Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 3, 467–480
DOI: 10.24193/subbmath.2023.3.01

Identification of induction curves

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Abstract. Induction curves (induction surfaces, induction sets in general) were recently introduced to provide a visual aid to examine the fractions defining the norm of a matrix, along with the discovery and description of p-eigenvectors. In our current investigation we delve into an inverse problem, the identification of induction curves. Namely: could the elements of the matrix and the used power parameter p be reconstructed given the induction curve, i.e. the case of 2×2 matrices is examined. The analytic solution is not possible in most cases already in this planar setting, therefore numerical approximation methods shall be applied.

Mathematics Subject Classification (2010): 15A83, 47A30, 65F20, 65F35.

Keywords: Matrix norms, power norm, p-norm, induction curves, identification, optimization, p-eigenvectors, Nelder–Mead method.

1. Introduction

A common way to define a norm of a matrix is to take the supremum of the fraction of the vector norms of the matrix-vector product and the non-zero vector, with respect to a given vector norm, i.e. the least upper bound for the norm of the vectors of the transformed unit sphere. Recently induction curves (induction surfaces, induction sets in general) were introduced to provide a visual aid to examine these fractions defining the norm of a matrix, along with the discovery and description of p-eigenvectors [12, 13]. The study of different phenomena in relation to various norms (most importantly with $p = 1$, 2 and ∞) is a traditional and still active topic [2, 3, 6, 8, 16].

This paper has been presented at 14th Joint Conference on Mathematics and Computer Science, Cluj-Napoca, November 24-27, 2022.

Received 01 March 2023; Accepted 14 August 2023.

In our current investigation we delve into the inverse problem, the identification of induction curves, posed in [12]. Namely: could the elements of the matrix and the used parameter p be reconstructed given the induction curve/surface/manifold, or a sampled subset of such an object. For now we restrict ourselves to induction curves, i.e. the case of 2×2 matrices.

The analytic solution is not possible in most cases already in this planar setting, therefore numerical approximation methods shall be applied. In this work our experiences using the well-known Nelder–Mead algorithm [14] are summarized. We have already successfully applied this method earlier to identification problems related to ECG curves and also examined a hyperbolic variant of it [7, 9, 11]. Of course several further optimization methods exist the application of which shall be also investigated for our problem at hand in the future. We have recently seen advances in related topics concerning e.g. Newton-type solvers, conjugate gradient (BFGS) and gradient projection methods [1, 5, 15, 17].

The software package of Matlab/Octave programs available at

`http://locsi.web.elte.hu/indsets/`

will be extended with new components for the task of identification.

2. Formulating the problem

Let us now consider a matrix $A \in \mathbb{R}^{n\times n}$. The p-norm of A is defined as

$$\|.\|_p : \mathbb{R}^{n\times n} \to \mathbb{R}, \qquad \|A\|_p = \sup_{x\neq 0} \frac{\|Ax\|_p}{\|x\|_p} \qquad (p \in [1,\infty]),$$

with the usual power norms for vectors $x \in \mathbb{R}^n$

$$\|.\|_p : \mathbb{R}^{n} \to \mathbb{R}, \qquad \|x\|_p = \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} \qquad (p \in [1,\infty))$$

and

$$\|x\|_\infty = \max_{k=1}^{n} |x_k| .$$

It is well known that $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$ $(x \in \mathbb{R}^n)$. Notable examples for the above matrix norms include the column norm for $p = 1$, the spectral norm for $p = 2$ and the row norm for $p = \infty$.

Definition 2.1. (c.f. [12], Def. 1.) Given a matrix $A \in \mathbb{R}^{n\times n}$ with $2 \le n \in \mathbb{N}$ and $p \in [1,\infty]$, the set of points

$$\mathcal{I}_p(A) := \left\{ \frac{\|Ax\|_p}{\|x\|_p} \cdot \frac{x}{\|x\|_2} \in \mathbb{R}^n \,:\, 0 \neq x \in \mathbb{R}^n \right\} \subset \mathbb{R}^n$$

is called the *induction set* of A with parameter p. The induction set may be called induction curve for $n = 2$, induction surface for $n = 3$, induction manifold in general.

