_\beta(z, p(z)) = 1 + \beta \frac{zp'(z)}{p^2(z)}.$$

Theorem 2.2. *Let* $\mathcal{Q}(z) \in \mathcal{P}$ *be defined by* (1.1) *and further*

$$\mathcal{L} = \int_{-1}^{0} \frac{e^{e^t-1} - 1}{t} dt \quad \text{and} \quad \mathfrak{U} = \int_{0}^{1} \frac{e^{e^t-1} - 1}{t} dt. \tag{2.1}$$

Assume p *to be an analytic function in* $\mathbb{D}$ *with* $p(0) = 1$. *If* $\Psi_\beta(z, p(z)) \prec \mathcal{Q}(z)$, *then*

(a) $p(z) \prec \phi_q(z)$ *for* $\beta \geq \frac{1}{\sqrt{2}}\mathfrak{U} \approx 1.49762$.

(b) $p(z) \prec \mathcal{Q}(z)$ *for* $\beta \geq \frac{1}{1-e^{(e^{-1}-1)}}\mathcal{L} \approx 1.446103$.

(c) $p(z) \prec \phi_c(z)$ *for* $\beta \geq \frac{1}{2}\mathfrak{U} \approx 1.05898$.

(d) $p(z) \prec \phi_0(z)$ *for* $\beta \geq \left(3 + 2\sqrt{2}\right)\mathcal{L} \approx 3.94906$.

(e) $p(z) \prec \phi_{\lim}(z)$ *for* $\beta \geq \frac{2}{2\sqrt{2}+1}\mathfrak{U} \approx 1.10643$.

(f) $p(z) \prec \phi_s(z)$ *for* $\beta \geq \frac{1}{\sin 1}\mathfrak{U} \approx 2.51696$.

(g) $p(z) \prec \phi_{SG}(z)$ *for* $\beta \geq \frac{e+1}{e-1}\mathfrak{U} \approx 4.583145$.

The bounds in each case are sharp.

Proof. The analytic function $q_\beta : \overline{\mathbb{D}} \to \mathbb{C}$ defined by

$$q_\beta(z) = 1 + \frac{1}{\beta} \int_0^z \frac{e^{e^t-1} - 1}{t} dt$$

is a solution of the first order linear differential equation $1 + \beta zq'_\beta(z) = e^{e^z-1}$. For $w \in \mathbb{C}$, define the functions $v(w) = 1$ and $\psi(w) = \beta$. Now, the function $Q : \overline{\mathbb{D}} \to \mathbb{C}$ defined by

$$Q(z) = zq'_\beta(z)\psi(q_\beta(z)) = \beta zq'_\beta(z) = e^{e^z-1} - 1$$

is starlike in $\mathbb{D}$. Also, note that by analytic characterization of starlike functions, the function $h : \overline{\mathbb{D}} \to \mathbb{C}$ defined by $h(z) := v(q_\beta(z)) + Q(z)$ satisfies the inequality

$$\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) = \operatorname{Re}\left(\frac{zQ'(z)}{Q(z)}\right) > 0.$$

Therefore, the subordination $1 + \beta zp'(z) \prec 1 + \beta zq'_\beta(z)$ implies $p \prec q_\beta$ by Lemma 2.1. For suitable $\mathcal{P}(z)$, as $r \to 1$, $q_\beta(z) \prec \mathcal{P}(z)$ holds if the following inequalities holds:

$$\mathcal{P}(-1) < q_\beta(-1) < q_\beta(1) < \mathcal{P}(1). \tag{2.2}$$

By the transitivity property, the required subordination $p(z) \prec \mathcal{P}(z)$ holds if $q_\beta(z) \prec \mathcal{P}(z)$. The condition (2.2) turns out to be both necessary and sufficient for the subordination $p \prec \mathcal{P}$ to hold.

(a) Consider $\mathcal{P}(z) = \phi_q(z)$. Then the inequalities

$$q_\beta(-1) > -1 + \sqrt{2} \text{ and } q_\beta(1) < 1 + \sqrt{2}$$

reduce to $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where

$$\beta_1 = \frac{1}{2-\sqrt{2}}\mathcal{L} \quad \text{and} \quad \beta_2 = \frac{1}{\sqrt{2}}\mathfrak{U}$$

respectively.
Thus, the subordination $q_\beta \prec \phi_q$ holds whenever $\beta \geq \max\{\beta_1, \beta_2\} = \beta_2$.

(b) For $\mathcal{P}(z) = \mathcal{Q}(z)$, the inequalities $q_\beta(-1) > \mathcal{Q}(-1)$ and $q_\beta(1) < \mathcal{Q}(1)$ give $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where

$$\beta_1 = \frac{1}{1-e^{e^{-1}-1}}\mathcal{L} \quad \text{and} \quad \beta_2 = \frac{1}{e^{e-1}-1}\mathfrak{U}$$

respectively. Therefore, $q_\beta \prec \mathcal{Q}$ whenever $\beta \geq \max\{\beta_1, \beta_2\} = \beta_1$.

