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Hardy-Littlewood-Stein-Weiss type theorems for Riesz potentials and their commutators in Morrey spaces

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Abstract. In this paper we consider weighted Morrey spaces $L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$. We prove the Hardy-Littlewood-Stein-Weiss type $L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ to $L_{q,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$ theorems for Riesz potential I^α and its commutators $[b, I^\alpha]$ and $|b, I^\alpha|$, where $0 < \alpha < n$, $0 \leq \lambda < n - \alpha$, $1 < p < \frac{n-\lambda}{\alpha}$, $-n + \lambda \leq \gamma < n(p-1) + \lambda$, $\mu = \frac{q\gamma}{p}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$, $b \in BMO(\mathbb{R}^n)$. As a result of these we obtain the conditions for the boundedness of the commutator $|b, I^\alpha|$ from Besov-Morrey spaces $B^s_{p,\theta,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ to $B^s_{q,\theta,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$. Furthermore, we consider the Schrödinger operator $-\Delta + V$ on $\mathbb{R}^n$ and obtain weighted Morrey $L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ estimates for the operators $V^s(-\Delta + V)^{-\beta}$ and $V^s\nabla(-\Delta + V)^{-\beta}$. Finally we apply our results to various operators which are estimated from above by Riesz potentials.

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1. Introduction

The well known Morrey spaces $\mathcal{L}^{p,\lambda}(\Omega)$ introduced by Charles Morrey (see [24]) in 1938 in relation to the study of partial differential equations, and presented in various books, see e.g. [11, 16, 39]. They were widely investigated during the last decades, including the study of classical operators of harmonic analysis maximal, singular and potential operators on Morrey spaces and their various generalizations

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have found wide applications in many problems of real analysis and partial differential equations. Morrey spaces are defined by the norm

$$\|f\|_{\mathcal{L}^{p,\lambda}} = \sup_{x,\, t>0} t^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,t))},$$

where $0 \le \lambda < n$, $1 \le p < \infty$ and $B(x,t)$ is the open ball in $\mathbb{R}^n$ of radius t centered at x. In the theory of partial differential equations, together with weighted Lebesgue spaces, Morrey spaces play an important role. Later, Morrey spaces found important applications to Navier-Stokes ([22], [39]) and Schrödinger ([28], [29], [30], [33], [34]) equations, elliptic problems with discontinuous coefficients ([5], [8]), and potential theory ([1], [2]).

The results on the boundedness of potential operators and classical Calderón-Zygmund singular operators go back to [1] and [27], respectively, while the boundedness of the maximal operator in the Euclidean setting was proved in [6].

Hardy-Littlewood-Stein-Weiss inequality in the Lebesgue spaces was proved by H.G. Hardy and J.E. Littlewood [12] in the one-dimensional case and by E.M. Stein and G. Weiss [37] in the case $n > 1$. In the Lebesgue and Morrey spaces with variable exponent the Hardy-Littlewood-Stein-Weiss inequality was proved by S.G. Samko [31] and J.J. Hasanov [13], respectively.

Let f be a locally integrable function on $\mathbb{R}^n$. The so-called fractional maximal function is defined by the formula

$$M^\alpha f(x) = \sup_{t>0} |B(x,t)|^{-1+\alpha/n} \int_{B(x,t)} |f(y)|dy, \ 0 \le \alpha < n,$$

where $|B(x,t)|$ is the Lebesgue measure of the ball $B(x,t)$ such that $|B(x,t)| = \omega_n t^n$ in which ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. It coincides with the Hardy-Littlewood maximal function $Mf \equiv M_0 f$. Maximal operators play an important role in the differentiability properties of functions, singular integrals and partial differential equations. They often provide a deeper and more simplified approach to understanding problems in these areas than other methods.

Fractional maximal operator is intimately related to the Riesz potential

$$I^\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|^{n-\alpha}}, \qquad 0 < \alpha < n,$$

such that

$$M^\alpha f(x) \le \omega_n^{\frac{\alpha}{n}-1} (I^\alpha |f|(x)).$$

The aim of this paper is to give the necessary and sufficient conditions for the boundedness of Riesz potential I^α and its commutators from weighted Morrey spaces $L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ to $L_{p,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$. We also obtain the necessary conditions for the boundedness of the commutator $|b, I^\alpha|$ from Besov-Morrey spaces $B^s_{p,\theta,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ to $B^s_{q,\theta,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$. Furthermore, we consider the Schrödinger operator $-\Delta + V$ on $\mathbb{R}^n$ and obtain weighted Morrey $L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ estimates for the operators $V^s(-\Delta+V)^{-\beta}$ and $V^s\nabla(-\Delta+V)^{-\beta}$. Finally we apply our results to various operators which are estimated from above by Riesz potentials.

Throughout the paper we use the letters c, C for positive constants, independent of appropriate parameters and not necessarily the same at each occurrence. If $A \le CB$ and $B \le CA$, we write $A \approx B$ and say that A and B are equivalent.

2. Preliminaries

We use the following notation. For $1 \le p < \infty$, $L_p(\mathbb{R}^n)$ is the space of all classes of measurable functions on $\mathbb{R}^n$ for which

$$\|f\|_{L_p} = \left(\int\limits_{\mathbb{R}^n} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty,$$

up to the equivalence of the norms

$$\|f\|_{L_p} \sim \sup_{\|g\|_{L^{p'}} \le 1} \left| \int_{\mathbb{R}^n} f(y)g(y)dy \right| \tag{2.1}$$

and also $WL_p(\mathbb{R}^n)$, the weak L_p space defined as the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{WL_p} = \sup_{r>0} r \left| \{x \in \mathbb{R}^n \ : \ |f(x)| > r\} \right|^{1/p} < \infty.$$

For $p = \infty$ the space $L_\infty(\mathbb{R}^n)$ is defined by means of the usual modification

$$\|f\|_{L_\infty} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} |f(x)|.$$

For $1 \le p < \infty$ let $L_{p,\omega}(\mathbb{R}^n)$ be the space of measurable functions on $\mathbb{R}^n$ such that

$$\|f\|_{L_{p,\omega}} = \|f\omega^{1/p}\|_{L_p(\mathbb{R}^n)} = \left(\int\limits_{\mathbb{R}^n} |f(x)|^p \omega(x) dx \right)^{1/p} < \infty,$$

and for $p = \infty$ the space $L_{\infty,\omega}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$.

