Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 2, 439–448
DOI: 10.24193/subbmath.2023.2.17

Four-dimensional Riemannian product manifolds with circulant structures

Iva Dokuzova

Abstract. A 4-dimensional Riemannian manifold equipped with an additional tensor structure, whose fourth power is the identity, is considered. This structure has a circulant matrix with respect to some basis, i.e. the structure is circulant, and it acts as an isometry with respect to the metric. The Riemannian product manifold associated with the considered manifold is studied. Conditions for the metric, which imply that the Riemannian product manifold belongs to each of the basic classes of Staikova-Gribachev's classification, are obtained. Examples of such manifolds are given.

Mathematics Subject Classification (2010): 53B20, 53C15, 15B05.

Keywords: Riemannian metric, almost product structure, circulant matrix.

1. Introduction

The study of Riemannian manifolds M with a metric g and an almost product structure P is initiated by K. Yano in [15]. The classification of the almost product manifolds (M, g, P) with respect to the covariant derivative of P is made by A. M. Naveira in [11]. The manifolds (M, g, P) with zero trace of P are classified with respect to the covariant derivative of P by M. Staikova and K. Gribachev in [14]. The basic classes of this classification are $\mathcal{W}_1, \mathcal{W}_2$ and $\mathcal{W}_3$. Their intersection is the class $\mathcal{W}_0$ of Riemannian P-manifolds. The class of the Riemannian product manifolds is $\mathcal{W}_1 \oplus \mathcal{W}_2$. It is formed by manifolds with an integrable structure P. The class $\mathcal{W}_1$ consists of the conformal Riemannian P-manifolds. Some of the recent studies of the Riemannian almost product manifolds are made in [4], [5], [7], [8] and [12].

Problems of differential geometry of a 4-dimensional Riemannian manifold M with a tensor structure Q of type $(1, 1)$, which satisfies $Q^4 = \mathrm{id}$, $Q^2 \neq \pm\mathrm{id}$, are

Received 16 May 2020; Accepted 14 July 2021.

considered in [1], [2], [3], [10] and [13]. The matrix of Q in some basis is circulant and Q is compatible with the metric g, so that an isometry is induced in any tangent space on M. Such a manifold (M, g, Q) is associated with a Riemannian almost product manifold (M, g, P), where $P = Q^2$ and $\mathrm{tr}P = 0$.

In the present work we continue studying the manifold (M, g, Q) and the associated manifold (M, g, P). Our purpose is to determine their position, among the well-known manifolds, by using the classifications in [11] and [14]. In Sect. 2, we recall some necessary facts about (M, g, Q) and about (M, g, P). In Sect. 3, we obtain the components of the fundamental tensor F, determined by the metric g and by the covariant derivative of P. We establish that (M, g, P) is a Riemannian product manifold, i.e. (M, g, P) belongs to the class $\mathcal{W}_1 \oplus \mathcal{W}_2$. We find necessary and sufficient conditions under which (M, g, P) belongs to each of the classes $\mathcal{W}_0$, $\mathcal{W}_1$ and $\mathcal{W}_2$. In Sect. 4, we give some examples of the considered Riemannian product manifolds.

2. Preliminaries

Let M be a 4-dimensional Riemannian manifold equipped with a tensor structure Q of type $(1, 1)$. The structure Q has a circulant matrix, with respect to some basis, as follows:

$$(Q_j^k) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{2.1}$$

Then Q has the properties

$$Q^4 = \mathrm{id}, \quad Q^2 \neq \pm \mathrm{id}.$$

Let the metric g and the structure Q satisfy

$$g(Qx, Qy) = g(x, y), \quad x, y \in \mathfrak{X}(M).$$

The above condition and (2.1) imply that the matrix of g has the form:

$$(g_{ij}) = \begin{pmatrix} A & B & C & B \\ B & A & B & C \\ C & B & A & B \\ B & C & B & A \end{pmatrix}. \tag{2.2}$$

Here A, B and C are smooth functions of an arbitrary point $p(x^1, x^2, x^3, x^4)$ on M. It is supposed that $A > C > B > 0$ in order g to be positive definite. The manifold (M, g, Q) is introduced in [13].