(c) On taking $\mathcal{P}(z) = \phi_c(z)$, a simple calculation shows that the inequalities $q_\beta(-1) > \phi_c(-1)$ and $q_\beta(1) < \phi_c(1)$ give $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where $\beta_1 = (3/2)\mathcal{L}$ and $\beta_2 = (1/2)\mathfrak{U}$ respectively. Therefore, $q_\beta \prec \phi_c$ holds whenever $\beta \geq \max\{\beta_1, \beta_2\} = \beta_2$.

(d) On substituting $\mathcal{P}(z) = \phi_0(z)$, the inequalities

$$q_\beta(-1) > \phi_0(-1) \text{ and } q_\beta(1) < \phi_0(1)$$

give $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where $\beta_1 = (1/(3-2\sqrt{2}))\mathcal{L}$ and $\beta_2 = \mathfrak{U}$ respectively. Therefore, the subordination $q_\beta \prec \phi_0$ holds if $\beta \geq \max\{\beta_1, \beta_2\} = \beta_1$.

(e) Take $\mathcal{P}(z) = \phi_{\lim}(z)$. Then the inequalities

$$q_\beta(-1) > \frac{3}{2} - \sqrt{2} \text{ and } q_\beta(1) < \frac{3}{2} + \sqrt{2}$$

reduce to $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where

$$\beta_1 = (2/(2\sqrt{2}-1)\mathcal{L} \text{ and } \beta_2 = 2/(2\sqrt{2}+1)\mathfrak{U}$$

respectively.
Thus, the required subordination $q_\beta \prec \phi_{\lim}$ holds if $\beta \geq \max\{\beta_1, \beta_2\} = \beta_2$.

(f) Take $\mathcal{P}(z) = \phi_s(z)$. Then the inequalities

$$q_\beta(-1) > 1 + \sin(-1) \text{ and } q_\beta(1) < 1 + \sin(1)$$

give $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where $\beta_1 = \mathcal{L}/\sin 1$ and $\beta_2 = \mathfrak{U}/\sin 1$ respectively. This shows that the subordination $q_\beta \prec \phi_s$ holds if $\beta \geq \max\{\beta_1, \beta_2\} = \beta_2$.

(g) Set $\mathcal{P}(z) = 2/(1+e^{-z})$. Then $q_\beta(-1) > 2/(e+1)$ and $q_\beta(1) < 2e/(e+1)$ gives

$$\beta_1 = \frac{e+1}{e-1}\mathcal{L} \quad \text{and} \quad \beta_2 = \frac{e+1}{e-1}\mathfrak{U}.$$

Hence, the subordination holds true for $\beta \geq \beta_2$ since $\max\{\beta_1, \beta_2\} = \beta_2$.

Thus, we get the required result. □

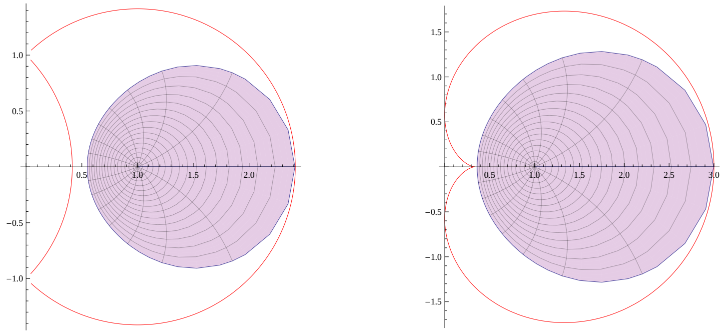

FIGURE 1. Sharpness for the case (a) and (b).

As an application of Theorem 2.2, we have the following sufficient conditions for starlikeness:

Corollary 2.3. *Set* $\mathfrak{M}(z) := 1 - zf'(z)/f(z) + zf''(z)/f'(z)$*. If the function* $f \in \mathcal{A}$ *satisfies* $1 + \beta\frac{zf'(z)}{f(z)}\mathfrak{M}(z) \prec \mathcal{Q}(z)$*, then*

(a) $f \in \mathcal{S}^*_q$ *if* $\beta \geq \left(1/\sqrt{2}\right)\mathfrak{U}$,
(b) $f \in \mathcal{S}^*_B$ *if* $\beta \geq \left(1/(1 - e^{(e^{-1}-1)})\right)\mathcal{L}$,
(c) $f \in \mathcal{S}^*_c$ *if* $\beta \geq (1/2)\,\mathfrak{U}$,
(d) $f \in \mathcal{S}^*_R$ *if* $\beta \geq \left(3 + 2\sqrt{2}\right)\mathcal{L}$,
(e) $f \in \mathcal{S}^*_{LC}$ *if* $\beta \geq \left(2/(2\sqrt{2}+1)\right)\mathfrak{U}$,
(f) $f \in \mathcal{S}^*_s$ *if* $\beta \geq (1/(\sin 1))\,\mathfrak{U}$
(g) $f \in \mathcal{S}^*_{SG}$ *if* $\beta \geq ((e+1)/(e-1))\,\mathfrak{U}$,

where $\mathfrak{U}$ *and* $\mathcal{L}$ *are given by* (2.1).