Remark 2.2. An induction set basically describes the effect of the multiplication with the matrix on the norm of the vectors in each direction (independent of the length of the vector). The properties of induction sets are discussed in detail in [12]. Here we recall the following. For each direction the set contains exactly one point and its distance from the origin depends continuously on the direction, so in case of 2×2 matrices the set is a closed curve around the origin. These sets are always symmetric with respect to the origin. The values $p \in (0, 1)$ may be also allowed. These sets are not to be confused with the transformed unit sphere by multiplication with the matrix.

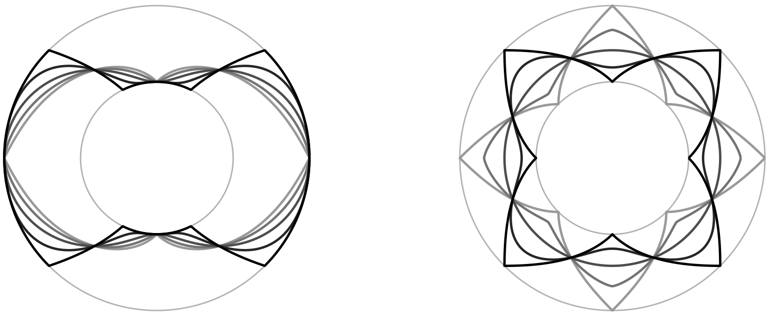

Figure 1. Some examples of induction curves.

Example 2.3. Fig. 1 shows examples of induction curves. On the left-hand side the diagonal matrix $\operatorname{diag}(2, 1)$ is used, on the right-hand side the rotation matrix (with scaling) $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. Shades of gray represent different p values, namely 1, 4/3, 2, 4 and ∞. Circles denote radial units 1 and 2.

Remark 2.4. Note the intersection points of the induction curves for different p values in case of a fixed matrix. These are common intersection points for all values $p \in [1, \infty]$. Since eigenvectors provide such directions, these are called p-eigenvectors. In some cases these can be expressed explicitly with the matrix elements, and in general they can be found by computing eigenvectors of the matrix with permuted rows as detailed in [12] and [13]. The case of p-eigenvectors should be considered also in our current task of identification, see Section 4.3.

2.1. The identification problem

Now our task is to identify an induction curve, i.e. given some points on the curve, can we find the elements of the matrix and the used parameter value p? As a motivation we provide one more example plot on Fig. 2 and will aim to identify it during this research.[1]

[1] This problem was posed by the Author on the presentation about [12] on the conference *Harmonic Analysis and Related Fields* in Visegrd, Hungary, June 11–13, 2019.

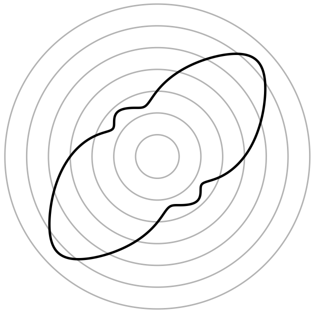

FIGURE 2. The originally posed problem to identify an induction curve. What could be the used matrix and p value?

Consider input data given in polar form, i.e. we are handed pairs (φ_i, r_i) $(i = 1, \ldots, k)$ for a fixed value $k \in \mathbb{N}$. Let us first formalize the problem in the simple case of diagonal 2×2 matrices with positive diagonal elements. Denote by

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \text{thus} \quad Ax = \begin{pmatrix} ax_1 \\ bx_2 \end{pmatrix},$$

with $0 < a, b \in \mathbb{R}$. Introduce $f \colon \mathbb{R} \to \mathbb{R}$ as

$$f_{a,b,p}(\varphi) := n(v(\varphi)) := \frac{\|A \cdot v(\varphi)\|_p}{\|v(\varphi)\|_p} \quad \text{where} \quad v(\varphi) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}. \tag{2.1}$$