Anywhere in this work, x, y, z will stand for arbitrary elements of the algebra $\mathfrak{X}(M)$ of the smooth vector fields on M. The Einstein summation convention is used, the range of the summation indices being always $\{1, 2, 3, 4\}$.

In [1], it is noted that the manifold (M, g, P), where $P = Q^2$, is a Riemannian manifold with an almost product structure P, because $P^2 = \mathrm{id}$, $P \neq \pm \mathrm{id}$ and $g(Px, Py) = g(x, y)$. Moreover $\mathrm{tr}P = 0$. For such manifolds Staikova-Gribachev's

classification is valid [14]. This classification was made with respect to the tensor F of type $(0,3)$ and the associated 1-form θ, which are defined by

$$F(x,y,z) = g((\nabla_x P)y, z), \quad \theta(x) = g^{ij}F(e_i, e_j, x). \tag{2.3}$$

Here ∇ is the Levi-Civita connection of g, and g^{ij} are the components of the inverse matrix of (g_{ij}) with respect to an arbitrary basis $\{e_i\}$. The tensor F has the following properties

$$F(x,z,y) = F(x,y,z), \quad F(x,Py,Pz) = -F(x,y,z). \tag{2.4}$$

The manifolds (M,g,P) with an integrable structure P are called Riemannian product manifolds and they form the class $\mathcal{W}_1 \oplus \mathcal{W}_2$. The characteristic conditions for the classes $\mathcal{W}_0$, $\mathcal{W}_1, \mathcal{W}_2$ and $\mathcal{W}_1 \oplus \mathcal{W}_2$ are the following:

i) $\mathcal{W}_0$:

$$F(x,y,z) = 0, \tag{2.5}$$

ii) $\mathcal{W}_1$:

$$\begin{aligned} F(x,y,z) = \frac{1}{4}\big((g(x,y)\theta(z) + g(x,z)\theta(y) \\ - g(x,Py)\theta(Pz) - g(x,Pz)\theta(Py)\big), \end{aligned} \tag{2.6}$$

iii) $\mathcal{W}_2$:

$$F(x,y,Pz) + F(y,z,Px) + F(z,x,Py) = 0, \quad \theta(z) = 0, \tag{2.7}$$

iv) $\mathcal{W}_1 \oplus \mathcal{W}_2$:

$$F(x,y,Pz) + F(y,z,Px) + F(z,x,Py) = 0. \tag{2.8}$$

In the next section, we obtain conditions under which (M,g,P) belongs to each of the classes i) – iv) and to some of their subclasses. Thus the following statements are useful for the completeness of our research.

Theorem 2.1. [10] *If the structure Q of (M,g,Q) satisfies $\nabla Q = 0$, then $\nabla P = 0$, i.e. (M,g,P) belongs to the class $\mathcal{W}_0$.*

Theorem 2.2. [13] *The structure Q of (M,g,Q) satisfies $\nabla Q = 0$ if and only if the following equalities are valid:*

$$\begin{aligned} &A_3 = C_1, \quad A_1 = C_3, \quad B_3 = B_1, \quad 2B_1 = C_4 + C_2, \\ &B_4 = B_2, \quad A_2 = C_4, \quad A_4 = C_2, \quad 2B_2 = C_1 + C_3, \end{aligned} \tag{2.9}$$

where $A_i = \frac{\partial A}{\partial x^i}$, $B_i = \frac{\partial B}{\partial x^i}$, $C_i = \frac{\partial C}{\partial x^i}$.

3. The fundamental tensor F on (M,g,P)

The components of Nijenhuis tensor N of the almost product structure P are determined by the equalities

$$N_{ij}^k = P_i^a(\partial_a P_j^k - \partial_j P_a^k) - P_j^a(\partial_a P_i^k - \partial_i P_a^k).$$

It is known from [14] that the vanishing of the Nijenhuis tensor N is equivalent to the condition (2.8).

Using (2.1), we get that the components of the almost product structure $P = Q^2$ on (M, g, P) are given by the matrix

$$(P_j^k) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{3.1}$$

Evidently the Nijenhuis tensor N of P vanishes, so we have the following

Lemma 3.1. *The manifold (M, g, P) belongs to the class $\mathcal{W}_1 \oplus \mathcal{W}_2$.*

Further, we will consider each of the cases when the fundamental tensor F on (M, g, P) satisfies the identities (2.5), (2.6) or (2.7), since F satisfies (2.8). First we calculate the components of F.