Definition 2.1. The weight function ω belongs to the class $A_p(\mathbb{R}^n)$ for $1 \le p < \infty$, if the following statement

$$\sup_{x \in \mathbb{R}^n, t>0} \frac{1}{|B(x,t)|} \int\limits_{B(x,t)} \omega(y) dy \left(\frac{1}{|B(x,r)|} \int\limits_{B(x,r)} \omega^{-\frac{1}{p-1}}(y) dy \right)^{p-1}$$

is finite and ω belongs to $A_1(\mathbb{R}^n)$, if there exists a positive constant C such that for any $x \in \mathbb{R}^n$ and $t > 0$

$$|B(x,t)|^{-1} \int\limits_{B(x,t)} \omega(y) dy \le C \operatorname*{ess\,sup}_{y \in B(x,t)} \frac{1}{\omega(y)}.$$

The following theorem was proved in [37].

Theorem 2.2. *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$, $\alpha p - n < \gamma < n(p-1)$, $\mu = \frac{q\gamma}{p}$. *Then the operators* M^α *and* I^α *are bounded from* $L_{p,|\cdot|^\gamma}(\mathbb{R}^n)$ *to* $L_{q,|\cdot|^\mu}(\mathbb{R}^n)$.

Theorem 2.3. [36] *Let $1 < p < \infty$ and $-n < \gamma < n(p-1)$. Then the operator M is bounded on $L_{p,|\cdot|^\gamma}(\mathbb{R}^n)$.*

Let $M^\sharp$ be the sharp maximal function defined by

$$M^\sharp f(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y) - f_{B(x,t)}|dy,$$

where $f_{B(x,t)}(x) = |B(x,t)|^{-1} \int_{B(x,t)} f(y)dy$.

Definition 2.4. We define the $BMO(\mathbb{R}^n)$ space as the set of all locally integrable functions f with finite norm

$$\|f\|_{BMO} = \sup_{x\in\mathbb{R}^n,\, t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y) - f_{B(x,t)}|dy$$

or

$$\|f\|_{BMO} = \inf_C \sup_{x\in\mathbb{R}^n,\, t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y) - C|dy.$$

Definition 2.5. We define the $BMO_{p,\omega}(\mathbb{R}^n)$ $(1 \le p < \infty)$ space as the set of all locally integrable functions f with finite norm

$$\|f\|_{BMO_{p,\omega}} = \sup_{x\in\mathbb{R}^n,\, t>0} \frac{\|(f(\cdot) - f_{B(x,t)})\chi_{B(x,t)}\|_{L_{p,\omega}(\mathbb{R}^n)}}{\|\chi_{B(x,t)}\|_{L_{p,\omega}(\mathbb{R}^n)}}.$$

Theorem 2.6. [14, Theorem 4.4] *Let $1 \le p < \infty$ and ω be a Lebesgue measurable function. If $\omega \in A_p(\mathbb{R}^n)$, then the norms $\|\cdot\|_{BMO_{p,\omega}}$ and $\|\cdot\|_{BMO}$ are mutually equivalent.*

We find it convenient to define the Morrey and weighted Morrey spaces in the form as follows.

Definition 2.7. Let $1 \le p < \infty$. Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ and weighted Morrey spaces $L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ are defined by the norms

$$\|f\|_{L_{p,\lambda}} = \sup_{x\in\mathbb{R}^n, t>0} t^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,t))}$$

and

$$\|f\|_{L_{p,\lambda,|\cdot|^\gamma}} = \sup_{x\in\mathbb{R}^n, t>0} t^{-\frac{\lambda}{p}} \|f\|_{L_{p,|\cdot|^\gamma}(B(x,t))},$$

respectively.

For $1 \le p, \theta \le \infty$ and $0 < s < 1$, Besov-Morrey space $B^s_{p,\theta,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ consists of all functions $f \in L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ such that

$$\|f\|_{B^s_{p,\theta,\lambda,|\cdot|^\gamma}} = \|f\|_{L_{p,\lambda,|\cdot|^\gamma}} + \left(\int_{\mathbb{R}^n} \frac{\|f(x-\cdot) - f(\cdot)\|^\theta_{L_{p,\lambda,|\cdot|^\gamma}}}{|x|^{n+s\theta}} dx \right)^{1/\theta} < \infty.$$

3. Riesz potential operator in the spaces $L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$

In this section we prove the Hardy-Littlewood-Stein-Weiss type $L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ to $L_{q,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$ -theorem for Riesz potential I^α, where $-n+\lambda \le \gamma < n(p-1)+\lambda$, $1 < p < \frac{n-\lambda}{\alpha}$, $\mu = \frac{q\gamma}{p}$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$.
First we give following theorems which we use while proving our main results.

Theorem 3.1. [25] *Let $1 < p < \infty$, then $M : L_{p,\varphi}(\mathbb{R}^n) \to L_{p,\varphi}(\mathbb{R}^n)$ if and only if $\varphi \in A_p(\mathbb{R}^n)$.*

Theorem 3.2. [15] *Let $1 < p < \infty$, $0 \le \lambda < n$, $\varphi \in A_p(\mathbb{R}^n)$, then $M : L_{p,\lambda,\varphi}(\mathbb{R}^n) \to L_{p,\lambda,\varphi}(\mathbb{R}^n)$.*

Theorem 3.3. *Let $0 < \alpha < n$, $0 \le \lambda < n-\alpha$, $1 < p < \frac{n-\lambda}{\alpha}$, $-n+\lambda \le \gamma < n(p-1)+\lambda$ and $\mu = \frac{q\gamma}{p}$. Then the operator I^α is bounded from $L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ to $L_{q,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$ if and only if $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$.*

Proof. Sufficiency: Let $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ and $f \in L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$. Then

$$\begin{aligned} |I^\alpha f(x)| &= \left(\int\limits_{B(x,t)} + \int\limits_{\mathbb{R}^n \setminus B(x,t)} \right) |f(y)||x-y|^{\alpha-n} dy \\ &\equiv F_1(x,t) + F_2(x,t). \end{aligned}$$

First we estimate $F_1(x,t)$. By using Hölder's inequality we have

$$\begin{aligned} F_1(x,t) &= \int\limits_{B(x,t)} |f(y)||x-y|^{\alpha-n} dy \\ &\le \sum_{j=-\infty}^{-1} \left(2^j t\right)^{\alpha-n} \int\limits_{B(x,2^{j+1}t)\setminus B(x,2^j t)} |f(y)| dy \\ &\le Ct^\alpha Mf(x). \end{aligned} \tag{3.1}$$