Thus

$$f(\varphi) = f_{a,b,p}(\varphi) = \frac{\left(|a \cos\varphi|^p + |b \sin\varphi|^p \right)^{1/p}}{\left(|\cos\varphi|^p + |\sin\varphi|^p \right)^{1/p}} \tag{2.2}$$

and we are to find parameters a, b and p such that $f(\varphi_i) = r_i$ $(i = 1, \ldots, k)$ holds with respect to the input data. Contemplating the formula for f we conclude that the problem is strongly non-linear (mostly with respect to p), but to find 3 parameters $k = 3$ should be minimally prescribed for a unique solution. Solving the problem analytically does not seem to be a promising path, therefore numerical methods shall be applied.

For numerical optimization consider the least squares problem

$$F(a, b, p) := \sum_{i=1}^{k} \left(f_{a,b,p}(\varphi_i) - r_i \right)^2 \quad \longrightarrow \quad \min_{a,b,p}.$$

Without noise at the exact solution the minimum $F(a, b, p) = 0$ could be achieved and is desirable. Add penalty terms to ensure non-negativity of parameters a and b, and tame also $p \in [1, +\infty]$ using $p = w(q)$ and $\mathrm{in}(q)$:

$$\Phi(a, b, q) := F(a, b, w(q)) + \mathrm{nn}(a) + \mathrm{nn}(b) + \mathrm{in}(q) \tag{2.3}$$

with

$$w(q) = \begin{cases} 1, & q < 1 \\ (q-1)^2 + 1, & 1 \le q \le 2 \\ 2/(3-q), & 2 < q < 3 \\ +\infty & q \ge 3 \end{cases}, \quad \mathrm{in}(q) = \begin{cases} (q-1)^2, & q < 1 \\ 0, & 1 \le q \le 3 \\ (q-3)^2, & q > 3 \end{cases}$$

and

$$\mathrm{nn}(x) = \begin{cases} x^2, & x < 0 \\ 0, & x \ge 0 \end{cases}.$$

The functions w and *in* used for p serve the purpose to transform the optimization of this variable from the domain $[1, \infty]$ to $[1, 3]$ which would be easier to handle for any numerical method considering constraints. Note that this way the extreme parameter values $p = 1$ (corresponding to $q = 1$) and $p = \infty$ ($q = 3$) can both be reached and will not be exceeded, furthermore $w(2) = 2$. The choice of $[1, 3]$ may be modified to a different compact interval. Observe that with this choice of w we don't have to put a constraint on q, and that the function w provides a spline-like smooth (continuously differentiable) map $q \mapsto p$ at least on $q \in (-\infty, 3)$. Fig. 3 illustrates functions w and *in*, the latter being basically a square penalty function, similarly to *nn*.

Therefore the task of identification is reduced to an unconstrained optimization problem with the objective function Φ of (2.3):

$$\Phi(a, b, q) = F(a, b, w(q)) + \mathrm{nn}(a) + \mathrm{nn}(b) + \mathrm{in}(q) \quad \longrightarrow \quad \min_{a,b,q}. \tag{2.4}$$

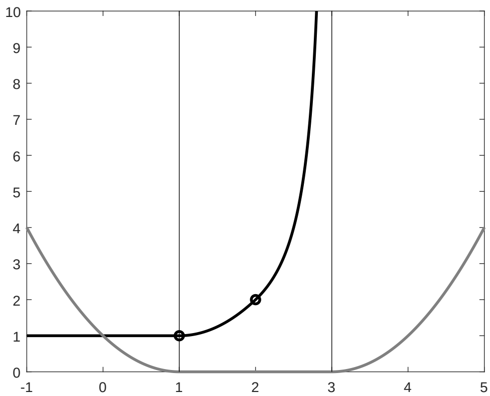

FIGURE 3. The functions w and *in* plotted on the interval $[-1, 5]$. These maps are used to reduce the optimization of parameter $p \in [1, +\infty]$ to $q \in [1, 3]$ in an unconstrained manner.