Theorem 2.4. *Let* $\mathfrak{U}$ *and* $\mathcal{L}$ *be given by* (2.1) *and* $\mathcal{Q}(z)$ *be given by* (1.1)*. Let* p *be an analytic function in* $\mathbb{D}$ *with* $p(0) = 1$*. If* $\Lambda_\beta(z, p(z)) \prec \mathcal{Q}(z)$*, then*

(a) $p(z) \prec \phi_q(z)$ *for* $\beta \geq \frac{1}{\log(1+\sqrt{2})}\mathfrak{U} \approx 2.40301.$
(b) $p(z) \prec \mathcal{Q}(z)$ *for* $\beta \geq \frac{1}{e-1}\mathfrak{U} \approx 1.23260.$
(c) $p(z) \prec \phi_c(z)$ *for* $\beta \geq \frac{1}{\log 3}\mathfrak{U} \approx 1.92784.$
(d) $p(z) \prec \phi_0(z)$ *for* $\beta \geq \frac{1}{\log\left(\frac{1+\sqrt{2}}{2}\right)}\mathcal{L} \approx 3.59966.$
(e) $p(z) \prec \phi_{\lim}(z)$ *for* $\beta \geq \frac{1}{\log(\sqrt{2}+3/2)}\mathfrak{U} \approx 1.98013.$
(f) $p(z) \prec \phi_s(z)$ *for* $\beta \geq \frac{1}{\log(1+\sin 1)}\mathfrak{U} \approx 3.4688.$
(g) $p(z) \prec \phi_{SG}(z)$ *for* $\beta \geq \frac{1}{1+\log 2-\log(1+e)}\mathfrak{U} \approx 5.57523.$

The bounds on β *are best possible.*

Proof. Consider the first order differential equation given by

$$1 + \beta\frac{z\breve{q}_\beta'(z)}{\breve{q}_\beta(z)} = e^{e^z-1}. \tag{2.3}$$

It is easy to verify that the analytic function $\breve{q}_\beta : \overline{\mathbb{D}} \to \mathbb{C}$ defined by

$$\breve{q}_\beta(z) = \exp\left(\frac{1}{\beta}\int_0^z \frac{e^{e^t-1}-1}{t}dt\right)$$

is a solution of differential equation (2.3). On taking $v(w) = 1$ and $\psi(w) = \beta/w$, the functions $Q, h : \overline{\mathbb{D}} \to \mathbb{C}$ reduces to

$$Q(z) = z\breve{q}'_\beta(z)\psi(\breve{q}_\beta(z)) = \beta z\breve{q}'_\beta(z)/\breve{q}_\beta(z) = e^{e^z-1} - 1$$

and

$$h(z) = v(\breve{q}_\beta(z)) + Q(z) = 1 + Q(z) = e^{e^z-1}.$$

It is seen that the function Q is starlike and $\operatorname{Re}(zh'(z)/Q(z)) > 0$, for $z \in \mathbb{D}$. Hence,

$$1 + \beta\frac{zp'(z)}{p(z)} \prec 1 + \beta\frac{z\breve{q}'_\beta(z)}{\breve{q}_\beta(z)} \quad \text{implies} \quad p(z) \prec \breve{q}_\beta(z)$$

which follows from Lemma 2.1. Proceeding as Theorem 2.2, proof is completed. □

Theorem 2.5. *Let* $\mathfrak{U}$ *and* $\mathcal{L}$ *be given by* (2.1) *and* $\mathcal{Q}(z)$ *be given by* (1.1). *Assume* p *to be an analytic function in* $\mathbb{D}$ *with* $p(0) = 1$. *If* $\Theta_\beta(z, p(z)) \prec \mathcal{Q}(z)$, *then each of the following subordination holds:*

