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An upper bound of the Hankel determinant of third order for the inverse of reciprocal of bounded turning functions

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Abstract. The objective of this paper is to obtain an upper bound of the third order Hankel determinant for the inverse of the function f, when f belongs to the reciprocal of bounded turning functions with new approach.

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Keywords: Bounded turning function, upper bound, Hankel determinant, positive real function.

1. Introduction

Let $\mathcal{A}$ denote the class of all analytic functions f of the form

$$f(z) = z + \sum_{n \geq 2} a_n z^n \tag{1.1}$$

in the open unit disc $\mathcal{U}_d = \{z \in \mathbb{C} : |z| < 1\}$ standardized by $f(0) = 0$, and $f'(0) = 1$. Let S be the subclass of $\mathcal{A}$, consisting of univalent functions. In 1985, Louis de Branges de Bourcia proved the Bieberbach conjecture also called as Coefficient conjecture, which states that for a univalent function its n^{th}-Taylor's coefficient is bounded by n (see [5]). The bounds of the coefficients for these functions give information about their geometric properties. A typical problem in geometric function theory is to study a functional made up of combination of the coefficients of the original function. The

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Hankel determinant of order q for the regular mapping f, was defined by Pommerenke [23], as follows.

$$H_{q,t}(f) = \begin{vmatrix} a_t & a_{t+1} & \cdots & a_{t+q-1} \\ a_{t+1} & a_{t+2} & \cdots & a_{t+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{t+q-1} & a_{t+q} & \cdots & a_{t+2q-2} \end{vmatrix}. \tag{1.2}$$

Here $a_1 = 1$, q and t are integers, positive in nature. The determinant given in (1.2) has been investigated by many authors, a few of them are cited here. Ehrenborg [8] studied the Hankel determinant of exponential polynomials. Noor [20] determined the rate of growth of $H_{q,t}$ as $t \to \infty$ for the functions in S with bounded boundary. The Hankel transform of an integer sequence and some of its features were studied by Layman (see [14]). For $q = 2$ and $t = 1$ in (1.2), we obtain $H_{2,1}(f)$, the Fekete-Szegö functional is the classical problem settled by Fekete-Szegö [9] is to find for each $\lambda \in [0, 1]$, the maximum value of the coefficient functional, defined by $\phi_\lambda(f) := |a_3 - \lambda a_2^2|$ over the class S and was proved by using Loewner method. Ali [1] found sharp bounds of the first four coefficients and sharp estimate for the Fekete-Szegö functional $|t_3 - \delta a_2^2|$, where δ is real, for the inverse function of f defined as

$$f^{-1}(w) = w + \sum_{n\geq 2} q_n w^n,$$

when $f^{-1} \in \widetilde{ST}(\alpha)$, the class of strongly starlike functions of order α with $\alpha \in (0, 1]$. In recent years, the research on Hankel determinants has focused on the estimation of $H_{2,2}(f)$, known as the second Hankel determinant obtained for $q = 2 = t$ in (1.2), given by

$$H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.$$

Many authors obtained results associated with estimation of an upper bound of the functional $H_{2,2}(f)$ for various subclasses of univalent and multivalent analytic functions. The exact (sharp) estimates of $H_{2,2}(f)$ for the subclasses of S namely, bounded turning, starlike and convex functions denoted by $\mathcal{R}$, S^* and $\mathcal{K}$ respectively in $\mathcal{U}_d$, i.e., functions satisfying the conditions

$$\text{Re} f'(z) > 0,\ \text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \text{ and } \text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0$$

were proved by Janteng et al. [11, 12] and determined the bounds as 4/9, 1 and 1/8 respectively. For the class of Ma-Minda starlike functions, the sharp bound of the second Hankel determinant was obtained by Lee et al. [16]. Choosing $q = 2$ and $t = p + 1$ in (1.2), we obtain the second Hankel determinant for the p-valent function (see [26]), as follows.