Lemma 3.2. *The nonzero components $F_{ijk} = F(e_i, e_j, e_k)$ of the fundamental tensor F on the manifold (M, g, P) are given by*

$$\begin{array}{ll}
F_{111} = -F_{133} = A_3 - C_1, & 2F_{112} = -2F_{134} = A_4 - B_1 - C_2 + B_3, \\
F_{122} = -F_{144} = B_4 - B_2, & 2F_{114} = -2F_{123} = A_2 - B_1 - C_4 + B_3, \\
F_{211} = -F_{233} = B_3 - B_1, & 2F_{212} = -2F_{234} = A_3 - B_2 - C_1 + B_4, \\
F_{222} = -F_{244} = A_4 - C_2, & 2F_{223} = -2F_{214} = A_1 + B_4 - C_3 - B_2, \\
F_{311} = -F_{333} = C_3 - A_1, & 2F_{334} = -2F_{312} = A_2 + B_1 - C_4 - B_3, \\
F_{322} = -F_{344} = B_4 - B_2, & 2F_{323} = -2F_{314} = A_4 + B_1 - C_2 - B_3, \\
F_{411} = -F_{433} = B_3 - B_1, & 2F_{434} = -2F_{412} = A_1 + B_2 - C_3 - B_4, \\
F_{422} = -F_{444} = C_4 - A_2, & 2F_{414} = -2F_{423} = A_3 - B_4 - C_1 + B_2.
\end{array} \tag{3.2}$$

Proof. The inverse matrix of (g_{ij}) has the form:

$$(g^{ik}) = \frac{1}{D} \begin{pmatrix} \overline{A} & \overline{B} & \overline{C} & \overline{B} \\ \overline{B} & \overline{A} & \overline{B} & \overline{C} \\ \overline{C} & \overline{B} & \overline{A} & \overline{B} \\ \overline{B} & \overline{C} & \overline{B} & \overline{A} \end{pmatrix}, \tag{3.3}$$

where

$$\begin{aligned}
&\overline{A} = A(A + C) - 2B^2,\ \overline{B} = B(C - A),\ \overline{C} = 2B^2 - C(A + C), \\
&D = (A - C)\big((A + C)^2 - 4B^2\big).
\end{aligned} \tag{3.4}$$

The next formula for the Christoffel symbols Γ of ∇ is well known:

$$2\Gamma_{ij}^k = g^{ak}(\partial_i g_{aj} + \partial_j g_{ai} - \partial_a g_{ij}).$$

Then, using (2.2), (3.3) and (3.4), we calculate these coefficients. They are given below:

$$\begin{aligned}
\Gamma^i_{ii} &= \frac{1}{2D}\big(B(C-A)(4B_i-A_j-A_s)+2B^2(2C_i-A_i-A_k)\\
&\qquad +A(A+C)A_i-C(A+C)(2C_i-A_k)\big),\\
\Gamma^j_{ii} &= \frac{1}{2D}\big(B(C-A)(A_i+2C_i-A_k)+2B^2(A_j-A_s)\\
&\qquad +A(A+C)(2B_i-A_j)-C(A+C)(2B_i-A_s)\big),\\
\Gamma^k_{ii} &= \frac{1}{2D}\big(B(C-A)(4B_i-A_j-A_s)+2B^2(A_i+A_k-2C_i)\\
&\qquad +A(A+C)(2C_i-A_k)-C(A+C)A_i\big),\\
\Gamma^i_{ij} &= \frac{1}{2D}\big(B(C-A)(A_i+C_i+B_j-B_s)+2B^2(B_i+C_j-B_k-A_j)\\
&\qquad +A(A+C)A_j-C(A+C)(B_i+C_j-B_k)\big),\\
\Gamma^k_{ij} &= \frac{1}{2D}\big(B(C-A)(A_i+C_i+B_j-B_s)+2B^2(B_k-C_j-B_i+A_j)\\
&\qquad +A(A+C)(B_i+C_j-B_k)-C(A+C)A_j\big),\\
\Gamma^j_{ik} &= \frac{1}{2D}\big(B(C-A)(A_i+A_k)+2B^2(C_j-C_s)\\
&\qquad +A(A+C)(B_i+B_k-C_j)-C(A+C)(B_i+B_k-C_s)\big),\\
\Gamma^i_{ik} &= \frac{1}{2D}\big(B(C-A)(2B_i+2B_k-C_j-C_s)+2B^2(A_i-A_k)\\
&\qquad +A(A+C)A_k-C(A+C)A_i)\big).
\end{aligned} \tag{3.5}$$

