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A new class of Bernstein-type operators obtained by iteration

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Abstract. A new class of Bernstein-type operators are obtained by applying an iterative method of modifications starting from the Bernstein operators. These operators have good properties of approximation of functions and of their derivatives.

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1. Introduction

Bernstein operators are defined by

$$B_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right) \tag{1.1}$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \tag{1.2}$$

for $f : [0,1] \to \mathbb{R}$, $n \in \mathbb{N}$, $x \in [0,1]$.

They are the source of a vast literature with a multitude of modifications and generalizations. In this article we propose a new construction of a sequence of linear positive operators recursively obtained by applying a modification method starting from the Bernstein operators.

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For integers $0 \le r < n$ consider the operator

$$T_n^r(f)(x) = \sum_{i=0}^{n-r} p_{n-r,i}(x) F_{n,i}^r(f),\ f : [0,1] \to \mathbb{R},\ x \in [0,1], \tag{1.3}$$

where the functionals $F_{n,i}^r$ are defined recursively by $F_{n,i}^0(f) = f\left(\frac{i}{n}\right)$, $0 \le i \le n$ and, for $r \ge 1$:

$$F_{n,i}^r(f) = \left(1 - \frac{i}{n-r}\right) F_{n,i}^{r-1}(f) + \frac{i}{n-r} F_{n,i+1}^{r-1}(f),\ 0 \le i \le n-r. \tag{1.4}$$

Note that for $r = 0$, T_n^r coincides with the Bernstein operator, B_n. Also, the operator T_n^1 can be put in connection with operators $T_{n,\alpha}$, defined by

$$T_{n,\alpha} = \alpha B_n + (1-\alpha) T_n^1,\ \text{for } \alpha \in [0,1]$$

and introduced by Chen et alt. [1]. The Chlodovsky variant of operators $T_{n,\alpha}$ was studied in [7].

For operators T_n^r we study in this paper the explicit representation, the moments, estimates of the degree of approximation in terms of moduli of continuity, the Voronoskaja-type theorem, the preservation of the convexity of higher order and the simultaneous approximation. There exists a partial analogy between the operators T_n^r and the iteration by composition of Bernstein operators:

$$(B_n)^r := B_n \circ \cdot \circ B_n, \quad (r \text{ times}).$$

2. Basic identities

For $p \in \mathbb{N}$ define the monomial function $e_p(t) = t^p$, $t \in [0,1]$. Let $B[0,1]$ be the space of bounded functions defined on interval $[0,1]$, $C[0,1]$ be the space of continuous functions defined on interval $[0,1]$ and $C^k[0,1]$, $k \ge 1$ be the space of functions with k continuous derivatives.

Lemma 2.1. *For integers $0 \le r < n$, $0 \le i \le n-r$ there hold:*

i) $F_{n,i}^r(e_0) = 1$,

ii) $F_{n,i}^r(e_1) = \frac{i}{n-r}$.

Proof. The relations follows immediately by induction. □

Corollary 2.2. *For integers $0 \le r < n$, and $x \in [0,1]$, the following relation are true:*

i) $T_n^r(e_0)(x) = 1$,

ii) $T_n^r(e_1)(x) = x$.

Proof. Corollary 2.2 follows from Lemma 2.1 using the properties of Bernstein operators. □

For $a \in \mathbb{R}$, and $n \in \mathbb{N} \cup \{0\}$ denote by $(a)_n$ the Pochhammer symbol, i.e. $(a)_0 = 1$ and $(a)_n = a(a+1)\ldots(a+n-1)$, for $n \ge 1$.

For $n, r, i, k \in \mathbb{N} \cup \{0\}$, $0 \le r \le n$, $0 \le i \le n-r$, $0 \le k \le r$ define

$$c_{n,r,i,k} = \binom{r}{k} (n-i-r)_{r-k} (i)_k. \tag{2.1}$$

Lemma 2.3. *For $f \in C[0,1]$, $n \in \mathbb{N}$, $r \in \mathbb{N} \cup \{0\}$, $0 \le r < n$, $0 \le i \le n-r$, we have*

$$F_{n,i}^{r}(f) = \frac{1}{(n-r)_r} \sum_{k=0}^{r} c_{n,r,i,k} f\left(\frac{i+k}{n}\right). \tag{2.2}$$

Proof. We prove by mathematical induction with regards to r. For $r = 0$ equation (2.2) is clear. Suppose (2.2) true for $r < n-1$. Then, for $0 \le i \le n-r-1$, and $f : [0,1] \to \mathbb{R}$:

$$\begin{aligned}
F_{n,i}^{r+1}(f) &= \left(1 - \frac{i}{n-r-1}\right) F_{n,i}^{r}(f) + \frac{i}{n-r-1} F_{n,i+1}^{r}(f) \\
&= \frac{n-r-i-1}{n-r-1} \cdot \frac{1}{(n-r)_r} \sum_{k=0}^{r} \binom{r}{k} (n-r-i)_{r-k} (i)_k f\left(\frac{i+k}{n}\right) \\
&\quad + \frac{i}{n-r-1} \cdot \frac{1}{(n-r)_r} \sum_{k=0}^{r} \binom{r}{k} (n-r-i-1)_{r-k} (i+1)_k \left(\frac{i+1+k}{n}\right) \\
&= \frac{1}{(n-r-1)_{r+1}} \left\{ \sum_{k=0}^{r} \binom{r}{k} (n-r-i)_{r-k} (i)_k (n-r-i-1) f\left(\frac{i+k}{n}\right) \right. \\
&\quad \left. + \sum_{k=0}^{r} \binom{r}{k} (n-r-i-1)_{r-k} (i+1)_k i \left(\frac{i+1+k}{n}\right) \right\} \\
&= \frac{1}{(n-r-1)_{r+1}} \left\{ \sum_{k=0}^{r} \binom{r}{k} (n-r-i-1)_{r-k+1} (i)_k f\left(\frac{i+k}{n}\right) \right. \\
&\quad \left. + \sum_{k=0}^{r} \binom{r}{k} (n-r-i-1)_{r-k} (i)_{k+1} f\left(\frac{i+1+k}{n}\right) \right\}.
\end{aligned} \tag{2.3}$$

Since

$$\begin{aligned}
&\sum_{k=0}^{r} \binom{r}{k} (n-r-i-1)_{r-k} (i)_{k+1} f\left(\frac{i+1+k}{n}\right) \\
&= \sum_{k=1}^{r+1} \binom{r}{k-1} (n-r-i-1)_{r-k+1} (i)_k f\left(\frac{i+k}{n}\right)
\end{aligned}$$

and

$$\binom{r}{k} + \binom{r}{k-1} = \binom{r+1}{k},$$

by adding the last two sums in (2.3) one obtains

$$\begin{aligned}
F_{n,i}^{r+1}(f) &= \frac{1}{(n-r-1)_{r+1}} \sum_{k=0}^{r+1} \binom{r+1}{k} (n-r-i-1)_{r-k+1} (i)_k f\left(\frac{i+k}{n}\right) \\
&= \frac{1}{(n-r-1)_{r+1}} \sum_{k=0}^{r+1} c_{n,r+1,i,k} f\left(\frac{i+k}{n}\right).
\end{aligned}$$

□

Remark 2.4. From Lemma 2.3 it follows that

$$T_n^{n-1}(f)(x) = (1-x)f(0) + xf(1),\ f:[0,1]\to\mathbb{R},\ n\in\mathbb{N},\ x\in[0,1].$$

This relation, shows that the operators T_n^r make a link between the operators B_n and B_1, similarly with the link made by $(B_n)^r$, for $r=1$ and the limit $r\to\infty$.

For any $n\in\mathbb{N}$ consider the operator

$$G_n(f)(t) = (1-t)f\left(\frac{n-1}{n}t\right) + tf\left(\frac{n-1}{n}t+\frac{1}{n}\right),\ f\in C[0,1],\ t\in[0,1]. \tag{2.4}$$

Lemma 2.5. *For $1\le r<n$ and $f\in C[0,1]$ there holds*

$$T_n^r(f)(x) = (T_{n-1}^{r-1}\circ G_n)(f)(x),\ x\in[0,1]. \tag{2.5}$$

Proof. From relations (2.1) and (2.2) one has

$$F_{n,i}^r(f) = \frac{1}{(n-r)_r}\sum_{k=0}^r \binom{r}{k}(n-r-i)_{r-k}(i)_k f\left(\frac{i+k}{n}\right).$$

We decompose this sum in two sums denoted U_1 and U_2 using formula

$$\binom{r}{k} = \binom{r-1}{k-1} + \binom{r-1}{k}.$$

By changing the index one obtains

$$\begin{aligned}
U_1 &= \frac{1}{(n-r)_r}\sum_{k=0}^r \binom{r-1}{k-1}(n-r-i)_{r-k}(i)_k f\left(\frac{i+k}{n}\right)\\
&= \frac{1}{(n-r)_r}\sum_{k=1}^r \binom{r-1}{k-1}(n-r-i)_{r-k}(i)_{k-1}(i+k-1) f\left(\frac{i+k}{n}\right)\\
&= \frac{1}{(n-r)_r}\sum_{k=0}^{r-1} \binom{r-1}{k}(n-r-i)_{r-1-k}(i)_k(i+k) f\left(\frac{i+k+1}{n}\right)\\
&= \frac{1}{n-1}\cdot\frac{1}{(n-r)_{r-1}}\sum_{k=0}^{r-1} c_{n-1,r-1,i,k}(i+k) f\left(\frac{i+k+1}{n}\right).
\end{aligned}$$

Also, there holds

$$\begin{aligned}
U_2 &= \frac{1}{(n-r)_r}\sum_{k=0}^r \binom{r-1}{k}(n-r-i)_{r-k}(i)_k f\left(\frac{i+k}{n}\right)\\
&= \frac{1}{(n-r)_r}\sum_{k=0}^{r-1} \binom{r-1}{k}(n-r-i)_{r-1-k}(i)_k(n-k-i-1) f\left(\frac{i+k}{n}\right)\\
&= \frac{1}{n-1}\cdot\frac{1}{(n-r)_{r-1}}\sum_{k=0}^{r-1} c_{n-1,r-1,i,k}(n-k-i-1) f\left(\frac{i+k}{n}\right).
\end{aligned}$$