Now we estimate $F_2(x,t)$. By using Hölder's inequality we get

$$\begin{aligned} F_2(x,t) &\le \int\limits_{\mathbb{R}^n \setminus B(x,t)} |f(y)||x-y|^{\alpha-n} dy \\ &\le \sum_{j=0}^{\infty} \left(2^j t\right)^{\alpha-n} \int\limits_{B(x,2^{j+1}t)\setminus B(x,2^j t)} |f(y)| dy \\ &\le \sum_{j=0}^{\infty} \left(2^j t\right)^{\alpha-n} \left\|\chi_{B(x,2^{j+1}t)}\right\|_{L_{p'(\cdot),|\cdot|^{\gamma/(1-p)}}} \left\|f\chi_{B(x,2^{j+1}t)}\right\|_{L_{p,|\cdot|^\gamma}} \\ &\le Ct^{\alpha-\frac{n-\lambda}{p}} |x|^{-\frac{\gamma}{p}} \|f\|_{L_{p,\lambda,|\cdot|^\gamma}} \sum_{j=0}^{\infty} 2^{j\left(\alpha-\frac{n-\lambda}{p}\right)} \\ &\le Ct^{\alpha-\frac{n-\lambda}{p}} |x|^{-\frac{\gamma}{p}} \|f\|_{L_{p,\lambda,|\cdot|^\gamma}} \end{aligned}$$

Thus

$$F_2(x,t) \le C t^{\alpha - \frac{n-\lambda}{p}} |x|^{-\frac{\gamma}{p}} \|f\|_{L_{p,\lambda,|\cdot|^\gamma}}. \tag{3.2}$$

Therefore from (3.1) and (3.2) we get

$$|I^\alpha f(x)| \le C t^\alpha Mf(x) + C t^{\alpha - \frac{n-\lambda}{p}} |x|^{-\frac{\gamma}{p}} \|f\|_{L_{p,\lambda,|\cdot|^\gamma}}.$$

Minimizing with respect to $t = \left[(Mf(x))^{-1} \|f\|_{L_{p,\lambda,|\cdot|^\gamma}} \right]^{\frac{p}{n-\lambda}} |x|^{-\frac{\gamma}{n-\lambda}}$ we arrive at

$$|I^\alpha f(x)| \le C \left(\frac{Mf(x)}{\|f\|_{L_{p,\lambda,|\cdot|^\gamma}}} \right)^{1-\frac{p\alpha}{n-\lambda}} |x|^{-\frac{\gamma\alpha}{n-\lambda}}.$$

It is obvious that

$$|x|^\gamma = |x|^{\mu - \frac{\gamma\alpha q}{n-\lambda}}.$$

From Theorem 3.2, taking $\varphi(x) = |x|^\gamma$ we get

$$\begin{aligned} \int\limits_{B(x,t)} |I^\alpha f(y)|^q |y|^\mu dy &\le C \|f\|^{q-p}_{L_{p,\lambda,|\cdot|^\gamma}} \int\limits_{B(x,t)} (Mf(y))^p |y|^\gamma dy \\ &\le C t^\lambda \|f\|^{q-p}_{L_{p,\lambda,|\cdot|^\gamma}} \|f\|^p_{L_{p,\lambda,|\cdot|^\gamma}} \\ &= C t^\lambda \|f\|^q_{L_{p,\lambda,|\cdot|^\gamma}}. \end{aligned}$$

Therefore $I^\alpha f \in L_{q,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$ and we obtain

$$\|I^\alpha f\|_{L_{q,\lambda,|\cdot|^\mu}} \le C \|f\|_{L_{p,\lambda,|\cdot|^\gamma}}.$$

Necessity: Let I^α be bounded from $L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ to $L_{q,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$, $1 < p < \frac{n-\lambda}{\alpha}$. Define $f_t(x) =: f(tx)$, $t > 0$. Then

$$\begin{aligned} \left(r^{-\lambda} \int_{B(x,r)} |f_t(y)|^p |y|^\gamma dy \right)^{1/p} &= t^{-\frac{n+\gamma}{p}} \left(r^{-\lambda} \int_{B(x,tr)} |f(y)|^p |y|^\gamma dy \right)^{1/p} \\ &= t^{-\frac{n-\lambda+\gamma}{p}} \left((tr)^{-\lambda} \int_{B(x,tr)} |f(y)|^p |y|^\gamma dy \right)^{1/p} \\ &\le t^{-\frac{n-\lambda+\gamma}{p}} \|f\|_{L_{p,\lambda,|\cdot|^\gamma}}. \end{aligned}$$

Therefore we get

$$\|f_t\|_{L_{p,\lambda,|\cdot|^\gamma}} \le t^{-\frac{n-\lambda+\gamma}{p}} \|f\|_{L_{p,\lambda,|\cdot|^\gamma}}.$$

Since

$$I^\alpha f_t(x) = t^{-\alpha} I^\alpha f(tx),$$

we obtain

$$\left(r^{-\lambda}\int_{B(x,r)}|I^{\alpha}f_t(y)|^q\,|y|^{\mu}dy\right)^{1/q}=t^{-\alpha}\left(r^{-\lambda}\int_{B(x,r)}|I^{\alpha}f(ty)|^q\,|y|^{\mu}dy\right)^{1/q}$$
$$=t^{-\alpha-\frac{n-\lambda+\mu}{q}}\left((tr)^{-\lambda}\int_{B(x,tr)}|I^{\alpha}f(y)|^q\,|y|^{\mu}dy\right)^{1/q}$$
$$\le t^{-\alpha-\frac{n-\lambda+\mu}{q}}\,\|I^{\alpha}f\|_{L_{q,\lambda,|\cdot|^{\mu}}}.$$

Therefore we get

$$\|I^{\alpha}f_t\|_{L_{q,\lambda,|\cdot|^{\mu}}}\le t^{-\alpha-\frac{n-\lambda+\mu}{q}}\,\|I^{\alpha}f\|_{L_{q,\lambda,|\cdot|^{\mu}}}.$$

Since the operator I^{α} is bounded from $L_{p,\lambda,|\cdot|^{\gamma}}(\mathbb{R}^n)$ to $L_{q,\lambda,|\cdot|^{\mu}}(\mathbb{R}^n)$, we have

$$\|I^{\alpha}f_t\|_{L_{q,\lambda,|\cdot|^{\mu}}}\le Ct^{-\alpha-\frac{n-\lambda+\mu}{q}+\frac{n-\lambda+\gamma}{p}}\|f\|_{L_{p,\lambda,|\cdot|^{\gamma}}},\tag{3.3}$$

where C depends on p,q,λ,γ,μ and n.
If $\frac{1}{p}>\frac{1}{q}+\frac{\alpha}{n-\lambda}$, from the inequality (3.3), $\|I^{\alpha}f_t\|_{L_{q,\lambda,|\cdot|^{\mu}}}=0$ for all $f\in L_{p,\lambda,|\cdot|^{\gamma}}(\mathbb{R}^n)$ as $t\to 0$.
If $\frac{1}{p}<\frac{1}{q}+\frac{\alpha}{n-\lambda}$, from the inequality (3.3), $\|I^{\alpha}f_t\|_{L_{q,\lambda,|\cdot|^{\mu}}}=0$ for all $f\in L_{p,\lambda,|\cdot|^{\gamma}}(\mathbb{R}^n)$ as $t\to\infty$. Therefore $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n-\lambda}$. □

Remark 3.4. The proof of the sufficiency part of Theorem 3.3 is also given with different methods in [26].