$$H_{2,p+1}(f) = \begin{vmatrix} a_{p+1} & a_{p+2} \\ a_{p+2} & a_{p+3} \end{vmatrix} = a_{p+1} a_{p+3} - a_{p+2}^2,$$

The case $q = 3$ appears to be much more difficult than the case $q = 2$. Very few papers have been devoted for the study of third order Hankel determinant denoted by $H_{3,1}(f)$, obtained for $q = 3$ and $t = 1$ in (1.2), namely

$$H_{3,1}(f) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}.$$

Expanding the determinant, we have

$$H_{3,1}(f) = a_1(a_3a_5 - a_4^2) + a_2(a_3a_4 - a_2a_5) + a_3(a_2a_4 - a_3^2), \tag{1.3}$$

$$\Leftrightarrow \; H_{3,1}(f) = H_{2,3}(f) + a_2J_2 + a_3H_{2,2}(f),$$

where $J_2 = (a_3a_4 - a_2a_5)$ and $H_{2,3}(f) = (a_3a_5 - a_4^2)$.
The concept of estimation of an upper bound of $H_{3,1}(f)$ was firstly introduced and studied by Babalola [3], who tried to estimate for this functional to the classes $\mathcal{R}$, S^* and $\mathcal{K}$, obtained as follows.

(i) $f \in S^* \Rightarrow |H_{3,1}(f)| \leq 16$.
(ii) $f \in \mathcal{K} \Rightarrow |H_{3,1}(f)| \leq 0.714$.
(iii) $f \in \mathcal{R} \Rightarrow |H_{3,1}(f)| \leq 0.742$.

As a result of this paper, Raza and Malik [24] obtained an upper bound of the third Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli. Sudharsan et al. [25] derived an upper bound of the third kind Hankel determinant for a subclass of analytic functions, namely

$$\mathcal{C}_\alpha^\beta = \operatorname{Re}\left\{\frac{(zf'(z) + \alpha z^2 f''(z))'}{f'(z)}\right\} > \beta,$$

where $(0 \leq \alpha \leq 1)$ and $(0 \leq \beta < 1)$. Bansal et al. [4] improved the upper bound of $H_{3,1}(f)$ for some of the classes estimated by Babalola [3] to some extent. Recently, Zaprawa [29] improved all the results obtained by Babalola [3]. Further, Orhan and Zaprawa [21] obtained an upper bound of the third kind Hankel determinant for the classes S^* and $\mathcal{K}$ functions of order α $(0 \leq \alpha < 1)$. Very recently, Kowalczyk et al. [13] calculated sharp upper bound of $H_{3,1}(f)$ for the class $\mathcal{K}$ of convex functions and showed as $|H_{3,1}(f)| \leq \frac{4}{135}$, which is more refined bound than the bound derived by Zaprawa [29]. Lecko et al. [15] determined sharp bound of the third order Hankel determinant for starlike functions of order 1/2. Arif et al. [2] estimated an upper bound of the Fourth Hankel determinant for the family of bounded turning functions. Motivated by the results obtained by different authors mentioned above and who are working in this direction (see [6,26]), in this paper, we are making an attempt to introduce a new subclass of analytic functions and obtain an upper bound of the functional $H_{3,1}(f^{-1})$, where f^{-1} is the inverse function for the function f belonging to this class, defined as follows.

Definition 1.1. A function $f(z) \in \mathcal{A}$ is said to be in the class $\widehat{RT}$, consisting of functions whose reciprocal derivative have a positive real part (also called reciprocal of bounded turning functions) (for the properties of bounded turning functions (see [19]), given by

$$\operatorname{Re}\left\{\frac{1}{f'(z)}\right\} > 0, \; z \in \mathcal{U}_d. \tag{1.4}$$

In proving our result, we require a few sharp estimates in the form of lemmas valid for functions with positive real part.
Let $\mathcal{P}$ denote the class of functions consisting of g, such that

$$g(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... = 1 + \sum_{n\geq 1} c_n z^n, \tag{1.5}$$

which are analytic in $\mathcal{U}_d$ and $\text{Re}g(z) > 0$ for $z \in \mathcal{U}_d$. Here g is called the Caratheodory function [7].