Then,

$$F_{n,i}^{r}(f) = U_1 + U_2$$
$$= \frac{1}{(n-r)_{r-1}} \sum_{k=0}^{r-1} c_{n-1,r-1,i,k} \Big[\frac{i+k}{n-1} f\left(\frac{i+k+1}{n}\right) + \frac{n-k-i-1}{n-1} f\left(\frac{i+k}{n}\right) \Big].$$

But:

$$\begin{aligned} & \frac{i+k}{n-1} f\left(\frac{i+k+1}{n}\right) + \frac{n-k-i-1}{n-1} f\left(\frac{i+k}{n}\right) \\ = \; & \frac{i+k}{n-1} f\left(\frac{n-1}{n}\frac{i+k}{n-1} + \frac{1}{n}\right) + \left(1 - \frac{k+i}{n-1}\right) f\left(\frac{n-1}{n} \cdot \frac{i+k}{n-1}\right) \\ = \; & G_n(f)\left(\frac{i+k}{n-1}\right). \end{aligned}$$

Then, for $0 \le i \le n-r$,

$$F_{n,i}^{r}(f) = \frac{1}{(n-r)_{r-1}} \sum_{k=0}^{r-1} c_{n-1,r-1,i,k} G_n(f)\left(\frac{i+k}{n-1}\right) = F_{n-1,i}^{r-1}(G_n(f)).$$

Finally,

$$\begin{aligned} T_n^r(f)(x) &= \sum_{i=0}^{n-r} p_{n-r,i}(x) F_{n,i}^{r}(f) = \sum_{i=0}^{n-r} p_{n-r,i}(x) F_{n-1,i}^{r-1}(G_n(f)) \\ &= T_{n-1}^{r-1}(G_n(f))(x). \end{aligned}$$

□

Corollary 2.6. *For integers $0 \le r < n$ there exists the representation*

$$T_n^r = B_{n-r} \circ G_{n-r+1} \circ G_{n-r+2} \circ \ldots \circ G_n. \tag{2.6}$$

3. The moments

Lemma 3.1. *For $n \in \mathbb{N}$, $p \in \mathbb{N}$ there holds*

$$G_n((e_1 - xe_0)^p)(t) = \sum_{j=0}^{p} (t-x)^j d_{n,p,j}(x), \; t, x \in [0,1], \tag{3.1}$$

where

$$\begin{aligned} d_{n,p,j}(x) &= \frac{1}{n^p}\binom{p}{j}(n-1)^j \Big[(1-x)(-x)^{p-j} + x(1-x)^{p-j}\Big] \\ &\quad + \frac{1}{n^p}\binom{p}{j-1}(n-1)^{j-1}\Big[x(-x)^{p-j} + (1-x)(1-x)^{p-j}\Big]. \end{aligned}$$

Proof. From the definition of G_n, grouping the terms with the same power of $t-x$ one obtains

$$\begin{aligned}
&G_n((e_1-xe_0)^p)(t)=(1-t)\left(\frac{n-1}{n}t-x\right)^p+t\left(\frac{n-1}{n}t+\frac{1}{n}-x\right)^p\\
=\;&(1-x+x-t)\left(\frac{n-1}{n}(t-x)-\frac{x}{n}\right)^p\\
&+(t-x+x)\left(\frac{n-1}{n}(t-x)+\frac{1}{n}(1-x)\right)^p\\
=\;&(1-x+x-t)\sum_{j=0}^{p}\binom{p}{j}\left(\frac{n-1}{n}\right)^j(t-x)^j\left(-\frac{x}{n}\right)^{p-j}\\
&+(t-x+x)\sum_{j=0}^{p}\binom{p}{j}\left(\frac{n-1}{n}\right)^j(t-x)^j\left(\frac{1-x}{n}\right)^{p-j}\\
=\;&\frac{1}{n^p}\sum_{j=0}^{p+1}(t-x)^j\Big[\binom{p}{j}(1-x)(n-1)^j(-x)^{p-j}\\
&\qquad-\binom{p}{j-1}(n-1)^{j-1}(-x)^{p+1-j}\Big]\\
&+\frac{1}{n^p}\sum_{j=0}^{p+1}(t-x)^j\Big[\binom{p}{j}x(n-1)^j(1-x)^{p-j}\\
&\qquad+\binom{p}{j-1}(n-1)^{j-1}(1-x)^{p+1-j}\Big].
\end{aligned}$$

Finally, equation (3.1) follows, because the coefficient of $(t-x)^{p+1}$ is null. □

Define the moments of order p of operators T_n^r, by

$$M^p[T_n^r](x)=T_n^r((e_1-xe_0)^p)(x),\ 0\le r<n,\ p\ge 0,\ x\in[0,1]. \tag{3.2}$$

From Lemma 2.5 and Lemma 3.1 we have the following relation of recurrence

Corollary 3.2.

$$M^p[T_n^r](x)=\sum_{j=0}^{p}d_{n,p,j}(x)M^j[T_{n-1}^{r-1}](x),\ 1\le r<n,\ p\ge 0,\ x\in[0,1]. \tag{3.3}$$

Lemma 3.3. *We have, for $x \in [0,1]$, $0 \le r < n$:*

$$M^0[T_n^r](x) = 1; \tag{3.4}$$

$$M^1[T_n^r](x) = 0; \tag{3.5}$$

$$M^2[T_n^r](x) = \frac{n+r+1}{n(n-r+1)}x(1-x); \tag{3.6}$$

$$M^3[T_n^r](x) = \frac{n^2+4nr+3n+r^2+3r+2}{n^2(n-r+1)(n-r+2)}x(1-x)(1-2x); \tag{3.7}$$

$$M^4[T_n^r](x) = x(1-x)a_{n,r}(x), \text{ with } |a_{n,r}(x)| \le C_r \cdot \frac{1}{n^2} \tag{3.8}$$

where C_r is independent on $n \in \mathbb{N}$, and $x \in [0,1]$.