(a) $p(z) \prec \phi_q(z)$ *for* $\beta \geq \frac{1}{2-\sqrt{2}}\mathfrak{U} \approx 3.61556.$
(b) $p(z) \prec \mathcal{Q}(z)$ *for* $\beta \geq \frac{e^{e-1}}{e^{e-1}-1}\mathfrak{U} \approx 2.58089.$
(c) $p(z) \prec \phi_c(z)$ *for* $\beta \geq \frac{3}{2}\mathfrak{U} \approx 3.17692.$
(d) $p(z) \prec \phi_0(z)$ *for* $\beta \geq 2\mathfrak{U} \approx 4.2359.$
(e) $p(z) \prec \phi_{\lim}(z)$ *for* $\beta \geq \frac{5+4\sqrt{2}}{7}\mathfrak{U} \approx 3.22438.$
(f) $p(z) \prec \phi_s(z)$ *for* $\beta \geq \frac{1+\sin 1}{\sin 1}\mathfrak{U} \approx 4.63491.$
(g) $p(z) \prec \phi_{SG}(z)$ *for* $\beta \geq \frac{2e}{e-1}\mathfrak{U} \approx 6.7011.$

The estimates on β *cannot be improved further.*

Proof. The function

$$\hat{q}_\beta(z) = \left(1 - \frac{1}{\beta}\int_0^z \frac{e^{e^t-1}-1}{t}dt\right)^{-1}$$

is the analytic solution of the differential equation

$$\beta\frac{z\hat{q}'_\beta(z)}{\hat{q}^2_\beta(z)} = e^{e^z-1} - 1.$$

Consider the functions $v(w) = 1$ and $\psi(w) = \beta/w^2$. Moreover, the function $Q(z) = z\hat{q}'_\beta(z)\psi(\hat{q}_\beta(z)) = e^{e^z-1} - 1$ is starlike in $\mathbb{D}$. Simple computation shows that the function $h(z) := 1 + Q(z)$ satisfies the inequality $\operatorname{Re}(zh'(z)/Q(z)) > 0, (z \in \mathbb{D})$. Now, by Lemma 2.1, we see that the subordination

$$1 + \beta\frac{zp'(z)}{p^2(z)} \prec 1 + \beta\frac{z\hat{q}'_\beta(z)}{\hat{q}^2_\beta(z)}$$

implies $p(z) \prec \hat{q}_\beta(z)$. Proceeding as Theorem 2.2, we conclude the proof. □

Theorem 2.6. *Let ϕ_{SG} be given by* (1.2) *and further*

$$I_- = \int_{-1}^{0} \frac{e^t - 1}{t(e^t + 1)} dt \quad and \quad I_+ = \int_{0}^{1} \frac{e^t - 1}{t(e^t + 1)} dt. \tag{2.4}$$

Assume p to be an analytic function in $\mathbb{D}$ with $p(0) = 1$. If the subordination

$$\Psi_\beta(z, p(z)) \prec \phi_{SG}(z)$$

holds, then each of the following subordination inclusion hold:

(a) $p(z) \prec \phi_q(z)$ *for* $\beta \geq \frac{1}{2-\sqrt{2}} I_- \approx 0.83117$.
(b) $p(z) \prec \phi_c(z)$ *for* $\beta \geq \frac{3}{2} I_- \approx 0.730335$.
(c) $p(z) \prec \phi_0(z)$ *for* $\beta \geq (3 + 2\sqrt{2}) I_- \approx 2.837797$.
(d) $p(z) \prec \mathcal{Q}(z)$ *for* $\beta \geq \frac{1}{1-e^{e^{-1}-1}} I_- \approx 1.039170$.
(e) $p(z) \prec \phi_{\lim}(z)$ *for* $\beta \geq \frac{2}{2\sqrt{2}-1} I_- \approx 0.53257$.
(f) $p(z) \prec \phi_s(z)$ *for* $\beta \geq \frac{1}{\sin 1} I_- \approx 0.578616$
(g) $p(z) \prec \phi_{SG}(z)$ *for* $\beta \geq \frac{e+1}{e-1} I_- \approx 1.05361$.

The bounds in each of the above case are sharp.

Proof. Consider the functions υ and ψ defined as in Theorem 2.2. Define the function $q_\beta : \overline{\mathbb{D}} \to \mathbb{C}$ by

$$q_\beta(z) = 1 + \frac{1}{\beta} \int_0^z \frac{e^t - 1}{t(e^t + 1)} dt$$

Note that the function $q_\beta(z)$ is analytic solution of the differential equation

$$1 + \beta z q'_\beta(z) = 2/(1 + e^{-z}).$$

The function $Q(z) = z q'_\beta(z) \psi(q_\beta(z)) = (e^z - 1)/(e^z + 1)$ is starlike in $\mathbb{D}$ and $h(z) = 1 + Q(z)$ satisfies the inequality $\operatorname{Re}(z h'(z)/Q(z)) > 0$, $z \in \mathbb{D}$. Thus, applying Lemma 2.1, it follows that the subordination $1 + \beta z p'(z) \prec 1 + \beta z q'_\beta(z)$ implies $p(z) \prec q_\beta(z)$. Each of the subordination $p(z) \prec \mathcal{P}(z)$, for appropriate $\mathcal{P}$, from (a) to (g) holds if $q_\beta(z) \prec \mathcal{P}(z)$ holds. This subordination holds provided

$$\mathcal{P}(-1) < q_\beta(-1) < q_\beta(1) < \mathcal{P}(1).$$

These inequalities yield necessary and sufficient condition for the required subordination.