This formalization of the problem treats the number of input data points $k \ge 3$ generally. Furthermore the method also generalizes straightforward to arbitrary 2×2

matrices with the main difference in the notation of the matrix A and the function f, namely (c.f. (2.1))

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad f_{A,p}(\varphi) = f_{a,b,c,d,p} := n(v(\varphi)) = \frac{\|A \cdot v(\varphi)\|_p}{\|v(\varphi)\|_p}.$$

In an actual implementation using a high-level programming language we don't need to expand the form of f such as in (2.2). This is left as an exercise to the Reader. Several further terms arise, the formula is much more complicated, but the non-linear nature of the problem still persists. However in case of non-diagonal matrices, the signs of the elements results in different induction curves (unlike for diagonal matrices), therefore the constraints to keep the parameters positive should be dropped.

2.2. Conditions for non-uniqueness

In the case of diagonal matrices it is trivial to observe that varying the signs of the diagonal elements would result in the same induction curve, i.e. with the notation of Def. 2.1. (parentheses simplified)

$$\mathcal{I}_p \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \mathcal{I}_p \begin{pmatrix} -a & 0 \\ 0 & b \end{pmatrix} = \mathcal{I}_p \begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix} = \mathcal{I}_p \begin{pmatrix} -a & 0 \\ 0 & -b \end{pmatrix}.$$

Therefore we only consider diagonal matrices with positive diagonal elements in the identification task.

But in the case of arbitrary 2×2 matrices we can not neglect the variations with signs since different induction curves arise which need to be identified. However we still experienced that a seemingly perfectly fitting approximation arises from a "completely" different matrix, which led to the following observation about the possible ill-posedness of the problem.

Proposition 2.5. *Let $A \in \mathbb{R}^{2\times 2}$. The matrix A and the new matrix that we get by switching the two rows of A or multiplying a row (or both rows) of A by -1 (or performing both operations) have the same induction curve.*

Proof. Consider the matrices with switched rows

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} c & d \\ a & b \end{pmatrix}.$$

Then following the definition of the induction sets, for all $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ and $p \in [1, \infty]$:

$$Ax = \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}, \qquad Bx = \begin{pmatrix} cx_1 + dx_2 \\ ax_1 + bx_2 \end{pmatrix}$$

therefore

$$\|Ax\|_p = \left(|ax_1 + bx_2|^p + |cx_1 + dx_2|^p \right)^{1/p} = \|Bx\|_p$$

and hence indeed $\mathcal{I}_p(A) = \mathcal{I}_p(B)$. Clearly multiplying a row with -1 does not effect the p-norm values either. $\square$

The above proposition can be generalized to arbitrary dimensions. Following the notation in [13] let $P \in S_n$ be an element of the symmetric group over n elements represented by a permutation matrix of $\mathbb{R}^{n\times n}$ and $I^{\pm} = \operatorname{diag}(\pm 1, \pm 1, \dots, \pm 1) \in \mathbb{R}^{n\times n}$.

Theorem 2.6. *Let $A \in \mathbb{R}^{n\times n}$, then with the above notations*

$$\mathcal{I}_p(A) = \mathcal{I}_p(I^{\pm}PA) \qquad (p \in [1, \infty]).$$

Proof. The vital observation for this proof is the same as for Proposition 2.5, that the p-norm of the matrix-vector product present in the definition of induction sets is unaffected by the operations of permuting the rows, or multiplying them with -1, i.e.

$$\|Ax\|_p = \|PAx\|_p = \left\|I^{\pm}Ax\right\|_p = \left\|I^{\pm}PAx\right\|_p \qquad (x \in \mathbb{R}^n)$$

Therefore the statement of the theorem holds. □

Example 2.7. The below matrices all have the same induction curve.

$$\begin{pmatrix} 1 & -2 \\ -3 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}, \begin{pmatrix} -3 & 4 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 3 & -4 \\ -1 & 2 \end{pmatrix}.$$

There would be 8 possibilities in this case.

Remark 2.8. The transformations of the matrix of the type $I^{\pm}PA$ were used in [13] to deduce the relation of p-eigenvectors the regular eigenvectors of the transformed matrices. It is left to examine the reason behind having the same transform behind two seemingly different but clearly closely related phenomena.