In the equalities (3.5) it is assumed that $i\neq j\neq k\neq s$ and the numbers in the pair (i,k) (resp. in (j,s)) are simultaneously even or odd.

The matrix of the associated metric $\tilde{g}$, determined by $\tilde{g}(x,y)=g(x,Py)$, is of the type:

$$(\tilde{g}_{ij})=\begin{pmatrix} C & B & A & B\\ B & C & B & A\\ A & B & C & B\\ B & A & B & C\end{pmatrix}. \tag{3.6}$$

Due to (2.3) the components of F are $F_{ijk}=\nabla_i\tilde{g}_{jk}$. The well-known are the following identities for a Riemannian metric:

$$\nabla_i\tilde{g}_{jk}=\partial_i\tilde{g}_{jk}-\Gamma^a_{ij}\tilde{g}_{ak}-\Gamma^a_{ik}\tilde{g}_{aj}. \tag{3.7}$$

Applying (3.5) and (3.6) into (3.7), and bearing in mind (2.3) and (2.4), we find the nonzero components of F, given in (3.2). □

Immediately, we have the following

Corollary 3.3. *The components $\theta_k = g^{ij}F(e_i, e_j, e_k)$ of the 1-form θ on the manifold (M, g, P) are expressed by the equalities*

$$\begin{aligned}
\theta_1 &= \frac{1}{D}\big(\overline{C}(2C_3 - 2A_1) + \overline{B}(4B_3 - 4B_1) + \overline{A}(2A_3 - 2C_1)\big),\\
\theta_2 &= \frac{1}{D}\big(\overline{A}(2A_4 - 2C_2) + \overline{B}(4B_4 - 4B_2) + \overline{C}(2C_4 - 2A_2)\big),\\
\theta_3 &= \frac{1}{D}\big(\overline{C}(2C_1 - 2A_3) + \overline{B}(4B_1 - 4B_3) + \overline{A}(2A_1 - 2C_3)\big),\\
\theta_4 &= \frac{1}{D}\big(\overline{A}(2A_2 - 2C_4) + \overline{B}(4B_2 - 4B_4) + \overline{C}(2C_2 - 2A_4)\big).
\end{aligned} \tag{3.8}$$

Proof. The equalities (3.8) follow by direct computations from (3.2), (3.3) and (3.4). □

Having in mind Lemma 3.1, Lemma 3.2 and Corollary 3.3 we obtain the next statements.

Theorem 3.4. *The manifold (M, g, P) belongs to the class $\mathcal{W}_0$ if and only if the following equalities are valid:*

$$A_3 = C_1, \quad A_1 = C_3, \quad B_3 = B_1, \quad B_4 = B_2, \quad A_2 = C_4, \quad A_4 = C_2. \tag{3.9}$$

Proof. Due to (3.2) we get that (2.5) is satisfied if and only if (3.9) holds true. □

Theorem 3.5. *The manifold (M, g, P) belongs to the class $\mathcal{W}_1$ if and only if the following equalities are valid:*

$$\begin{aligned}
(A + C)(B_4 - B_2) &= B(A_4 - C_2 + C_4 - A_2),\\
(A + C)(B_3 - B_1) &= B(A_3 - C_1 + C_3 - A_1).
\end{aligned} \tag{3.10}$$

Proof. Using (2.2), (3.1), (3.2), (3.6), (3.8) and (3.10) we obtain

$$F_{kij} = \frac{1}{4}\big(g_{kj}\theta_i + g_{ki}\theta_j - \tilde{g}_{kj}\tilde{\theta}_i - \tilde{g}_{ki}\tilde{\theta}_j\big), \quad \tilde{\theta}_i = P_i^a\theta_a, \tag{3.11}$$

which is equivalent to (2.6).