Proof. Relations (3.4) and (3.5) can be obtained directly from Corollary 2.2.
For the moment $M^2[T_n^r](x)$, first note that for $r = 0$ and $n \ge 1$, equality (3.6) becomes

$$M^2[T_n^0](x) = \frac{x(1-x)}{n},$$

which is known, from the property of Bernstein operators. For $r \ge 1$, from Corollary 3.2 and equations (3.4) and (3.5) one obtains

$$\begin{aligned} M^2[T_n^r](x) &= \frac{n^2-1}{n^2}M^2[T_{n-1}^{r-1}](x) + \frac{1-2x}{n^2}M^1[T_{n-1}^{r-1}](x) \\ &\quad + \frac{x(1-x)}{n^2}M^0[T_{n-1}^{r-1}](x) \\ &= \frac{n^2-1}{n^2}M^2[T_{n-1}^{r-1}](x) + \frac{x(1-x)}{n^2}. \end{aligned}$$

Then, equation (3.6) follows by induction since

$$\frac{n+r+1}{n(n-r+1)}x(1-x) = \frac{n^2-1}{n^2} \cdot \frac{n+r-1}{(n-1)(n-r+1)}x(1-x) + \frac{x(1-x)}{n^2}.$$

Equation (3.7) for $r = 0$, $n \in \mathbb{N}$ reads $M^3[T_n^0](x) = \frac{x(1-x)(1-2x)}{n^2}$, which coincides with the moment of order 3 of Bernstein operators. For $r \ge 1$, suppose that (3.7) is true for $r-1$ and $n-1$. From relations (3.3), (3.4), (3.5), (3.6) it follows after certain computations:

$$\begin{aligned} M^3[T_n^r](x) &= \frac{(n-1)^2(n+2)}{n^3}M^3[T_{n-1}^{r-1}](x) + 3\frac{n-1}{n^3}(1-2x)M^2[T_{n-1}^{r-1}](x) \\ &\quad + \frac{3nx(1-x)+1-6x+6x^2}{n^3}M^1[T_{n-1}^{r-1}](x) + \frac{x(1-x)(1-2x)}{n^3}M^0[T_{n-1}^{r-1}](x) \\ &= \frac{(n-1)^2(n+2)}{n^3} \cdot \frac{n^2+4nr+r^2-3n-3r+2}{(n-1)^2(n-r+1)(n-r+2)}x(1-x)(1-2x) \\ &\quad + 3\frac{n-1}{n^3}(1-2x)\frac{n+r-1}{(n-1)(n-r+1)}x(1-x) + \frac{1}{n^3}x(1-x)(1-2x) \\ &= \frac{n^2+4nr+3n+r^2+3r+2}{n^2(n-r+1)(n-r+2)}x(1-x)(1-2x). \end{aligned}$$

Finally, it is known that $B_n((e_1 - xe_0)^4)(x) = \mathrm{O}\left(\frac{1}{n^2}\right)$. Hence equation (3.8) is true for $r = 0$, $n \in \mathbb{N}$. For $1 \le r < n$ equation (3.3) yields

$$\begin{aligned} M^4[T_n^r](x) &= \frac{(n-1)^3(n+3)}{n^4} M^4[T_{n-1}^{r-1}](x) + 6\frac{(n-1)^2(1-2x)}{n^4} M^3[T_{n-1}^{r-1}](x) \\ &+ \frac{(n-1)(6(n-3)x(1-x)+4)}{n^4} M^2[T_{n-1}^{r-1}](x) \\ &+ \frac{4(n-1)x(1-x)(1-2x) - 4x^3 + 6x^2 - 4x + 1}{n^4} M^1[T_{n-1}^{r-1}](x) \\ &+ \frac{x(1-x)(3x^2-3x+1)}{n^4} M^0[T_{n-1}^{r-1}](x). \end{aligned}$$

From this relation, from (3.4), (3.5), (3.6), (3.7) and supposing that

$$M^4[T_{n-1}^{r-1}](x) = x(1-x)\mathrm{O}\left(\frac{1}{n^2}\right)$$

it follows that

$$M^4[T_n^r](x) = x(1-x)\mathrm{O}\left(\frac{1}{n^2}\right).$$

So, relation (3.8) follows by induction. □

Lemma 3.4. *For integers n, r, p, with $n > r + p$ we have the representation*

$$T_n^r(e_p)(x) = \left(\frac{n-r}{n}\right)^p B_{n-r}(e_p)(x) + R_{n,p,r}(x), \tag{3.9}$$

where $R_{n,p}(x)$ is a polynomial with degree at most p having all the coefficients of type $\mathrm{O}\left(\frac{1}{n}\right)$, depending on p and r.