Remark 2.9. Obviously matrices with the same induction curve cannot be told apart using any identification technique.

Remark 2.10. A related fact in linear algebra is that the only matrices that preserve the p-norm of a real vector (for any p) are also the *signed permutation matrices* $I^{\pm}P$ as in the above theorem [4, 10].

3. Optimization method

For the numerical optimization now we have used the *Nelder–Mead simplex method* [14]. This is a general unconstrained, derivative-free method for the optimization of an arbitrary objective function. Unfortunately it has very few proven convergence properties, but is widely used in practice which is highlighted by the fact that it is method behind the `fminsearch` command of the Matlab software package for mathematical modeling, programming and numerical computation.

We already have significant experience using this algorithm [7, 9, 11] and we have our own implementation which allows us e.g. to create animations to examine the progress of the optimization. Fig. 4. illustrates how this method works in two dimensions, basically relying on the function values at the vertices of a simplex and applying the steps of reflection, expansion, (inner and outer) shrink and contraction.

In the problem of induction curve identification we have used the starting parameters $(a, b, q) = (1, 1, 2)$ in case of diagonal matrices and $(a, b, c, d, q) = (1, 1, 1, 1, 2)$

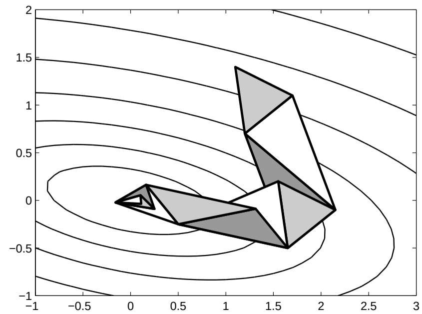

FIGURE 4. The progress of the Nelder–Mead method in case of the optimization of a quadratic function of two variables.

with slight variation in the parameters for further vertices of the simplex. The optimization process was terminated if the mean objective function value at the vertices comes below a prescribed $\varepsilon = 10^{-6}$ or 10^{-8} threshold (or a step count limit has been reached).

A direction of future research can be the investigation of further optimization methods applied to our problem at hand.

4. Results and experiences

In this section we will summarize the results of the identification process carried out according to the formalization and optimization method discussed in Sections 2 and 3. Furthermore we discuss some findings with respect to the case of p-eigenvectors.

4.1. Diagonal matrices

We have many options to analyze the efficiency of the identification already in the case of diagonal matrices. We have the parameters a and b as positive numbers, the parameters q (or equivalently p) and also the effect of the number of input points can be considered. Furthermore the angles of the points can be chosen randomly or uniformly distributed. Many experiments were carried out for various parameter settings.

To present an overall impression of the identification results we have chosen the following method. First we have selected a number of random points from induction curves generated with parameters a and b randomly chosen from a uniform distribution on $[1, 8]$ (these already provide a wide range of possible induction curves) and q also similarly chosen from $[1, 3]$. Since $a = b$ would result in a multiple of the identity matrix generating a circle as induction curve independent of q we required that $|a - b| > 0.1$. We used the values $k = 3, 5, 10, 20$ for the number of points sampled from the induction curves. Furthermore we ensured that the angle difference of the samples are at least $0.2\pi, 0.2\pi, 0.1\pi, 0.05\pi$ respectively to avoid points too close to each other.

For each value of k we performed $N = 1000$ tests with random a, b and q values as detailed above. Denote by a', b' and q' the approximated values given by the optimization method, we collected the values $\|(a, b) - (a', b')\|_2$ and $|q - q'|$. Finally we plotted the sorted approximation errors on a logarithmic scale as seen on Fig. 5. On the left-hand side one can observe the errors of (a, b), on the right-hand side the errors of q. (The results are very similar.) Shades of gray correspond to the values of k, the number of input points, the lightest for $k = 3$, the darkest for $k = 20$.