Corollary 3.5. [26] *Let $0<\alpha<n$, $0\le\lambda<n-\alpha$, $1<p<\frac{n-\lambda}{\alpha}$, $-n+\lambda\le\gamma<n(p-1)+\lambda$, $\mu=\frac{q\gamma}{p}$ and $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n-\lambda}$. Then the operator M^{α} is bounded from $L_{p,\lambda,|\cdot|^{\gamma}}(\mathbb{R}^n)$ to $L_{q,\lambda,|\cdot|^{\mu}}(\mathbb{R}^n)$.*

4. Commutators of the Riesz potential operator in the spaces $L_{p,\lambda,|\cdot|^{\gamma}}(\mathbb{R}^n)$

It is well-known that the commutator is an important integral operator and it plays a key role in harmonic analysis. In this section we consider commutators of the Riesz potential defined by the following equality

$$[b,I^{\alpha}]f(x)=\int_{\mathbb{R}^n}(b(x)-b(y))|x-y|^{\alpha-n}f(y)dy,\quad 0<\alpha<n.$$

Given a measurable function b the operator $|b,I^{\alpha}|$ is defined by

$$|b,I^{\alpha}|f(x)=\int_{\mathbb{R}^n}|b(x)-b(y)|\,|x-y|^{\alpha-n}\,|f(y)|dy,\quad 0<\alpha<n.$$

The following statement holds:

Lemma 4.1. [9] *Let $1 < s < \infty$ and $b \in BMO(\mathbb{R}^n)$. Then there exists a positive constant C, independent of f and x, such that*

$$M^\sharp([b, I^\alpha]f(x)) \le C\|b\|_{BMO}\left[(M|I^\alpha f(x)|^s)^{\frac{1}{s}} + (M^{s\alpha}|f(x)|^s)^{\frac{1}{s}}\right].$$

Proposition 4.2. *([36], Lemma 3.5) Let $1 < p < \infty$. Then for all $f \in L^p(\mathbb{R}^n)$ and $g \in L^{p'}(\mathbb{R}^n)$ there exists a positive constant C such that*

$$\left|\int_{\mathbb{R}^n} f(y)g(y)dy\right| \le C\left|\int_{\mathbb{R}^n} M^\sharp f(y)Mg(y)dy\right|.$$

The following lemma is valid.

Lemma 4.3. *Let $1 < p < \infty$, $\varphi \in A_p(\mathbb{R}^n)$. Then there exists a positive constant C, independent of f, such that*

$$\|f\varphi^{\frac{1}{p}}\|_{L_p(\mathbb{R}^n)} \le C\|\varphi^{\frac{1}{p}}M^\sharp f\|_{L_p(\mathbb{R}^n)}.$$

Proof. By (2.1) we have

$$\|f\varphi^{\frac{1}{p}}\|_{L_p(\mathbb{R}^n)} \le C \sup_{\|g\|_{L_{p'}(\mathbb{R}^n)}\le 1}\left|\int_{\mathbb{R}^n} f(y)g(y)\varphi^{\frac{1}{p}}(y)dy\right|.$$

According to Proposition 4.2,

$$\|f\varphi^{\frac{1}{p}}\|_{L_p(\mathbb{R}^n)} \le C \sup_{\|g\|_{L_{p'}(\mathbb{R}^n)}\le 1}\left|\int_{\mathbb{R}^n} M^\sharp f(y)M(g\varphi^{\frac{1}{p}})(y)dy\right|.$$

From Hölder inequality and Theorem 3.1, we obtain

$$\|f\varphi^{\frac{1}{p}}\|_{L_p(\mathbb{R}^n)} \le C \sup_{\|g\|_{L_{p'}(\mathbb{R}^n)}\le 1}\|\varphi^{\frac{1}{p}}M^\sharp f\|_{L_p(\mathbb{R}^n)}\|\varphi^{-\frac{1}{p}}M(g\varphi^{\frac{1}{p}})\|_{L_{p'}(\mathbb{R}^n)}$$

$$\le C \sup_{\|g\|_{Lp'(\mathbb{R}^n)}\le 1}\|\varphi^{\frac{1}{p}}M^\sharp f\|_{L_p(\mathbb{R}^n)}\|g\|_{L_{p'}(\mathbb{R}^n)} \le C\|\varphi^{\frac{1}{p}}M^\sharp f\|_{L_p(\mathbb{R}^n)}.$$ □

Corollary 4.4. *Let $1 < p < \infty$, $\varphi = \psi|\cdot|^\gamma \in A_p(\mathbb{R}^n)$. Then there exists a positive constant C, independent of f, such that*

$$\|f\psi^{\frac{1}{p}}\|_{L_{p,|\cdot|^\gamma}(\mathbb{R}^n)} \le C\|\psi^{\frac{1}{p}}M^\sharp f\|_{L_{p,|\cdot|^\gamma}(\mathbb{R}^n)}.$$

Lemma 4.5. *Let $1 < p < \infty$, $0 \le \lambda < n$. Then the following inequality holds*

$$\|f\|_{L_{p,\lambda,|\cdot|^\gamma}} \le C\left\|M^\sharp f\right\|_{L_{p,\lambda,|\cdot|^\gamma}}.$$

Proof. If $0 < \theta < 1$, $\psi(x) = (M\chi_{B(x,r)})^\theta \in A_p(\mathbb{R}^n)$, from Lemma 4.3 we have

$$\|f\|_{L_{p,|\cdot|^\gamma}(B(x,r))} \le \|f\psi^{\frac{1}{p}}\|_{L_{p,|\cdot|^\gamma}(\mathbb{R}^n)} \le C\|\psi^{\frac{1}{p}}M^\sharp f\|_{L_{p,|\cdot|^\gamma}(\mathbb{R}^n)} \le C\|M^\sharp f\|_{L_{p,|\cdot|^\gamma}(B(x,r))}.$$

Therefore we get

$$\|f\|_{L_{p,\lambda,|\cdot|^\gamma}} = \sup_{x\in\mathbb{R}^n, r>0} r^{-\frac{\lambda}{p}}\|f\|_{L_{p,|\cdot|^\gamma}(B(x,t))}$$

$$\le C \sup_{x\in\mathbb{R}^n, r>0} r^{-\frac{\lambda}{p}}\|M^\sharp f\|_{L_{p,|\cdot|^\gamma}(B(x,r))} = C\|M^\sharp f\|_{L_{p,\lambda,|\cdot|^\gamma}}.$$

Thus the lemma has been proved. □

In the following theorem we give the necessary and sufficient conditions for the boundedness of the commutator $[b, I^\alpha]$ from $L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ to $L_{q,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$.