Lemma 1.2. ([10]) *If $g \in \mathcal{P}$, then the sharp estimate $|c_n - \mu c_k c_{n-k}| \leq 2$, holds for $n, k \in \mathbb{N} = \{1, 2, 3...\}$, with $n > k$ and $\mu \in [0, 1]$.*

Lemma 1.3. ([18]) *If $g \in \mathcal{P}$, then the sharp estimate $|c_n - c_k c_{n-k}| \leq 2$, holds for $n, k \in \mathbb{N}$, with $n > k$.*

Lemma 1.4. ([22]) *If $g \in \mathcal{P}$ then $|c_k| \leq 2$, for each $k \geq 1$ and the inequality is sharp for the mobious transformation $g(z) = \frac{1+z}{1-z}$, $z \in \mathcal{U}_d$.*

In order to obtain our result, we referred to the classical method devised by Libera and Zlotkiewicz [17], used by several authors.

2. Main result

Theorem 2.1. *If $f \in \widehat{RT}$ and $f^{-1}(w) = w + \sum_{n\geq 2} q_n w^n$ near the origin i.e., $w = 0$ is the inverse function of f, given in (1.1) then*

$$|H_{3,1}(f^{-1})| \leq \frac{527}{540}.$$

Proof. For the function $f \in \widehat{RT}$, by virtue of Definition 1.1, there exists a holomorphic function $g \in \mathcal{P}$ in $\mathcal{U}_d$ with $g(0) = 1$ and $\text{Re}g(z) > 0$ such that

$$\frac{1}{f'(z)} = g(z) \Leftrightarrow \ 1 = g(z)f'(z). \tag{2.1}$$

Replacing f' and g with their series expressions in (2.1), upon simplification, we get

$$\begin{aligned}
a_2 &= -\frac{c_1}{2}; \\
a_3 &= -\frac{1}{3}(c_2 - c_1^2); \\
a_4 &= -\frac{1}{4}(c_3 - 2c_1c_2 + c_1^3); \\
a_5 &= -\frac{1}{5}(c_4 - 2c_1c_3 + 3c_1^2c_2 - c_2^2 - c_1^4).
\end{aligned} \tag{2.2}$$

According to Koebe's $(\frac{1}{4})^{th}$- theorem, also known as one-quarter theorem every holomorphic and univalent function ϖ in $\mathcal{U}_d$ possesses an inverse denoted by ϖ^{-1}, satisfying

$$z = \left\{\varpi^{-1}(\varpi(z))\right\}, \ z \in \mathcal{U}_d$$

and

$$\varpi\left\{\varpi^{-1}(w)\right\}=w,\quad\left(|w|<\rho_0(f);\rho_0(f)\geq\frac{1}{4}\right).$$

Consider

$$w=\varpi\left\{\varpi^{-1}(w)\right\}=\left\{\varpi^{-1}(w)\right\}+\sum_{n\geq 2}a_n\left\{\varpi^{-1}(w)\right\}^n$$

$$\Leftrightarrow w=\left\{w+\sum_{n\geq 2}q_nw^n\right\}+\sum_{n\geq 2}a_n\left\{w+\sum_{n\geq 2}q_nw^n\right\}^n.$$

By simple computation, we get

$$\Big[(q_2+a_2)w^2+(q_3+2a_2q_2+a_3)w^3+(q_4+2a_2q_3+a_2q_2^2+3a_3q_2+a_4)w^4$$
$$+(q_5+2a_2q_4+2a_2q_2t_3+3a_3q_3+3a_3q_2^2+4a_4q_2+a_5)w^5+\ldots\Big]=0. \quad (2.3)$$