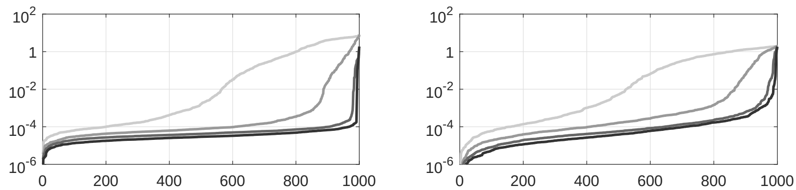

FIGURE 5. Measurement results about the identification of induction curves of diagonal matrices. The error is plotted on a logarithmic scale versus the measurement number (results are sorted). Darker lines correspond to higher number of input points.

On one hand we can conclude that in case of only $k = 3$ input points, in about half of the test cases the identification errors are below 10^{-2}. Such few points may not prove sufficient to identify the matrix and the parameter for this algorithm, although theoretically the solution is unique. It is known that the Nelder–Mead algorithm may also get stuck in local minima, here this phenomena would correspond to very similar induction curves considering only 3 given points.

On the other hand if at least $k = 10$ points are given, the approximation error rises above 10^{-4} only in very few cases. So in practice (when we could observe many points of the curve) already our current method performs very well.

4.2. General matrices

Since in case of arbitrary matrices the generating matrix may be significantly different then the matrix resulting from the optimization process, a representative of their equivalence classes (based on induction curves) must be chosen to measure the approximation results. A representative is chosen based on the signs and ordering of matrix elements.

With the above in sight we have carried out very similar measurements to those in case of diagonal matrices described in Section 4.1, and the presentation of the results is also analogous as seen on Fig. 6.

In this case the matrix elements were all randomly chosen from a uniform distribution on the interval $[-10, 10]$, q again from $[1, 3]$. Now we did not rule out any special matrices (such as possible multiples of the identity matrix). The values for the number of sample points k were $5, 10, 20, 30$ with the minimal angle differences

$0.2\pi, 0.1\pi, 0.05\pi, 0.01\pi$ respectively. We measured $\|A - A'\|_2$ (seen on the left-hand side of the figure) and again $|q - q'|$ (right-hand side).

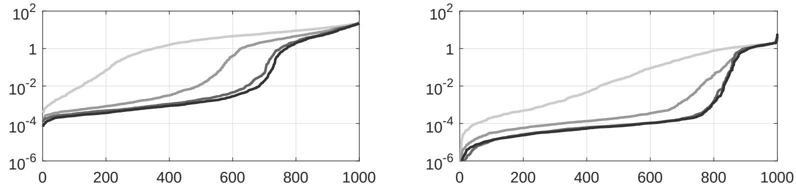

FIGURE 6. Measurement results about the identification of induction curves of general 2×2 matrices.

In this case the approximation results are not as good as in case of diagonal matrices, but still acceptable. Again the results in case of $k = 5$ are not good, in many cases the difference is considerable. But in case of at least $k = 20$ points about 70% of the tests the difference in the matrix is less than 10^{-2}, and the error in identifying q is less than 10^{-3} in about 80% of the tests.

Possible directions of improvement include using different optimization methods, maybe even creating a dictionary for better starting points based on some similarity measure on induction curves. Also it would be interesting to examine the problematic cases (matrices and parameters) in more detail.

Finally in this section on Fig. 7 we present some steps of the optimization progress in case of the original example for the identification problem as on Fig. 2. The sample points in the amount of $k = 20$ were selected randomly with a minimum angle difference of 0.05π. The images also show the induction curves corresponding to the vertices of the simplex, and the matrix and parameter by the centroid of the simplex is written on the lower right parts rounded to 2 decimal digits. Steps $1, 20, 50, 100, 200$ and 270 are shown. We have arrived at the result

$$A' = \begin{pmatrix} -1 & 2 \\ 4 & 3 \end{pmatrix} \quad \text{and} \quad p' = 7,$$

which is correct in light of Proposition 2.5 and Theorem 2.6. The original parameters for generating Fig. 2 and the sample points for the optimization were

$$A = \begin{pmatrix} 4 & 3 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad p = 7.$$

4.3. On the case of p-eigenvectors

A corner case of induction curve identification is when we are given exactly the common intersection points for the p values. In this case any value for p is good and will fit. E.g. in case of the diagonal matrix $\operatorname{diag}(2, 1)$ of Fig. 1 we are handed polar values $(0, 2)$, $(\pi/2, 1)$ and $(\operatorname{atan}\sqrt{2}, \sqrt{2})$ (c.f. [12], Ex. 2.).