(a) Take $\mathcal{P}(z) = \phi_q(z)$. Then, the inequalities $q_\beta(-1) > -1 + \sqrt{2}$ and $q_\beta(1) < 1 + \sqrt{2}$ reduce to $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where

$$\beta_1 = \frac{1}{2 - \sqrt{2}} I_- \quad \text{and} \quad \beta_2 = \frac{1}{\sqrt{2}} I_+$$

respectively. Therefore, $q_\beta \prec \phi_q$ whenever $\beta \geq \max\{\beta_1, \beta_2\} = \beta_1$.

(b) Consider $\mathcal{P}(z) = \phi_c(z)$. A simple calculation shows that the inequalities $q_\beta(-1) > \phi_c(-1)$ and $q_\beta(1) < \phi_c(1)$ gives $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where

$$\beta_1 = \frac{3}{2} I_- \quad \text{and} \quad \beta_2 = \frac{1}{2} I_+$$

respectively.
Therefore, the subordination $q_\beta \prec \phi_c$ holds if $\beta \geq \max\{\beta_1, \beta_2\} = \beta_1$.

(c) On taking $\mathcal{P}(z) = \phi_0(z)$, the inequalities $q_\beta(-1) > \phi_0(-1)$ and $q_\beta(1) < \phi_0(1)$ give $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where

$$\beta_1 = \frac{1}{3 - 2\sqrt{2}} I_- \quad \text{and} \quad \beta_2 = I_+$$

respectively. Therefore, $q_\beta \prec \phi_0$ if $\beta \geq \max\{\beta_1, \beta_2\} = \beta_1$.

(d) Consider $\mathcal{P}(z) = \mathcal{Q}(z)$. From the inequalities $q_\beta(-1) > \mathcal{Q}(-1)$ and $q_\beta(1) < \mathcal{Q}(1)$, we note that $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where

$$\beta_1 = \frac{1}{1 - e^{e^{-1}-1}} I_- \quad \text{and} \quad \beta_2 = \frac{1}{e^{e-1} - 1} I_+$$

respectively. Thus, $q_\beta \prec \mathcal{Q}$ if $\beta \geq \max\{\beta_1, \beta_2\}$.

(e) Take $\mathcal{P}(z) = \phi_{\lim}(z)$. Then, the inequalities $q_\beta(-1) > \frac{3}{2} - \sqrt{2}$ and $q_\beta(1) < \frac{3}{2} + \sqrt{2}$ reduce to $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where

$$\beta_1 = \frac{2}{2\sqrt{2} - 1} I_- \quad \text{and} \quad \beta_2 = \frac{2}{2\sqrt{2} + 1} I_+$$

respectively. Thus, $q_\beta \prec \phi_{\lim}$ whenever $\beta \geq \max\{\beta_1, \beta_2\} = \beta_1$.

(f) Take $\mathcal{P}(z) = \phi_s(z)$. Then, the inequalities $q_\beta(-1) > 1 + \sin(-1)$ and $q_\beta(1) < 1 + \sin(1)$ give $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where

$$\beta_1 = \frac{1}{\sin 1} I_- \quad \text{and} \quad \beta_2 = \frac{1}{\sin 1} I_+$$

respectively.
Therefore, the subordination $q_\beta \prec \phi_s$ holds if $\beta \geq \max\{\beta_1, \beta_2\} = \beta_1$.

(g) Let $\mathcal{P}(z) = \phi_{SG}(z)$. On simplifying the inequalities $q_\beta(-1) > 2/(e+1)$ and $q_\beta(1) < 2e/(e+1)$, we get β_1 and β_2, where

$$\beta_1 = \frac{e+1}{e-1} I_- \quad \text{and} \quad \beta_2 = \frac{e+1}{e-1} I_+$$

respectively and thus, $q_\beta \prec \phi_{SG}$ whenever $\beta \geq \max\{\beta_1, \beta_2\} = \beta_1$.