Theorem 4.6. *Let* $0 < \alpha < n$, $0 \le \lambda < n-\alpha$, $1 < p < \frac{n-\lambda}{\alpha}$, $-n+\lambda \le \gamma < n(p-1)+\lambda$, $\mu = \frac{q\gamma}{p}$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$. *Then the commutator* $[b, I^\alpha]$ *is bounded from* $L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ *to* $L_{q,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$ *if and only if* $b \in BMO$.

Proof. Let $f \in L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ and $b \in BMO(\mathbb{R}^n)$. From Lemma 4.5, we have

$$\|[b, I^\alpha]f\|_{L_{q,\lambda,|\cdot|^\mu}} \le C_1 \|M^\sharp([b, I^\alpha]f)\|_{L_{q,\lambda,|\cdot|^\mu}}.$$

From Lemma 4.1, we get

$$\|M^\sharp([b, I^\alpha]f)\|_{L_{q,\lambda,|\cdot|^\mu}} \le C_2 \|b\|_{BMO} \left\| (M|I^\alpha f|^s)^{\frac{1}{s}} + (M^{\alpha s}|f|^s)^{\frac{1}{s}} \right\|_{L_{q,\lambda,|\cdot|^\mu}}$$
$$\le C_3 \|b\|_{BMO} \left[\left\| (M|I^\alpha f|^s)^{\frac{1}{s}} \right\|_{L_{q,\lambda,|\cdot|^\mu}} + \left\| (M^{\alpha s}|f|^s)^{\frac{1}{s}} \right\|_{L_{q,\lambda,|\cdot|^\mu}} \right].$$

From Theorem 3.2 and Theorem 3.3, we have

$$\left\| (M|I^\alpha f|^s)^{\frac{1}{s}} \right\|_{L_{q,\lambda,|\cdot|^\mu}} = \|M|I^\alpha f|^s\|^{\frac{1}{s}}_{L_{\frac{q}{s},\lambda,|\cdot|^\mu}}$$
$$\le C \, \||I^\alpha f|^s\|^{\frac{1}{s}}_{L_{\frac{q}{s},\lambda,|\cdot|^\mu}} = C \, \|I^\alpha f\|_{L_{q,\lambda,|\cdot|^\mu}} \le C \, \|f\|_{L_{p,\lambda,|\cdot|^\mu}}.$$

Similarly it can be shown that

$$\left\| (M^{\alpha s}|f|^s)^{\frac{1}{s}} \right\|_{L_{q,\lambda,|\cdot|^\mu}} \le C \, \|f\|_{L_{p,\lambda,|\cdot|^\gamma}}.$$

Therefore we obtain

$$\|[b, I^\alpha]f\|_{L_{q,\lambda,|\cdot|^\mu}} \le C_2 \|b\|_{BMO} \, \|f\|_{L_{p,\lambda,|\cdot|^\gamma}}.$$

$(i) \Rightarrow (ii)$ Now, let us prove the "only if" part. Let $[b, I^\alpha]$ be bounded from $L_{p,\lambda,|\cdot|^\gamma}$ to $L_{q,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$, $1 < p < \frac{n-\lambda}{\alpha}$. Now we consider $f = \chi_{B(x,r)}$. It is easy to compute that

$$\left\|\chi_{B(x,r)}\right\|_{L_{p,\lambda,|\cdot|^\gamma}} \approx \sup_{t>0,\, x\in\mathbb{R}^n} \left(t^{-\lambda} \int\limits_{B(y,t)} \chi_{B(x,r)}(y)|y|^\gamma dy \right)^{1/p}$$
$$\approx \sup_{B(y,t)\subset B(x,r)} \left(t^{-\lambda} \int\limits_{B(y,t)} |y|^\gamma dy \right)^{1/p} \approx r^{\frac{n-\lambda+\gamma}{p}}.$$

Then

$$\frac{1}{|B(x,t)|}\int_{B(x,t)}|b(z)-b_{B(x,t)}|dz$$

$$=\frac{1}{|B(x,t)|}\int\limits_{B(x,t)}\left|b(z)-\frac{1}{|B(x,t)|}\int\limits_{B(x,t)}b(y)dy\right|dz$$

$$\leq\frac{1}{|B(x,t)|^{1+\frac{\alpha}{n}}}\int\limits_{B(x,t)}\frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}}\left|\int\limits_{B(x,t)}(b(z)-b(y))\,dy\right|dz$$

$$\leq\frac{1}{|B(x,t)|^{1+\frac{\alpha}{n}}}\int\limits_{B(x,t)}\left|\int\limits_{B(x,t)}(b(z)-b(y))\,|x-y|^{\alpha-n}dy\right|dz$$

$$\leq\frac{1}{|B(x,t)|^{1+\frac{\alpha}{n}}}\int\limits_{B(x,t)}\left|[b,I^{\alpha}]\chi_{B(x,t)}(z)\right|dz$$

$$\leq Ct^{-n-\alpha+\lambda}\|[b,I^{\alpha}]\chi_{B(x,t)}\|_{L_{q,\lambda,|\cdot|^{\mu}}}\|\chi_{B(x,t)}\|_{L_{q',\lambda,|\cdot|^{\frac{\mu}{1-q}}}}$$

$$\leq Ct^{-n-\alpha+\frac{n-\lambda+\gamma}{p}+n-\frac{n-\lambda+\mu}{q}}\leq C.$$

Hence we get

$$|B(x,t)|^{-1}\int_{B(x,t)}|b(y)-b_{B(x,t)}|dy\leq C,$$

which shows that $b\in BMO(\mathbb{R}^n)$.
Thus the theorem has been proved. □

Theorem 4.7. *Let $0<\alpha<n$, $0\leq\lambda<n-\alpha$, $1<p<\frac{n-\lambda}{\alpha}$, $-n+\lambda\leq\gamma<n(p-1)+\lambda$, $\mu=\frac{q\gamma}{p}$ and $b\in BMO$. Then the commutator $|b,I^{\alpha}|$ is bounded from $L_{p,\lambda,|\cdot|^{\gamma}}(\mathbb{R}^n)$ to $L_{q,\lambda,|\cdot|^{\mu}}(\mathbb{R}^n)$ if and only if $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n-\lambda}$.*