Plotting the objective function values $\Phi(a, b, q)$ as in (2.3) for fixed $(a, b) = (2, 1)$, $q \in [1, 3]$, two input points fixed at $(0, 2)$, $(\pi/2, 1)$ but the third input point moving

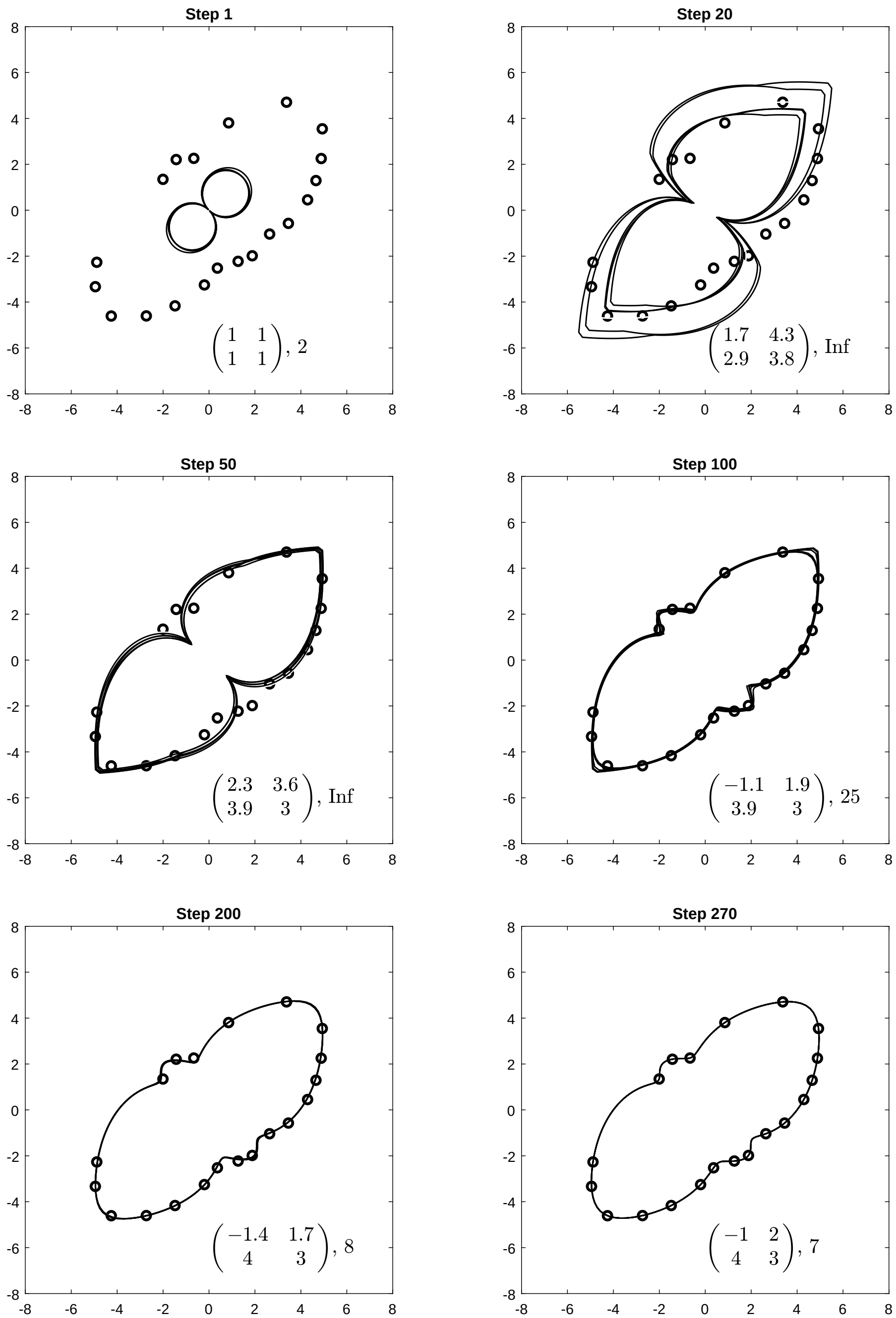

FIGURE 7. Some steps of the optimization progress in case of a sampled version of the originally posed identification problem as seen on Fig. 2.

along a short curve segment passing through $(\mathrm{atan}\sqrt{2}, \sqrt{2})$ with varying φ and r we get a result as depicted on Fig. 8.

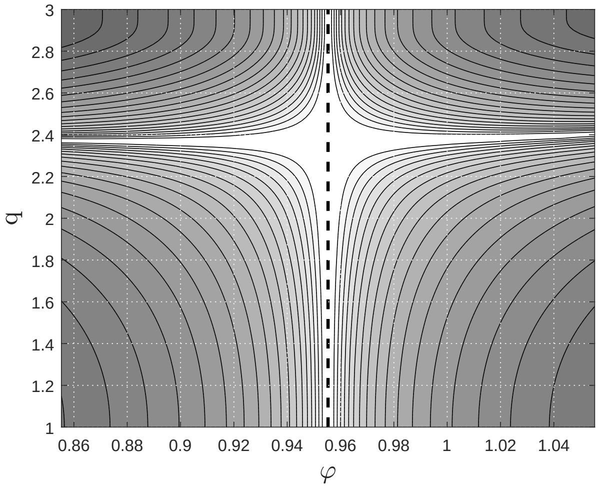

FIGURE 8. An illustration of the singularity near p-eigenvectors. If only the common intersection points are given as input, then the objective function does not have a unique minimizer, the parameter value p cannot be decided.

This contour plot confirms our expectations: At the critical value of φ corresponding to the p-eigenvector the q (and p) parameter values all give the same minimal objective function value 0 (as shown along the black dashed line). But already slightly away from the critical point with φ — where the induction curves for different p values start to spread — the optimal q parameter is unique, the farther from the critical angle the easier to identify.

In practice we would usually have far more points to start the identification process with. So this phenomena would not cause problems when considering an actual induction curve plot. It is just of theoretical importance.

5. Conclusions and further research