Vice versa, if (3.11) holds true, then (2.2), (3.1), (3.2), (3.6) and (3.8) imply (3.10). □

In [14], it is proved that $\mathcal{W}_1 = \overline{\mathcal{W}}_3 \oplus \overline{\mathcal{W}}_6$, where $\overline{\mathcal{W}}_3$ and $\overline{\mathcal{W}}_6$ are two basic classes of Naveira's classification. These classes have the following characteristic conditions ([6], [9]):

$$\begin{aligned}
\overline{\mathcal{W}}_3:\ F(x,y,z) = \frac{1}{4}\Big(&\big(g(x,y) + g(x,Py)\big)\theta(z)\\
&+ \big(g(x,z) + g(x,Pz)\big)\theta(y)\Big),\ \theta(Pz) = -\theta(z).
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
\overline{\mathcal{W}}_6:\ F(x,y,z) = \frac{1}{4}\Big(&\big(g(x,y) - g(x,Py)\big)\theta(z)\\
&+ \big(g(x,z) - g(x,Pz)\big)\theta(y)\Big),\ \theta(Pz) = \theta(z).
\end{aligned} \tag{3.13}$$

Corollary 3.6. *The manifold (M, g, P) belongs to the class $\overline{\mathcal{W}}_3$ if and only if the following equalities are valid:*

$$\begin{array}{ll} A_4 - C_2 = C_4 - A_2, & (A+C)(B_4 - B_2) = 2B(A_4 - C_2), \\ A_3 - C_1 = C_3 - A_1, & (A+C)(B_3 - B_1) = 2B(A_3 - C_1). \end{array} \tag{3.14}$$

Proof. The local form of (3.12) is

$$F_{kij} = \frac{1}{4}\big((g_{ki} + \tilde{g}_{ki})\theta_j + (g_{kj} + \tilde{g}_{kj})\theta_i\big), \quad \tilde{\theta}_i = -\theta_i. \tag{3.15}$$

Taking into account (3.8) and (3.10), we get that (3.15) is satisfied if and only if (3.14) holds true. □

Corollary 3.7. *The manifold (M, g, P) belongs to the class $\overline{\mathcal{W}}_6$ if and only if the following equalities are valid:*

$$A_4 - C_2 = A_2 - C_4, \quad B_4 = B_2, \quad A_3 - C_1 = A_1 - C_3, \quad B_3 = B_1. \tag{3.16}$$

Proof. The local form of (3.13) is

$$F_{kij} = \frac{1}{4}\big((g_{ki} - \tilde{g}_{ki})\theta_j + (g_{kj} - \tilde{g}_{kj})\theta_i\big), \quad \tilde{\theta}_i = \theta_i. \tag{3.17}$$

Having in mind (3.8) and (3.10), we get that (3.17) is satisfied if and only if (3.16) holds true. □

Theorem 3.8. *The manifold (M, g, P) belongs to the class $\mathcal{W}_2$ if and only if the following equalities are valid:*

$$\begin{array}{ll} (A+C)(C_3 - A_1) = 2B(B_3 - B_1), & C_3 - A_1 = A_3 - C_1, \\ (A+C)(C_4 - A_2) = 2B(B_4 - B_2), & C_4 - A_2 = A_4 - C_2. \end{array} \tag{3.18}$$

Proof. From (2.7) and (3.8) we have

$$\begin{aligned} &\overline{C}(2C_3 - 2A_1) + \overline{B}(4B_3 - 4B_1) + \overline{A}(2A_3 - 2C_1) = 0, \\ &\overline{A}(2A_4 - 2C_2) + \overline{B}(4B_4 - 4B_2) + \overline{C}(2C_4 - 2A_2) = 0, \\ &\overline{C}(2C_1 - 2A_3) + \overline{B}(4B_1 - 4B_3) + \overline{A}(2A_1 - 2C_3) = 0, \\ &\overline{A}(2A_2 - 2C_4) + \overline{B}(4B_2 - 4B_4) + \overline{C}(2C_2 - 2A_4) = 0. \end{aligned} \tag{3.19}$$

The equalities (3.4) and (3.19) imply (3.18).