Equating the coefficients of w^2, w^3, w^4 and w^5 in (2.3), upon simplification, we obtain

$$q_2=-a_2;\ q_3=\left\{-a_3+2a_2^2\right\};\ q_4=\left\{-a_4+5a_2a_3-5a_2^3\right\};$$

$$q_5=\left\{-a_5+6a_2a_4-21a_2^2a_3+3a_3^2+14a_2^4\right\}. \quad (2.4)$$

Simplifying the expressions (2.2) and (2.4), we get

$$q_2=\frac{c_1}{2};\ q_3=\frac{1}{6}\left\{2c_2+c_1^2\right\};\ q_4=\frac{1}{24}\left\{6c_3+8c_1c_2+c_1^3\right\};$$

$$q_5=\frac{1}{120}\left\{24c_4+42c_1c_3+22c_1^2c_2+16c_2^2+c_1^4\right\}. \quad (2.5)$$

At this juncture, based on the determinant $H_{3,1}(f)$ given in (1.3), the third order Hankel determinant for the inverse function of f, namely

$$f^{-1}(w)=w+\sum_{n\geq 2}q_nw^n$$

near the origin i.e., $w=0$, can be defined as

$$H_{3,1}(f^{-1})=\begin{vmatrix}q_1&q_2&q_3\\q_2&q_3&q_4\\q_3&q_4&q_5\end{vmatrix}(q_1=1). \quad (2.6)$$

Expanding the determinant, we get

$$H_{3,1}(f^{-1})=q_1(q_3q_5-q_4^2)+q_2(q_3q_4-q_2q_5)+q_3(q_2q_4-q_3^2). \quad (2.7)$$

Putting the values of q_2, q_3, q_4 and q_5 from (2.5) in the functional given in (2.7), it simplifies to

$$H_{3,1}(f^{-1}) = \Big[\frac{1}{15}c_2c_4 + \frac{1}{135}c_2^3 - \frac{1}{16}c_3^2 - \frac{1}{60}c_1^2c_4 + \frac{1}{30}c_1c_2c_3 - \frac{1}{180}c_1^2c_2^2$$

$$+ \frac{1}{720}c_1^4c_2 - \frac{1}{120}c_1^3c_3 - \frac{1}{8640}c_1^6\Big]. \quad (2.8)$$

Upon grouping the terms in the expression (2.8), we have

$$H_{3,1}(f^{-1}) = \Big[\frac{1}{60}c_4(c_2 - c_1^2) - \frac{1}{16}c_3(c_3 - \frac{16}{60}c_1c_2) - \frac{1}{135}c_2(c_4 - c_2^2) - \frac{1}{60}c_2(c_4 - c_1c_3)$$

$$+ \frac{1}{720}c_1^4(c_2 - \frac{1}{12}c_1^2) + \frac{2}{27}c_2c_4 - \frac{1}{120}c_1^3c_3 - \frac{1}{180}c_1^2c_2^2\Big]. \quad (2.9)$$

Applying the triangle inequality in (2.9), we obtain

$$\Big|H_{3,1}(f^{-1})\Big| \leq \Big[\frac{1}{60}|c_4||c_2 - c_1^2| + \frac{1}{16}|c_3||c_3 - \frac{16}{60}c_1c_2| + \frac{1}{135}|c_2||c_4 - c_2^2| + \frac{1}{60}|c_2||c_4 - c_1c_3|$$

$$+ \frac{1}{720}|c_1^4||c_2 - \frac{1}{12}c_1^2| + \frac{2}{27}|c_2||c_4| + \frac{1}{120}|c_1^3||c_3| + \frac{1}{180}|c_1^2||c_2^2|\Big]. \quad (2.10)$$

Upon using the lemmas given in 1.2, 1.3 and 1.4 in the inequality (2.10), after simplifying, we get

$$\Big|H_{3,1}(f^{-1})\Big| \leq \frac{527}{540}. \quad (2.11)$$

□

Remark 2.2. The result, obtained in (2.11) is far better than the result obtained by the authors (see [28]).

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