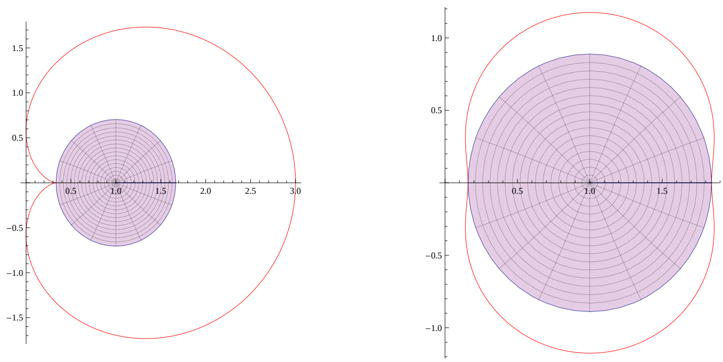

FIGURE 2. Sharpness for the case (b) and (f).

Hence, the result holds. □

As an application of Theorem 2.6, we have the following sufficient conditions for starlikeness:

Corollary 2.7. *Let $f \in \mathcal{A}$ be analytic function which satisfies*

$$1 + \beta \frac{zf'(z)}{f(z)} \mathfrak{M}(z) \prec \phi_{SG}(z).$$

Then,

(a) $f \in \mathcal{S}_q^*$ *if* $\beta \geq \left(1/(2-\sqrt{2})\right) I_-$,
(b) $f \in \mathcal{S}_c^*$ *if* $\beta \geq (3/2)\, I_-$,
(c) $f \in \mathcal{S}_R^*$ *if* $\beta \geq (3+2\sqrt{2}) I_-$,
(d) $f \in \mathcal{S}_B^*$ *if* $\beta \geq \left(1/(1-e^{e^{-1}-1})\right) I_-$,
(e) $f \in \mathcal{S}_{LC}^*$ *if* $\beta \geq \left(2/(2\sqrt{2}-1)\right) I_-$,
(f) $f \in \mathcal{S}_s^*$ *if* $\beta \geq (1/(\sin 1))\, I_-$,

where $\mathfrak{M}(z)$ is defined in Corollary 2.3.

Theorem 2.8. *Let I_+ and I_- be given by 2.4 and ϕ_{SG} be given by* (1.2)*. Assume p to be an analytic function in $\mathbb{D}$ with $p(0) = 1$. If $\Lambda_\beta(z, p(z)) \prec \phi_{SG}(z)$, then each of the following holds.*

(a) $p(z) \prec \phi_q(z)$ *for* $\beta \geq \frac{1}{\log(1+\sqrt{2})} I_- \approx 0.55242$.
(b) $p(z) \prec \phi_c(z)$ *for* $\beta \geq \frac{1}{\log 3} I_- \approx 0.443185$.
(c) $p(z) \prec \phi_0(z)$ *for* $\beta \geq \frac{1}{\log\left(\frac{1+\sqrt{2}}{2}\right)} I_- \approx 2.58671$.
(d) $p(z) \prec \mathcal{Q}(z)$ *for* $\beta \geq \frac{1}{1-e^{-1}} I_- \approx 0.77024$.
(e) $p(z) \prec \phi_{\lim}(z)$ *for* $\beta \geq \frac{1}{\log(\sqrt{2}+3/2)} I_+ \approx 0.455206$.
(f) $p(z) \prec \phi_s(z)$ *for* $\beta \geq \frac{1}{\log(1+\sin 1)} I_+ \approx 0.79744$.
(g) $p(z) \prec \phi_{SG}(z)$ *for* $\beta \geq \frac{1}{1+\log 2-\log(1+e)} I_+ \approx 1.28167$.

The estimates on β are best possible.

Proof. Let the functions v and ψ be defined as in Theorem 2.4. Define the analytic function $\breve{q}_\beta : \overline{\mathbb{D}} \to \mathbb{C}$ by

$$\breve{q}_\beta(z) = \exp\left(\frac{1}{\beta}\int_0^z \frac{e^t-1}{t(e^t+1)}dt\right),$$

which satisfies the differential equation

$$\frac{d\breve{q}'_\beta(z)}{dz} = \frac{1}{\beta z}\left(\frac{1-e^{-z}}{1+e^{-z}}\right)\breve{q}_\beta(z).$$

Now, observe that the function $Q(z) = z\breve{q}'_\beta(z)\psi(\breve{q}_\beta(z)) = \frac{1-e^{-z}}{1+e^{-z}}$ is starlike in $\mathbb{D}$. Also, it can be easily seen that the function h defined by $h(z) := v(\breve{q}_\beta(z)) + Q(z) = 1 + Q(z)$ satisfies the inequality $\operatorname{Re}(zh'(z)/Q(z)) > 0$, $z \in \mathbb{D}$. Therefore, the Lemma 2.1 states that the subordination $1 + \beta\frac{zp'(z)}{p(z)} \prec 1 + \beta\frac{z\breve{q}'_\beta(z)}{\breve{q}_\beta(z)}$ implies $p(z) \prec \breve{q}_\beta(z)$. As in the proof of Theorem 2.6, we conclude the result. □