In this paper we have shown that the automatic identification of induction curves based on a sampled subset of them is possible using the Nelder–Mead simplex method in an appropriate setting. The accuracy in case of diagonal matrices is very high, a bit lower in case of general matrices. Some mathematical reasons behind non-unique identification were uncovered in general: the signed permutations of the rows of a matrix results in the same induction curve. Experiments were also made near and in the extreme case of providing input values only in p-eigendirections.

As possible directions of further research we list:

- Experiment with further optimization methods in hope to improve identification accuracy and speed.
- Detailed analysis of the cases when this method does not give a proper approximation.
- Identification of induction surfaces (or their 2D projections). Even higher dimensional identification problems.
- The effect of noise in the input data on the precision of identification.
- Describing induction curves with exactly 2 common points.
- A possible topic may be the development of an *induction curve identification application* for mobile devices, such that a user can easily identify a curve of this family using the camera of the device.

Acknowledgements. The main concepts of this paper were presented on the *14th Joint Conference on Mathematics and Computer Science* (MaCS) in Cluj-Napoca, Romania, November 24–27, 2022. The author is thankful for the organizers at Babeş-Bolyai University.

An introduction to the topic of induction curves and p-eigenvectors was also recently presented on a poster on the Conference *Strobl22, Applied Harmonic Analysis and Friends* in Strobl, Austria, June 19–25, 2022. Thanks for the organizers at the Numerical Harmonic Analysis Group (NuHAG), University of Vienna and the Acoustics Research Institute, Austrian Academy of Sciences.

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