Proof. 1) The sufficiency follows from Theorem 4.6.
Necessity: Let $1<p<\frac{n-\lambda}{\alpha}$ and $|b,I^{\alpha}|$ be bounded from $L_{p,\lambda,|\cdot|^{\gamma}}(\mathbb{R}^n)$ to $L_{q,\lambda,|\cdot|^{\mu}}(\mathbb{R}^n)$. Define $f_t(x)=:f(tx)$, $t>0$. Then

$$\left(r^{-\lambda}\int_{B(x,r)}|f_t(y)|^p|y|^{\gamma}dy\right)^{1/p}=t^{-\frac{n+\gamma}{p}}\left(r^{-\lambda}\int_{B(x,tr)}|f(y)|^p|y|^{\gamma}dy\right)^{1/p}$$

$$=t^{-\frac{n-\lambda+\gamma}{p}}\left((tr)^{-\lambda}\int_{B(x,tr)}|f(y)|^p|y|^{\gamma}dy\right)^{1/p}$$

$$\leq t^{-\frac{n-\lambda+\gamma}{p}}\|f\|_{L_{p,\lambda,|\cdot|^{\gamma}}}.$$

Therefore we get

$$\|f_t\|_{L_{p,\lambda,|\cdot|^{\gamma}}}\leq t^{-\frac{n-\lambda+\gamma}{p}}\|f\|_{L_{p,\lambda,|\cdot|^{\gamma}}}.$$

Since

$$|b,I^\alpha|f_t(x)=t^{-\alpha}|b,I^\alpha|f(tx),$$

we obtain

$$\begin{aligned}&\left(r^{-\lambda}\int_{B(x,r)}\left[||b,I^\alpha|f_t|\right]^q(y)|y|^\mu dy\right)^{1/q}\\&=t^{-\alpha}\left(r^{-\lambda}\int_{B(x,r)}\left[||b,I^\alpha|f|\right]^q(ty)|y|^\mu dy\right)^{1/q}\\&=t^{-\alpha-\frac{n-\lambda+\mu}{q}}\left((tr)^{-\lambda}\int_{B(x,tr)}\left[||b,I^\alpha|f|\right]^q(y)|y|^\mu dy\right)^{1/q}\\&\le t^{-\alpha-\frac{n-\lambda+\mu}{q}}\left\||b,I^\alpha|f\right\|_{L_{q,\lambda,|\cdot|^\mu}}.\end{aligned}$$

Therefore we get

$$\left\||b,I^\alpha|f_t\right\|_{L_{q,\lambda,|\cdot|^\mu}}\le t^{-\alpha-\frac{n-\lambda+\mu}{q}}\left\||b,I^\alpha|f\right\|_{L_{q,\lambda,|\cdot|^\mu}}.$$

Since the operator $|b,I^\alpha|$ is bounded from $L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ to $L_{q,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$, we have

$$\left\||b,I^\alpha|f_t\right\|_{L_{q,\lambda,|\cdot|^\mu}}\le Ct^{-\alpha-\frac{n-\lambda+\mu}{q}+\frac{n-\lambda+\gamma}{p}}\|b\|_{BMO}\|f\|_{L_{p,\lambda,|\cdot|^\gamma}}, \tag{4.1}$$

where C depends on p,q,λ,γ,μ and n.
If $\frac{1}{p}>\frac{1}{q}+\frac{\alpha}{n-\lambda}$, from the inequality (4.1), $\left\||b,I^\alpha|f_t\right\|_{L_{q,\lambda,|\cdot|^\mu}}=0$ for all $f\in L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ as $t\to 0$.
If $\frac{1}{p}<\frac{1}{q}+\frac{\alpha}{n-\lambda}$, from the inequality (4.1), $\left\||b,I^\alpha|f_t\right\|_{L_{q,\lambda,|\cdot|^\mu}}=0$ for all $f\in L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ as $t\to\infty$. Therefore $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n-\lambda}$. □

The following theorem gives the conditions for the boundedness of the commutator $|b,I^\alpha|$ from $B^s_{p,\theta,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ to $B^s_{q,\theta,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$.

Theorem 4.8. *Let $0<\alpha<n$, $0\le\lambda<n-\alpha$, $1<p<\frac{n-\lambda}{\alpha}$, $-n+\lambda\le\gamma<n(p-1)+\lambda$, $\mu=\frac{q\gamma}{p}$, $0<s<1$, $1\le\theta\le\infty$, $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n-\lambda}$ and $b\in BMO(\mathbb{R}^n)$. Then the commutator $|b,I^\alpha|$ is bounded from $B^s_{p,\theta,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ to $B^s_{q,\theta,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$.*

Proof. From the definition of the Besov-Morrey type spaces it suffices to show that

$$\left\||b,I^\alpha|f(x-\cdot)-|b,I^\alpha|f(\cdot)\right\|_{L_{p,\lambda,|\cdot|^\gamma}}\le C\|b\|_{BMO}\|f(x-\cdot)-f(\cdot)\|_{L_{p,\lambda,|\cdot|^\gamma}}.$$

Hence we have

$$|[b,I^\alpha]f(x-\cdot)-|b,I^\alpha|f|\le|b,I^\alpha|(|f(x-\cdot)-f|).$$

Taking $L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ norm of both sides of the above inequality, from the boundedness of $|b,I^\alpha|$ from $L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ to $L_{q,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$, we obtain the desired result. Thus Theorem 4.8 has been proved. □

5. The weighted Morrey estimates for the operators $V^s(-\Delta+V)^{-\beta}$ and $V^s\nabla(-\Delta+V)^{-\beta}$

In this section we consider the Schrödinger operator $-\Delta+V$ on $\mathbb{R}^n$, where the nonnegative potential V belongs to the reverse Hölder class $B_q(\mathbb{R}^n)$ for some $q_1 \geq n$. We obtain weighted Morrey $L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ estimates for the operators $V^s(-\Delta+V)^{-\beta}$ and $V^s\nabla(-\Delta+V)^{-\beta}$.