Vice versa. We apply (3.18) into (3.8) and we get that (2.7) holds true. □

4. Examples of manifolds (M, g, P)

In this section we give a solution of each system of differential equations (3.9), (3.10), (3.14), (3.16) and (3.18), in order to get examples of (M, g, P) of the classes considered.

4.1. An example in $\mathcal{W}_0$

Let (M, g, P) be a manifold with

$$\begin{aligned} &A = (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2,\ B = x^1 + x^2 + x^3 + x^4, \\ &C = 2x^1x^3 + 2x^2x^4, \end{aligned} \tag{4.1}$$

where $x^i > 1$.

Evidently $A > C > B > 0$ are valid. We check directly that the functions (4.1) and their derivatives satisfy the equalities (3.9).

Thus we have the following

Proposition 4.1. *The manifold (M, g, P) with (4.1) belongs to $\mathcal{W}_0$.*

Remark 4.2. We note that the functions (4.1) do not satisfy (2.9). Then we have $\nabla Q \neq 0$ for (M, g, P), where $P = Q^2$. An example of a manifold (M, g, Q) with $\nabla Q = 0$ is given in [13].

4.2. An example in $\overline{\mathcal{W}}_3$

Let (M, g, P) be a manifold with

$$\begin{aligned} &A = a(x^1 + x^2 - x^3 - x^4), \quad B = b(x^1 + x^2 - x^3 - x^4), \\ &C = c(x^1 + x^2 - x^3 - x^4), \end{aligned} \tag{4.2}$$

where $a, c, b \in \mathbb{R},\ a > c > b > 0,\ x^1 + x^2 - x^3 - x^4 > 0$.

The inequalities $A > C > B > 0$ hold true. The functions (4.2) and their derivatives satisfy the equalities (3.14) and do not satisfy the conditions (3.9).

Therefore, we establish the following

Proposition 4.3. *The manifold (M, g, P) with (4.2) belongs to $\overline{\mathcal{W}}_3$ but does not belong to $\mathcal{W}_0$.*

4.3. An example in $\overline{\mathcal{W}}_6$

Let (M, g, P) be a manifold with

$$\begin{aligned} &A = a(x^1 + x^2 + x^3 + x^4), \quad B = b(x^1 + x^2 + x^3 + x^4), \\ &C = c(x^1 + x^2 + x^3 + x^4), \end{aligned} \tag{4.3}$$

where $a, c, b \in \mathbb{R},\ a > c > b > 0,\ x^1 + x^2 + x^3 + x^4 > 0$.

The inequalities $A > C > B > 0$ are satisfied. The functions (4.3) and their derivatives satisfy the equalities (3.16) and do not satisfy the conditions (3.9).

Immediately, we state the following

Proposition 4.4. *The manifold (M, g, P) with (4.3) belongs to $\overline{\mathcal{W}}_6$ but does not belong to $\mathcal{W}_0$.*

4.4. An example in $\mathcal{W}_1$

Let (M, g, P) be a manifold with

$$\begin{aligned} A =& a\exp(x^1 - x^2),\ C = c\exp(x^4 - x^3),\\ B =& a\exp(x^1 - x^2) - c\exp(x^4 - x^3), \end{aligned} \tag{4.4}$$

where $a, c \in \mathbb{R}^+$, $\ln\dfrac{c}{a} < x^1 - x^2 + x^3 - x^4 < \ln\dfrac{2c}{a}$.

Then $A > C > B > 0$ are valid. The functions (4.4) satisfy the equalities (3.10) but do not satisfy any of the conditions (3.14) and (3.16).

Therefore, we establish the following

Proposition 4.5. *The manifold (M, g, P) with (4.4) belongs to $\mathcal{W}_1$ but does not belong to $\overline{\mathcal{W}}_3$ or to $\overline{\mathcal{W}}_6$.*

4.5. An example in $\mathcal{W}_2$

Let (M, g, P) be a manifold with

$$\begin{aligned} &A = \exp(x^1 + x^2 - x^3 - x^4), \quad B = \sinh(x^1 + x^2 - x^3 - x^4),\\ &C = \exp(x^3 + x^4 - x^1 - x^2), \end{aligned} \tag{4.5}$$

where $0 < x^1 + x^2 - x^3 - x^4 < \ln\sqrt{3}$.