Proof. We have

$$G_n(e_p)(t) = (1-t)\left(\frac{n-1}{n}t\right)^p + t\left(\frac{n-1}{n}t + \frac{1}{n}\right)^p.$$

From this it follows that $G_n(e_p)(t) = \left(\frac{n-1}{n}t\right)^p + P_{n,p}(t)$, where $P_{n,p}(t)$ is a polynomial of degree at most p in variable t and all the coefficients of $P_{n,p}(t)$ are positive and of type $\mathrm{O}\left(\frac{1}{n}\right)$. Then, by induction we deduce that

$$(G_{n-r+1} \circ G_{n-r+2} \circ \cdots \circ G_n)(e_p) = \left(\frac{n-r}{n}\right)^p + \tilde{P}_{n,p,r}(t),$$

where $\tilde{P}_{n,p,r}(t)$ is a polynomial of degree at most p having all the coefficients of type $\mathrm{O}\left(\frac{1}{n}\right)$.

Using formula (2.6) we obtain

$$T_n^r(e_p) = \left(\frac{n-r}{n}\right)^p B_{n-r}(e_p) + B_{n-r}(\tilde{P}_{n,p,r}).$$

Denoting $R_{n,p,r}(x) = B_{n-r}(\tilde{P}_{n,p,r})(x)$ it follows that $R_{n,p}(x)$ satisfies the conditions from this lemma, because the Bernstein polynomials B_{n-r} preserve the degree of polynomials of degree up to $n - r$. □

4. Estimations of the degree of approximation by operators T_n^r.

In this section we deduce estimates of order of approximation using the first order modulus of continuity, the usual second order modulus of continuity and the second Ditzian-Totik modulus, which are given bellow, for a generic function $g \in B[0,1]$ and $h > 0$, respectively by

$$\begin{aligned}\omega_1(g,h) &= \sup\{|g(u)-g(v)|,\ u,v\in[0,1],\ |u-v|\le h\};\\ \omega_2(g,h) &= \sup\{|g(x-\rho)-2g(x)+g(x+\rho)|,\ x\pm\rho\in[0,1],\ |\rho|\le h\};\\ \omega_2^\varphi(g,h) &= \sup\{|g(x-\rho)-2g(x)+g(x+\rho)|,\ x\pm\rho\in[0,1],\ |\rho|\le h\varphi(x)\},\\ &\quad\ \text{where } \varphi(x)=\sqrt{x(1-x)}.\end{aligned}$$

Theorem 4.1. *For $f \in C[0,1]$, $x \in [0,1]$ and integers $0 \le r < n$ the following estimates are true:*

$$|T_n^r(f)(x)-f(x)| \le 2\omega_1\left(f,\mu_{n,r}(x)\right), \tag{4.1}$$
$$|T_n^r(f)(x)-f(x)| \le \frac{1}{2}\mu_{n,r}(x)\omega_1\left(f',2\mu_{n,r}(x)\right), \tag{4.2}$$
$$|T_n^r(f)(x)-f(x)| \le \frac{3}{2}\omega_2\left(f,\mu_{n,r}(x)\right), \tag{4.3}$$
$$|T_n^r(f)(x)-f(x)| \le \frac{5}{2}\omega_2^\varphi\left(f,\sqrt{\frac{n+r+1}{n(n-r+1)}}\right), \tag{4.4}$$

where $\mu_{n,r}(x)=\sqrt{\frac{(n+r+1)x(1-x)}{n(n-r+1)}}$ and additionally, in inequality (4.2) we suppose that $f\in C^1[0,1]$, in inequality (4.3) we suppose that $\sqrt{\frac{(n+r+1)x(1-x)}{n(n-r+1)}}\le\frac{1}{2}$ and in inequality (4.4) we suppose that $\sqrt{\frac{n+r+1}{n(n-r+1)}}\le\frac{1}{2}$.

Proof. Inequality (4.1) follows from the general estimate of Mond [4]. For the rest of the estimates we can apply the estimates obtained in [5] for general operators in terms of the moments. So, inequality (4.2) follows from [5]- Cor. 2.3.2, inequality (4.3) follows from [5]- Cor. 2.2.1, and inequality (4.4) follows from [5]- Th. 2.5.1. □

Corollary 4.2. *For any $f \in C[0,1]$ and integer $r \ge 0$ there holds:*

$$\lim_{n\to\infty}\|T_n^r(f)-f\|=0, \tag{4.5}$$

where $\|\cdot\|$ denotes the sup-norm.