Theorem 2.9. *Let I_+ and I_- be given by* (2.4)*. Assume p to be an analytic function in $\mathbb{D}$ with $p(0) = 1$. If $\Theta_\beta(z, p(z)) \prec \phi_{SG}(z)$, then*

(a) $p(z) \prec \phi_q(z)$ *for* $\beta \geq \frac{1}{2-\sqrt{2}} I_+ \approx 0.83117$.
(b) $p(z) \prec \phi_c(z)$ *for* $\beta \geq \frac{3}{2} I_+ \approx 0.73033$.
(c) $p(z) \prec \phi_0(z)$ *for* $\beta \geq (2 + 2\sqrt{2}) I_- \approx 2.35090$.
(d) $p(z) \prec \mathcal{Q}(z)$ *for* $\beta \geq \frac{e^{e-1}}{e^{e-1}-1} I_+ \approx 0.59331$.
(e) $p(z) \prec \phi_{\lim}(z)$ *for* $\beta \geq \frac{5+4\sqrt{2}}{7} I_+ \approx 0.74124$.
(f) $p(z) \prec \phi_s(z)$ *for* $\beta \geq \frac{1+\sin 1}{\sin 1} I_+ \approx 1.06550$.
(g) $p(z) \prec \phi_{SG}(z)$ *for* $\beta \geq \frac{2e}{e-1} I_+ \approx 1.54049$.

All these estimates are sharp.

Proof. The function $\hat{q}_\beta : \overline{\mathbb{D}} \to \mathbb{C}$ defined by

$$\hat{q}_\beta(z) = \left(1 - \frac{1}{\beta} \int_0^z \frac{e^t - 1}{t(e^t + 1)} dt\right)^{-1}$$

is clearly analytic in $\mathbb{D}$. It is noted that the function $\hat{q}_\beta(z)$ is a solution of the differential equation

$$1 + \beta \frac{z\hat{q}_\beta'(z)}{\hat{q}_\beta^2(z)} = \frac{2}{1 + e^{-z}}.$$

We take the functions v and ψ as in Theorem 2.5. Note that the function Q defined by $Q(z) = z\hat{q}_\beta'(z)\psi(\hat{q}_\beta(z)) = (1 - e^{-z})/(1 + e^{-z})$ is starlike in $\mathbb{D}$ and the function h defined as $h(z) := v(\hat{q}_\beta(z)) + Q(z) = 1 + Q(z)$ follows the inequality $\operatorname{Re}(zh'(z)/Q(z)) = \operatorname{Re}(zQ'(z)/Q(z)) > 0$. Therefore, as in view of Lemma 2.1, the subordination

$$1 + \beta \frac{zp'(z)}{p^2(z)} \prec 1 + \beta \frac{z\hat{q}_\beta'(z)}{\hat{q}_\beta^2(z)}$$

implies $p(z) \prec \hat{q}_\beta(z)$. Proceeding as in Theorem 2.6, proof is completed. □

Theorem 2.10. *Let p be an analytic function in $\mathbb{D}$ with $p(0) = 1$. Then each of the following subordination implies $p(z) \prec \mathcal{Q}(z) := e^{e^z - 1}$:*

(a) $\Psi_\beta(z, p(z)) \prec \phi_c(z)$ *if* $\beta \geq \frac{1}{1 - e^{(e^{-1}-1)}} \approx 2.13430$.
(b) $\Lambda_\beta(z, p(z)) \prec \phi_c(z)$ *if* $\beta \geq \frac{e}{e-1} \approx 1.581976$.
(c) $\Theta_\beta(z, p(z)) \prec \phi_c(z)$ *if* $\beta \geq \frac{5e^{e-1}}{3(e^{e-1}-1)} \approx 2.030970$.

The bounds in each case are sharp.

Proof. (a) Define the analytic function $q_\beta : \overline{\mathbb{D}} \to \mathbb{C}$ by

$$q_\beta(z) = 1 + \frac{1}{\beta}\left(\frac{4z}{3} + \frac{z^2}{3}\right)$$

It is easy to see that the function q_β satisfies the differential equation $\beta z q'(z) = \phi_c(z) - 1$. Proceeding as similar lines in Theorem 2.2, the required subordination holds if and only if,

$$e^{e^{-1}-1} < q_\beta(-1) < q_\beta(1) < e^{e-1}. \tag{2.5}$$

Simplifying the condition (2.5), we obtain the inequalities

$$\beta \geq \frac{1}{1-e^{(e^{-1}-1)}} = \beta_1 \quad \text{and} \quad \beta \geq \frac{5e}{3\,(e^e - e)} = \beta_2.$$

Thus, the required subordination holds if $\beta \geq \max\{\beta_1, \beta_2\} = \beta_1$.