Schrödinger operators on the Euclidean space $\mathbb{R}^n$ with nonnegative potentials which belong to the reverse Hölder class have been studied by many authors (see [10, 32, 40]). Shen [32] studied the Schrödinger operator $-\Delta+V$, assuming the nonnegative potential V belongs to the reverse Hölder class $B_q(\mathbb{R}^n)$ for $q \geq n/2$ and he proved the L_p boundedness of the operators $(-\Delta+V)^{is}$, $\nabla^2(-\Delta+V)^{-1}$, $\nabla(-\Delta+V)^{-\frac{1}{2}}$ and $\nabla(-\Delta+V)^{-1}$. Kurata and Sugano generalized Shens' results to uniformly elliptic operators in [18]. Sugano [38] also extended some results of Shen to the operator $V^s(-\Delta+V)^{-\beta}$, $0 \leq s \leq \beta \leq 1$ and $V^s\nabla(-\Delta+V)^{-\beta}$, $0 \leq s \leq \frac{1}{2} \leq \beta \leq 1$ and $\beta - s \geq \frac{1}{2}$. Later, Lu [21] and Li [19] investigated the Schrödinger operators in a more general setting.
We investigate the weighted Morrey $L_{p,\lambda,|\cdot|^\gamma} - L_{q,\lambda,|\cdot|^\mu}$ boundedness of the operators

$$T_1 = V^s(-\Delta+V)^{-\beta},\ 0 \leq s \leq \beta \leq 1,$$

$$T_2 = V^s\nabla(-\Delta+V)^{-\beta},\ 0 \leq s \leq \frac{1}{2} \leq \beta \leq 1,\ \beta - s \geq \frac{1}{2}.$$

Note that the operators $V(-\Delta+V)^{-1}$ and $V^{\frac{1}{2}}\nabla(-\Delta+V)^{-1}$ in [19] are the special case of T_1 and T_2, respectively.

It is worth pointing out that we need to establish pointwise estimates for T_1, T_2 and their adjoint operators by using the estimates of fundamental solution for the Schrödinger operator on $\mathbb{R}^n$ in [19]. And we give the Morrey estimates by using $L_{p,\lambda,|\cdot|^\gamma} - L_{q,\lambda,|\cdot|^\mu}$ boundedness of the fractional maximal operators.

Definition 5.1. 1) A nonnegative locally L_p integrable function V on $\mathbb{R}^n$ is said to belong to the reverse Hölder class B_p $(1 < p < \infty)$ if there exists a positive constant C such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V(x)^p dx\right)^{\frac{1}{p}} \leq \frac{C}{|B|}\int_B V(x)dx$$

holds for every ball B in $\mathbb{R}^n$.

2) Let $V \geq 0$. We say $V \in B_\infty$, if there exists a positive constant C such that the inequality

$$\|V\|_{L_\infty(B)} \leq \frac{C}{|B|}\int_B V(x)dx$$

holds for every ball B in $\mathbb{R}^n$.

Clearly, $B_\infty \subset B_p$ for $1 < p < \infty$. But it is important that the B_p class has a property of "self-improvement"; that is, if $V \in B_p$, then $V \in B_{p+\varepsilon}$ for some $\varepsilon > 0$ (see [19]).

The following two pointwise estimates for T_1 and T_2 were proved in [40] with the potential $V \in B_\infty$.

Theorem A. *Suppose $V \in B_\infty$ and $0 \le s \le \beta \le 1$. Then there exists a positive constant C such that*

$$|T_1 f(x)| \le C M^\alpha f(x), \ f \in C_0^\infty(\mathbb{R}^n),$$

where $\alpha = 2(\beta - s)$.

Theorem B. *Suppose $V \in B_\infty$, $0 \le s \le \frac{1}{2} \le \beta \le 1$ and $\beta - s \ge \frac{1}{2}$. Then there exists a positive constant C such that*

$$|T_2 f(x)| \le C M^\alpha f(x), \ f \in C_0^\infty(\mathbb{R}^n),$$

where $\alpha = 2(\beta - s) - 1$.

Note that the similar estimates for the adjoint operators T_1^* and T_2^* with the potential $V \in B_{q_1}$ for some $q_1 > \frac{n}{2}$ are also valid (see [20]).

Theorem C. *Suppose $V \in B_{q_1}$ for some $q_1 > \frac{n}{2}$, $0 \le s \le \beta \le 1$ and let $\frac{1}{q_2} = 1 - \frac{\alpha}{q_1}$. Then there exists a positive constant C such that*

$$|T_1^* f(x)| \le C \left(M_{\alpha q_2}\left(|f|^{q_2}\right)(x)\right)^{\frac{1}{q_2}}, \ f \in C_0^\infty(\mathbb{R}^n),$$

where $\alpha = 2(\beta - s)$.

Theorem D. *Suppose $V \in B_{q_1}$ for some $q_1 > \frac{n}{2}$, $0 \le s \le \frac{1}{2} \le \beta \le 1$ and $\beta - s \ge \frac{1}{2}$. And let*

$$\frac{1}{q_1} = \begin{cases} 1 - \frac{s}{q_1}, & \text{if } q_1 > n, \\ 1 - \frac{\alpha+1}{q_1} + \frac{1}{n}, & \text{if } \frac{n}{2} < q_1 < n. \end{cases}$$

Then there exists a positive constant C such that

$$|T_2^* f(x)| \le C \left(M_{\alpha q_2}\left(|f|^{q_2}\right)(x)\right)^{\frac{1}{q_2}}, \ f \in C_0^\infty(\mathbb{R}^n),$$

where $\alpha = 2(\beta - s) - 1$.

The above theorems will yield the weighted Morrey estimates for T_1 and T_2.

Corollary 5.2. *Assume that $V \in B_\infty$, and $0 \le s \le \beta \le 1$. Let $1 < p < \frac{n}{s}$, $-n + \lambda \le \gamma < n(p-1) + \lambda$, $\mu = \frac{q\gamma}{p}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ and $0 \le \lambda < n$, where $\alpha = 2(\beta - s) < n$.*

Then for any $f \in C_0^\infty(\mathbb{R}^n)$ there exists a positive constant C such that

$$\|T_1 f\|_{L_{q,\lambda,|\cdot|^\mu}} \le C\|f\|_{L_{p,\lambda,|\cdot|^\gamma}}.$$

Corollary 5.3. *Let $V \in B_\infty$, $0 \le s \le \frac{1}{2} \le \beta \le 1$, $\beta - s \ge \frac{1}{2}$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$, $-n + \lambda \le \gamma < n(p-1) + \lambda$, $\mu = \frac{q\gamma}{p}$ and $0 \le \lambda < n$, where $\alpha = 2(\beta - s) - 1 < n$.*

Then for any $f \in C_0^\infty(\mathbb{R}^n)$ there exists a positive constant C such that

$$\|T_2 f\|_{L_{q,\lambda,|\cdot|^\mu}} \le C\|f\|_{L_{p,\lambda,|\cdot|^\gamma}}.$$

Corollary 5.4. *Assume that $V \in B_{q_1}$ for $q_1 > \frac{n}{2}$, and $0 \le s \le \beta \le 1$.*

Let $\frac{1}{q_2} = 1 - \frac{\alpha}{q_1}$, $1 < p < \frac{1}{\frac{\alpha}{q_1} + \frac{\alpha}{n}}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\frac{n}{q_2} - \lambda}$, $-n + \lambda \le \gamma < n(p-1) + \lambda$, $\mu = \frac{q\gamma}{p}$ and $0 \le \lambda < nq_2$, where $\alpha = 2(\beta - s) < n$.