We give now a quantitative version of the Voronovskaja theorem. For this we use the least concave majorant of the first modulus of continuity, given for a function $f \in B[a,b]$ and $h > 0$ by

$$\tilde{\omega}_1(f,h)=\begin{cases}\sup\limits_{\substack{0\le x\le h\le y\le b\\ x\ne y}}\frac{(h-x)\omega_1(f,y)+(y-h)\omega_1(f,x)}{y-x}, & 0<h\le b-a\\ \omega_1(f,1), & h>b-a.\end{cases} \tag{4.6}$$

Theorem 4.3. *If $f \in C^2[0,1]$, $r \geq 0$ is an integer and $x \in [0,1]$, then we have*

$$\left| T_n^r(f)(x) - f(x) - \frac{1}{2} \cdot \frac{(n+r+1)x(1-x)}{n(n-r+1)} \cdot f''(x) \right| \leq \tilde{C}_r \frac{x(1-x)}{n} \tilde{\omega}_1 \left(f'', \frac{1}{\sqrt{n}} \right), \tag{4.7}$$

where $\tilde{C}_r > 0$ is a constant independent on f, n and x.

Proof. Using the estimate given in Gonska [2]-Th. 3.2 one obtains:

$$\left| T_n^r(f)(x) - f(x) - \frac{1}{2} \cdot \frac{(n+r+1)x(1-x)}{n(n-r+1)} \cdot f''(x) \right| \leq \frac{1}{2} T_n^r((e_1 - xe_0)^2)(x) \tilde{\omega}_1 \left(f'', \frac{1}{3} \cdot \frac{T_n^r(|e_1 - xe_0|^3)(x)}{T_n^r((e_1 - xe_0)^2)(x)} \right).$$

From the Cauchy-Schwartz inequality it results

$$\frac{T_n^r(|e_1 - xe_0|^3)(x)}{T_n^r((e_1 - xe_0)^2)(x)} \leq \sqrt{\frac{T_n^r((e_1 - xe_0)^4)(x)}{T_n^r((e_1 - xe_0)^2)(x)}}$$

Using Lemma 3.3 there is a constant C_r, independent on n and x such that

$$\frac{T_n^r((e_1 - xe_0)^4)(x)}{T_n^r((e_1 - xe_0)^2)(x)} \leq \frac{C_r \frac{x(1-x)}{n^2}}{\frac{(n+r+1)x(1-x)}{n(n-r+1)}} \leq \frac{C_r}{n}.$$

From the above relations it follows that there exists a constant $\tilde{C}_r$ such that relation (4.7) holds. □

5. Convexity of higher order. Simultaneous approximation

A function $f : I \to \mathbb{R}$, I interval, is named *convex of order* s, $s \geq -1$, or s-convex, in the sense of T. Popoviciu [6] if for any distinct points $x_0, x_1, \ldots x_{s+1}$ in I the inequality $[f; x_0, x_1, \ldots x_{s+1}] \geq 0$, holds, where $[f; x_0, x_1, \ldots x_{s+1}] \geq 0$ is the divided difference of function f. In particular, if f is convex of order s, then $\Delta_h^{s+1} f(x) \geq 0$, for any $x \in I$, $h > 0$, such that $x + (s+1)h \in I$, where $\Delta_h^{s+1} f(x) = \sum_{i=0}^{r+1} (-1)^{s+1+i} \binom{s+1}{i} f(x + ih)$ is the finite difference of order $s+1$ of f. So that f is convex of order -1 iff it is positive, f is convex of order 0 iff f is increasing, f is convex of order 1, if it is usual convex and so on. Denote by D the derivative operator, and by $D^s := D \circ D \circ D \circ \cdots \circ D$, ($s$-times), the operator of derivative of order s. If $f \in C^{s+1}(I)$, then f is convex of order s if and only if $D^{s+1} f(x) \geq 0$, for all $x \in I$. An operator which transforms each s-convex function in a s-convex function is named convex operator of order s.

Lemma 5.1. *For $f \in C[0,1]$, and integers $0 \leq r < n$, $0 \leq s < n - r$ we have*

$$D^s T_n^r(f)(x) = \frac{(n-r-s+1)_s}{(n-r)_r} \sum_{i=0}^{n-r-s} p_{n-r-s,i}(x) \sum_{k=0}^{r} c_{n+s,r,i+s,k} \Delta_{\frac{1}{n}}^s f\left(\frac{i+k}{n} \right). \tag{5.1}$$

Proof. We prove by induction with regard to s. For $s=0$ it results from Lemma 2.3. Now suppose that (5.1) is true for s and prove it for $s+1$. We have

$$\begin{aligned}
&D^{s+1}T_n^r(f)(x)\\
=\;& \frac{(n-r-s+1)_s}{(n-r)_r}\sum_{i=0}^{n-r-s}\frac{\mathrm{d}}{\mathrm{d}x}p_{n-r-s,i}(x)\sum_{k=0}^{r}c_{n+s,r,i+s,k}\Delta_{\frac{1}{n}}^{s}f\left(\frac{i+k}{n}\right)\\
=\;& \frac{(n-r-s+1)_s}{(n-r)_r}\sum_{i=0}^{n-r-s}(n-r-s)(p_{n-r-s-1,i-1}(x)-p_{n-r-s-1,i}(x))\times\\
&\qquad\times\sum_{k=0}^{r}c_{n+s,r,i+s,k}\Delta_{\frac{1}{n}}^{s}f\left(\frac{i+k}{n}\right)\\
=\;& \frac{(n-r-s)_{s+1}}{(n-r)_r}\sum_{i=0}^{n-r-s-1}p_{n-r-s-1,i}(x)\times\\
&\times\Big[\sum_{k=0}^{r}c_{n+s,r,i+s+1,k}\Delta_{\frac{1}{n}}^{s}f\left(\frac{i+k+1}{n}\right)-\sum_{k=0}^{r}c_{n+s,r,i+s,k}\Delta_{\frac{1}{n}}^{s}f\left(\frac{i+k}{n}\right)\Big]\\
=\;& \frac{(n-r-s)_{s+1}}{(n-r)_r}\sum_{i=0}^{n-r-s-1}p_{n-r-s-1,i}(x)\sum_{j=0}^{r+1}\Delta_{\frac{1}{n}}^{s}f\left(\frac{i+j}{n}\right)\times\\
&\qquad\times\Big[c_{n+s,r,i+s+1,j-1}-c_{n+s,r,i+s,j}\Big],
\end{aligned}$$