(b) Define the analytic function $\breve{q}_\beta(z)$ by,

$$\breve{q}_\beta(z) = \exp\left(\frac{1}{\beta}\left(\frac{4z}{3} + \frac{z^2}{3}\right)\right)$$

which is a solution of the equation

$$\frac{d\breve{q}'_\beta(z)}{dz} = \frac{2(2+z)}{3\beta}\breve{q}_\beta(z).$$

Proceeding as similar lines in Theorem 2.4, the subordination $p(z) \prec e^{e^z-1}$ holds if $\beta \geq \max\{\breve{\beta}_1, \breve{\beta}_2\}$, where

$$\breve{\beta}_1 = \frac{e}{e-1} \text{ and } \breve{\beta}_2 = \frac{5}{3(e-1)}$$

are obtained from the inequalities $\breve{q}_\beta(-1) > e^{e^{-1}-1}$ and $\breve{q}_\beta(1) < e^{e-1}$ respectively.

(c) The differential equation

$$\frac{d\hat{q}'_\beta(z)}{dz} = \frac{2(2+z)}{3\beta}\hat{q}^2_\beta(z)$$

has an analytic solution

$$\hat{q}_\beta(z) = \left(1 - \frac{1}{\beta}\left(\frac{4z}{3} + \frac{z^2}{3}\right)\right)^{-1}$$

in $\mathbb{D}$. Therefore, proceeding as in Theorem 2.5, the required subordination $p(z) \prec e^{e^z-1}$ holds if $\beta \geq \max\{\hat{\beta}_1, \hat{\beta}_2\} = \hat{\beta}_2$, where

$$\hat{\beta}_1 = \frac{e^{\frac{1}{e}-1}}{1-e^{\frac{1}{e}-1}} \text{ and } \hat{\beta}_2 = \frac{5e^{e-1}}{3(e^{e-1}-1)}.$$

□

Corollary 2.11. *Let $f \in \mathcal{A}$ be given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. If one of the following subordinations holds*

(a) $1 + \beta\frac{zf'(z)}{f(z)}\mathfrak{M}(z) \prec \phi_c(z)$ *for* $\beta \geq \frac{1}{1-e^{(e^{-1}-1)}}$,

(b) $1 + \beta\mathfrak{M}(z) \prec \phi_c(z)$ *for* $\beta \geq \frac{e}{e-1}$,

(c) $1 + \beta\left(\frac{zf'(z)}{f(z)}\right)^{-1}\mathfrak{M}(z) \prec \phi_c(z)$ *for* $\beta \geq \frac{5e^{e-1}}{3(e^{e-1}-1)}$,

then $f \in \mathcal{S}^*_B$, where $\mathfrak{M}(z)$ is defined in Corollary 2.3.

The next results provide best possible bound on β so that the subordination $1+\beta zp'(z)/p^j(z) \prec \phi_c(z)$, $\phi_0(z)(j = 0, 1, 2)$ implies the subordination $p(z) \prec \phi_{SG}(z)$. Proofs of the following results are omitted as similar to the previous Theorem 2.10.

Theorem 2.12. *Let p be an analytic function in $\mathbb{D}$ with $p(0) = 1$. Then the following subordinations hold for $p(z) \prec \phi_{SG}(z) := 2/(1 + e^{-z})$.*

(a) $\Psi_\beta(z, p(z)) \prec \phi_0(z)$ *if* $\beta \geq \frac{(e+1)(1-\sqrt{2}-2\log(2-\sqrt{2}))}{e-1} \approx 1.418226$.

(b) $\Lambda_\beta(z, p(z)) \prec \phi_0(z)$ *if* $\beta \geq \frac{1-\sqrt{2}-2\log(2-\sqrt{2})}{1+\log 2-\log(1+e)} \approx 1.725221$.

(c) $\Theta_\beta(z, p(z)) \prec \phi_0(z)$ *if* $\beta \geq \frac{2e(1-\sqrt{2}-2\log(2-\sqrt{2}))}{e-1} \approx 2.073612$.

The bounds on β in each case are sharp.

Theorem 2.13. *Let p be an analytic function in $\mathbb{D}$ which satisfies $p(0) = 1$. Then each of the following subordination is sufficient for $p(z) \prec \phi_{SG}(z)$.*

(a) $\Psi_\beta(z, p(z)) \prec \phi_c(z)$ *if* $\beta \geq \frac{5(e+1)}{3(e-1)} \approx 3.60659$.

(b) $\Lambda_\beta(z, p(z)) \prec \phi_c(z)$ *if* $\beta \geq \frac{5}{3(1+\log 2-\log(1+e))} \approx 4.387286$.

(c) $\Theta_\beta(z, p(z)) \prec \phi_c(z)$ *if* $\beta \geq \frac{10e}{3(e-1)} \approx 5.27326$.

The bounds on β in each case are sharp.

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