The inequalities $A > C > B > 0$ are satisfied. The functions from (4.5) give a solution to (3.18) but do not give a solution to (3.9).

Consequently we have the following

Proposition 4.6. *The manifold (M, g, P) with (4.5) belongs to $\mathcal{W}_2$ but does not belong to $\mathcal{W}_0$.*

5. Conclusion

In this paper we classify the 4-dimensional Riemannian product manifolds (M, g, P), associated with the Riemannian manifolds with circulant structures (M, g, Q), using the well-known classifications in [11] and [14].

References

[1] Dokuzova, I., *Curvature properties of 4-dimensional Riemannian manifolds with a circulant structure*, J. Geom., **108**(2017), no. 2, 517–527.

[2] Dokuzova, I., *Almost conformal transformation in a four-dimensional Riemannian manifold with an additional structure*, Int. Electron. J. Geom., **6**(2013), no. 2, 9-13.

[3] Dokuzova, I., Razpopov, D., Manev, M., *Two types of Lie Groups as 4-dimensional Riemannian manifolds with circulant structure*, Math. Educ. Math., In: Proc. 47th Spring Conf. of UBM, Borovets, (2018), 115-120. Preprint arXiv:1801.08307

[4] Druţă-Romaniuc, S.L., *Natural diagonal Riemannian almost product and para-Hermitian cotangent bundles*, Czechoslovak Math. J., **62(137)**(2012), no. 4, 937-949.

[5] Gribacheva, D., *A natural connection on a basic class of Riemannian product manifolds*, Int. J. Geom. Methods Mod. Phys., **9**(2012), no. 7. 1250057, 14pp.

[6] Gribacheva, D., *Curvature properties of two Naveira classes of Riemannian product manifolds*, Plovdiv. Univ. Paisiĭ Khilendarski Nauchn. Trud. Mat., **39**(2012), no. 3, 31-42. Preprint arXiv:1204.5838.

[7] Gribacheva, D., *Natural connections on Riemannian product manifolds*, C.R. Acad. Bulgare Sci., **64**(2011), no. 6, 799-806.

[8] Gribacheva, D., Mekerov, D., *Natural connections on conformal Riemannian P-manifolds*, C.R. Acad. Bulgare Sci., **65**(2012), no. 5, 581-590.

[9] Gribacheva, D., Mekerov, D., *Conformal Riemannian P-manifolds with connections whose curvature tensors are Riemannan P-tensors*, J. Geom., **105**(2014), no. 2, 273-286.

[10] Gribacheva, D., Razpopov, D., *Riemannian almost product manifolds generated by a circulant structure*, J. Tech. Univ. Plovdiv, Fundam. Sci. Appl., **23**(2017), 171-174. Preprint arXiv:1703.07988.

[11] Naveira, A.M., *A classification of Riemannian almost-product manifolds*, Rend. Mat. Appl., **7**(1983), no. 3, 577-592.

[12] Pušić, N., *On some connections on locally product Riemannian manifolds – Part I*, Novi Sad J. Math., **41**(2011), no. 2, 29-40.

[13] Razpopov, D., *Four-dimensional Riemannian manifolds with two circulant structures*, Math. Educ. Math., In: Proc. of 44th Spring Conf. of UBM, SOK Kamchia, Bulgaria, 2015, 179-185. Preprint arXiv:1409.7622.

[14] Staikova, M., Gribachev, K., *Canonical connections and their conformal invariants on Riemannian almost-product manifolds*, Serdica, **18**(1992), no. 3-4, 150-161.

[15] Yano, K., *Differential Geometry on Complex and Almost Complex Spaces*, International Series of Monographs in Pure and Applied Mathematics, **49**, A Pergamon Press Book, The Macmillan Co. New York, 1965.

Iva Dokuzova
Department of Algebra and Geometry,
Faculty of Mathematics and Informatics,
University of Plovdiv Paisii Hilendarski,
24 Tzar Asen, 4000 Plovdiv, Bulgaria
e-mail: dokuzova@uni-plovdiv.bg