where $c_{n+s,r,i+s+1,-1}=0$ and $c_{n+s,r,i+s,r+1}=0$.
For n,r,i fixed, denote $\alpha_j=c_{n+s,r,i+s+1,j-1}-c_{n+s,r,i+s,j}$, $0\le j\le r+1$.
In order to prove the induction step it suffices to show for $0\le i\le n-r-s-1$:

$$\sum_{j=0}^{r+1}\Delta_{\frac{1}{n}}^{s}f\left(\frac{i+j}{n}\right)\alpha_j=\sum_{k=0}^{r}c_{n+s+1,r,i+s+1,k}\Delta_{\frac{1}{n}}^{s+1}f\left(\frac{i+k}{n}\right). \tag{5.2}$$

Then

$$\begin{aligned}
&\sum_{j=0}^{r+1}\Delta_{\frac{1}{n}}^{s}f\left(\frac{i+j}{n}\right)\alpha_j\\
=\;&\alpha_{r+1}\left[\Delta_{\frac{1}{n}}^{s}f\left(\frac{i+r+1}{n}\right)-\Delta_{\frac{1}{n}}^{s}f\left(\frac{i+r}{n}\right)\right]\\
&+(\alpha_r+\alpha_{r+1})\left[\Delta_{\frac{1}{n}}^{s}f\left(\frac{i+r}{n}\right)-\Delta_{\frac{1}{n}}^{s}f\left(\frac{i+r-1}{n}\right)\right]+\ldots\\
&+(\alpha_1+\alpha_2+\ldots+\alpha_{r+1})\left[\Delta_{\frac{1}{n}}^{s}f\left(\frac{i+1}{n}\right)-\Delta_{\frac{1}{n}}^{s}f\left(\frac{i}{n}\right)\right]\\
&+(\alpha_0+\alpha_1+\ldots+\alpha_{r+1})\Delta_{\frac{1}{n}}^{s}f\left(\frac{i}{n}\right).
\end{aligned}$$

Using Lemma 2.3 and then Lemma 2.1-i) we have

$$\begin{aligned}\sum_{j=0}^{r+1}\alpha_j &= \sum_{j=0}^{r+1} c_{n+s,r,i+s+1,j-1} - \sum_{j=0}^{r+1} c_{n+s,r,i+s,j}\\ &= \sum_{j=0}^{r} c_{n+s,r,i+s+1,j} - \sum_{j=0}^{r} c_{n+s,r,i+s,j}\\ &= (n+s-r)_r F^r_{n+s,i+s+1}(e_0) - (n+s-r)_r F^r_{n+s,i+s}(e_0)\\ &= (n+s-r)_r - (n+s-r)_r = 0.\end{aligned}$$

Therefore, it results

$$\sum_{j=0}^{r+1}\Delta^s_{\frac{1}{n}} f\left(\frac{i+j}{n}\right)\alpha_j = \sum_{k=0}^{r}\sum_{j=k+1}^{r+1}\alpha_j\cdot\Delta^{s+1}_{\frac{1}{n}} f\left(\frac{i+k}{n}\right). \tag{5.3}$$

In order to obtain relation (5.2) it suffices to prove for $0\le k\le r,\ 0\le i\le n-r-s-1$ that:

$$\sum_{j=k+1}^{r+1}\alpha_j = c_{n+s+1,r,i+s+1,k}. \tag{5.4}$$

Fix i. We prove relation (5.4) by descending induction with regard to k. For $k=r$ we have

$$\begin{aligned}\sum_{j=r+1}^{r+1}\alpha_j &= \alpha_{r+1} = c_{n+s,r,i+s+1,r} = \binom{r}{r}(n-1-r)_0(i+s+1)_r\\ &= \binom{r}{r}(n-r)_0(i+s+1)_r = c_{n+s+1,r,i+s+1,r}.\end{aligned}$$