Then for any $f \in C_0^\infty(\mathbb{R}^n)$ there exists a positive constant C such that

$$\|T_1 f\|_{L_{q,\lambda,|\cdot|^\mu}} \le C\|f\|_{L_{p,\lambda,|\cdot|^\gamma}}.$$

Corollary 5.5. *Assume that* $V \in B_{q_1}$ *for* $q_1 > \frac{n}{2}$, *and*

$$\begin{cases} 0 \le s \le \frac{1}{2} \le \beta \le 1, & \text{if } q_1 > n, \\ 0 \le s \le \frac{1}{2} < \beta \le 1, & \text{if } \frac{n}{2} < q_1 < n. \end{cases}$$

Let $\alpha = 2(\beta - s) - 1 < n$ *and* $\beta - s \ge \frac{1}{2}$, *and let* $1 < p < \frac{1}{\frac{\alpha}{q_1} + \frac{\alpha}{n}}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\frac{n}{q_2} - \lambda}$, $\frac{1}{q_2} = 1 - \frac{\alpha}{q_1}$, $-n + \lambda \le \gamma < n(p-1) + \lambda$, $\mu = \frac{q\gamma}{p}$ *and* $0 \le \lambda < nq_2$, *where*

$$\frac{1}{p_1} = \begin{cases} \frac{\alpha}{q_1}, & \text{if } q_1 > n, \\ \frac{\alpha+1}{q_1} + \frac{1}{n}, & \text{if } \frac{n}{2} < q_1 < n. \end{cases}$$

Then for any $f \in C_0^\infty(\mathbb{R}^n)$ *there exists a positive constant* C *such that*

$$\|T_2 f\|_{L_{q,\lambda,|\cdot|^\mu}} \le C \|f\|_{L_{p,\lambda,|\cdot|^\gamma}}.$$

6. Some applications

The theorems of the Section 3 can be applied to various operators which are estimated from above by Riesz potentials. Now we give some examples.

Suppose that L is a linear operator on L_2 which generates an analytic semigroup e^{-tL} with the kernel $p_t(x,y)$ satisfying a Gaussian upper bound, that is,

$$|p_t(x,y)| \le \frac{c_1}{t^{n/2}} e^{-c_2 \frac{|x-y|^2}{t}} \tag{6.1}$$

for $x, y \in \mathbb{R}^n$ and all $t > 0$.

For $0 < \alpha < n$, the fractional powers $L^{-\alpha/2}$ of the operator L are defined by

$$L^{-\alpha/2} f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-tL} f(x) \frac{dt}{t^{-\alpha/2+1}}.$$

Note that if $L = -\triangle$ is the Laplacian on $\mathbb{R}^n$, then $L^{-\alpha/2}$ is the Riesz potential I^α. (See, for example, Chapter 5 in [36].)

Theorem 6.1. *Let* $0 < \alpha < n$, $0 \le \lambda < n - \alpha$, $1 < p < \frac{n-\lambda}{\alpha}$, $-n + \lambda \le \gamma < n(p-1) + \lambda$, $\mu = \frac{q\gamma}{p}$ *and condition* (6.1) *be satisfied. Then condition* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ *is sufficient for the boundedness of* $L^{-\alpha/2}$ *from* $L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ *to* $L_{q,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$.

Proof. Since the semigroup e^{-tL} has the kernel $p_t(x,y)$ which satisfies condition (6.1), it follows that

$$|L^{-\alpha/2} f(x)| \le C I^\alpha |f|(x)$$

for all $x \in \mathbb{R}^n$ (see [7]). Therefore from the aforementioned theorems we have

$$\|L^{-\alpha/2} f\|_{L_{q,\lambda,|\cdot|^\mu}} \le C \|I^\alpha |f|\|_{L_{q,\lambda,|\cdot|^\mu}} \le C \|f\|_{L_{p,\lambda,|\cdot|^\gamma}}.$$ □

Large classes of differential operators satisfies condition (6.1). Now we investigate two of them:

(i) Let us consider a magnetic potential $\vec{a}$, i. e., a real-valued vector potential $\vec{a} = (a_1, a_2, \ldots, a_n)$, and an electric potential V. Assume that for any $k = 1, 2, \ldots, n$, $a_k \in L_2^{loc}$ and $0 \le V \in L_1^{loc}$. The magnetic Schrödinger operator, L, is defined by

$$L = -(\nabla - i\vec{a})^2 + V(x).$$

From the well-known diamagnetic inequality (see [35], Theorem 2.3) we have the following pointwise estimate. For any $t > 0$ and $f \in L_2$,

$$|e^{-tL} f| \le e^{-t\triangle}|f|,$$

which implies that the semigroup e^{-tL} has the kernel $p_t(x, y)$ that satisfies upper bound (6.1).

(ii) Let $A = (a_{ij}(x))_{1\le i,j\le n}$ be an $n \times n$ matrix with complex-valued entries $a_{ij} \in L_\infty$ satisfying

$$\mathrm{Re} \sum_{i,j=1}^{n} a_{ij}(x)\zeta_i\zeta_j \ge \lambda|\zeta|^2$$

for all $x \in \mathbb{R}^n, \zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in \mathbb{C}^n$ and some $\lambda > 0$. Consider the divergence form operator

$$Lf \equiv -\mathrm{div}(A\nabla f),$$

which is interpreted in the usual weak sense via the appropriate sesquilinear form.

It is known that the Gaussian bound (6.1) for the kernel of e^{-tL} holds when A has real-valued entries (see, for example, [3]), or when $n = 1, 2$ in the case of complex-valued entries (see [4, Chapter 1]).

Finally we note that under the appropriate assumptions (see [23]; [36], Chapter 5; [4], pp. 58-59) one can obtain results similar to Theorem 6.1 for a homogeneous elliptic operator L in L_2 of order $2m$ in the divergence form

$$Lf = (-1)^m \sum_{|\alpha|=|\beta|=m} D^\alpha \left(a_{\alpha\beta} D^\beta f\right).$$

In this case estimate (6.1) should be replaced by

$$|p_t(x, y)| \le \frac{c_3}{t^{n/2m}} e^{-c_4\left(\frac{|x-y|}{t^{1/(2m)}}\right)^{2m/(2m-1)}}$$

for all $t > 0$ and all $x,\ y \in \mathbb{R}^n$.

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