Now, suppose that (5.4) is true for $k+1$, $0\le k\le r-1$ and prove it for k. One obtains

$$\begin{aligned}&\sum_{j=k+1}^{r+1}\alpha_j\\ &= \alpha_{k+1} + \sum_{j=k+2}^{r+1}\alpha_j\\ &= \alpha_{k+1} + c_{n+s+1,r,i+s+1,k+1}\\ &= c_{n+s,r,i+s+1,k} - c_{n+s,r,i+s,k+1} + c_{n+s+1,r,i+s+1,k+1}\\ &= \binom{r}{k}(n-i-r-1)_{r-k}(i+s+1)_k - \binom{r}{k+1}(n-i-r)_{r-k-1}(i+s)_{k+1}\\ &\quad + \binom{r}{k+1}(n-i-r)_{r-k-1}(i+s+1)_{k+1}\end{aligned}$$

$$\begin{aligned}
&= (i+s+1)_k(n-i-r)_{r-k-1}\Big[\binom{r}{k}(n-i-r-1)-\binom{r}{k+1}(i+s) \\
&\quad +\binom{r}{k+1}(i+s+k+1)\Big] \\
&= (i+s+1)_k(n-i-r)_{r-k-1}\Big[\binom{r}{k}(n-i-r-1)+\binom{r}{k+1}(k+1)\Big] \\
&= \binom{r}{k}(i+s+1)_k(n-i-r)_{r-k-1}[(n-i-r-1)+(r-k)] \\
&= \binom{r}{k}(i+s+1)_k(n-i-r)_{r-k} \\
&= c_{n+s+1,r,i+s+1,k}.
\end{aligned}$$

Then equality (5.4) is true and consequently relation (5.2) is true. □

Theorem 5.2. *Let integers n, r be such that $n > r$. Then operator T_n^r is convex of order s for each integer $s \geq -1$ such that $n > r+s$.*

Proof. If f is s-convex, then $\Delta_{\frac{1}{n}}^{s+1} f\left(\frac{i+k}{n}\right) \geq 0$, for $0 \leq i \leq n-r-s-1$. From relation (5.1) with $s+1$, instead of s it follows that $\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{s+1} T_n^r(f)(x) \geq 0$, i.e. $T_n^r(f)$ is s-convex. □

With the aid of this fact we can deduce the property of simultaneous approximation of operators T_n^r.

Theorem 5.3. *For any integers $0 \leq r < n$ and $0 \leq s < n-r$ we have*

$$\lim_{n\to\infty} \|(D^s \circ T_n^r)(f) - D^s f\| = 0 \tag{5.5}$$

Proof. It suffices to take $s \geq 1$. Let $n \in \mathbb{N}$ sufficiently large, such that $n > r+s$. Consider s-Kantorovich operator associated to the operator T_n^r, defined by

$$K_{n,r}^s = D^s \circ T_n^r \circ I_s,$$

where I_s is operator defined by

$$I_s(g)(x) = \int_0^x \frac{(x-t)^{s-1}}{(s-1)!} g(t)dt, \ x \in [0,1], \ g \in C[0,1],$$

for $s \geq 1$ and I_0 is the identical operator. Because operator T_n^r is convex of order $s-1$ if follows that $K_{n,r}^s$ is a linear positive operator. Note that

$$(D^s \circ T_n^r)(f) = K_{n,r}^s(D^s f).$$

So that, in order to prove relation (5.5) it is sufficient to prove that the sequence of operators $(K_{n,r}^s)_n$ satisfies the conditions in the theorem of Korovkin. In Knopp and Pottinger [3]- Korollar 2.2 it is shown that the necessary and sufficient condition for this is the following conditions

$$\lim_{n\to\infty} \|D^s e_{s+i} - (D^s \circ T_n^r)(e_{s+i})\| = 0, \text{ for } i = 0,1,2. \tag{5.6}$$

hold. From Lemma 3.4 we obtain

$$(D^s \circ T_n^r)(e_{s+i}) = \left(\frac{n-r}{n}\right)^{s+i} D^s B_{n-r}(e_{s+i}) + D^s R_{n,s+i,r},\ i = 0,1,2$$

Because the sequence $(B_{n-r})_n$ has the property of simultaneous approximation, we infer

$$\lim_{n\to\infty} \left\| D^s e_{s+i} - \left(\frac{n-r}{n}\right)^{s+i} (D^s \circ B_{n-r})(e_{s+i}) \right\| = 0,\ i = 0,1,2.$$

Also, from properties of polynomials $R_{n,s+i,r}$ we obtain

$$\lim_{n\to\infty} \|D^s R_{n,s+i,r}\| = 0,\ i = 0,1,2. \qquad \square$$

References

[1] Chen, X., Tan, J., Liu, Z., Xie, J., *Approximation of functions by a new family of generalized Bernstein operators*, J. Math. Anal. Appl., **450**(2017), 244-261.

[2] Gonska, H., *On the degree of approximation in Voronovskaja's theorem*, Stud. Univ. Babeş-Bolyai. Math., **52**(2007), no. 3, 103-115.

[3] Knopp, H.P., Pottinger, P., *Ein Satz vom Korovkin-Typ für C^k-Räume*, Math. Z., **148**(1976), 23-32.

[4] Mond, B., *Note: On the degree of approximation by linear positive operators*, J. Approx. Theory, **18**(1976), 304-306.

[5] Păltănea, R., *Approximation Theory Using Positive Linear operators*, Birkhäuser, Boston, 2004.

[6] Popoviciu, T., *Les Fonctions Convexes* (French), Actualités Sci. Ind., **992**, Hermann et Cie, Paris, 1944.

[7] Smuc, M., *On a Chlodovsky variant of α-Bernstein operator*, Bull. Transilv. Univ. Braşov Ser. III. Math. Inform. Phys., **10(59)**(2017), no. 1, 165-178.

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