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Generalized and numerical solution for a quasilinear parabolic equation with nonlocal conditions

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Abstract. In this paper we study the one dimensional mixed problem with nonlocal boundary conditions, for the quasilinear parabolic equation. We prove an existence, uniqueness of the weak generalized solution and also continuous dependence upon the data of the solution are shown by using the generalized Fourier method. We construct an iteration algorithm for the numerical solution of this problem. We analyze computationally convergence of the iteration algorithm, as well as on test example.

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1. Introduction

In this study, we consider the following mixed problem

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x,t,u), \;\; D := \{0 < x < 1, 0 < t < T\} \tag{1.1}$$

$$u(0,t) = u(1,t), \qquad t \in [0,T] \tag{1.2}$$

$$u_x(1,t) = 0, \qquad t \in [0,T] \tag{1.3}$$

$$u(x,0) = \varphi(x), \qquad x \in [0,1] \tag{1.4}$$

for a quasilinear parabolic equation with the nonlinear source term $f = f(x,t,u)$.

The functions $\varphi(x)$ and $f(x,t,u)$ are given functions on $[0,1]$ and $\bar{D} \times (-\infty, \infty)$, respectively.

Denote the solution of the problem (1.1)-(1.4) by $u = u(x,t)$.

This problem was investigated with different boundary conditions by various researchers by using Fourier or different methods [2, 4].

In this study, we consider the initial-boundary value problem (1.1)-(1.4) with nonlocal boundary conditions (1.2)-(1.3). The periodic nature of (1.2)-(1.3) type boundary conditions is demonstrated in [10]. In this study, we prove the existence, uniqueness, convergence of the weak generalized solution continuous dependence upon the data of the solution and we construct an iteration algorithm for the numerical solution of this problem. We analyze computationally convergence of the iteration algorithm, as well as on test example. We demonstrate a numerical procedure for this problem on concrete examples, and also we obtain numerical solution by using the implicit finite difference algorithm [11].

We will use the weak solution approach from [3] for the considered problem (1.1)-(1.4).

According to [1, 5] assume the following definitions.

Definition 1.1. *The function $v(x,t) \in C^2(\bar{D})$ is called test function if it satisfies the following conditions:*

$$v(x,T) = 0,\ v(0,t) = v(1,t),\ v_x(1,t) = 0, \forall t \in [0,T] \text{ and } \forall x \in [0,1].$$

Definition 1.2. *The function $u(x,t) \in C(\bar{D})$ satisfying the integral identity*

$$\int_0^T \int_0^1 \left[\left(\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} \right) u - f(x,t,u)v \right] dxdt$$

$$-\int_0^T \left[u(0,t)v_x(0,t) - v(0,t)u_x(0,t) \right] dt - \int_0^1 \varphi(x)v(x,0)dx = 0, \tag{1.5}$$

for arbitrary test function $v = v(x,t)$, is called a generalized (weak) solution of the problem (1)-(4).

2. Reducing the problem to countable system of integral equations

Consider the following system of functions on the interval $[0,1]$:

$$X_0(x) = 2,\ X_{2k-1}(x) = 4\cos(2\pi kx),\ X_{2k}(x) = 4(1-x)\sin(2\pi kx),\ k = 1,2,\ldots, \tag{2.1}$$

$$Y_0(x) = x,\ Y_{2k-1}(x) = x\cos(2\pi kx),\ Y_{2k}(x) = \sin(2\pi kx),\ k = 1,2,\ldots \tag{2.2}$$

The system of functions (2.1) and (2.2) arise in [6] for the solution of a nonlocal boundary value problem in heat conduction.

It is easy to verify that the system of function (2.1) and (2.2) are biorthonormal on $[0,1]$. They are also Riesz bases in $L_2[0,1]$ (see [7, 8]).

We will use the Fourier series representation of the weak solution to transform the initial-boundary value problem to the infinite set of nonlinear integral equations.

Any solution of the equation (1.1)-(1.4) can be represented as

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t) X_k(x), \tag{2.3}$$

where the functions $u_k(t)$, $k = 0, 1, 2, \ldots$ satisfy the following system of equations:

$$\begin{aligned} u_0(t) &= \varphi_0 + \int_0^t f_0(\tau)d\tau, \\ u_{2k}(t) &= \varphi_{2k}\, e^{-(2\pi k)^2 t} + \int_0^t f_{2k}(\tau)e^{-(2\pi k)^2(t-\tau)}d\tau, \\ u_{2k-1}(t) &= (\varphi_{2k-1} - 4\pi k\varphi_{2k})e^{-(2\pi k)^2 t} \\ &\quad + \int_0^t e^{-(2\pi k)^2(t-\tau)}[f_{2k-1}(\tau) - 4\pi k(t-\tau)f_{2k}(\tau)]d\tau, \end{aligned} \tag{2.4}$$

where

$$\varphi_k = \int_0^1 \varphi(x)Y_k(x)dx,$$

$$f_k(x) = \int_0^1 f(x,t,u)Y_k(x)dx.$$

Definition 2.1. *Denote the set*

$$\{u(t)\} = \{u_0(t), u_{2k}(t), u_{2k-1}(t),\ k = 1, 2, \ldots, \},$$

of continuous on $[0, T]$ *satisfying the following condition*

$$underset0 \le t \le T\max 2\,|u_0(t)| + 4\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}|u_{2k}(t)| + \max_{0\le t\le T}|u_{2k-1}(t)|\right) < \infty,$$

by B*. Let*

$$\|u(t)\| = \max_{0\le t\le T} 2\,|u_0(t)| + 4\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}|u_{2k}(t)| + \max_{0\le t\le T}|u_{2k-1}(t)|\right),$$

be the norm in B*. It can be shown that* B *is the Banach space* [9].

We denote the solution of the nonlinear system (2.4) by $\{u(t)\}$.

Theorem 2.2. *a) Let the function* $f(x,t,u)$ *is continuous with respect to all arguments in* $\bar{D} \times (-\infty, \infty)$ *and satisfies the following condition*

$$|f(x,t,u) - f(x,t,\tilde{u})| \le b(x,t)\,|u - \tilde{u}|,$$

where $b(x,t) \in L_2(D)$, $b(x,t) \ge 0$,

b) $f(x,t,0) \in C^2[0,1]$, $t \in [0,1]$,

c) $\varphi(x) \in C^2[0,1]$.

Then the system (2.4) has a unique solution in D.

Proof. For $N = 0, 1, \ldots$ let's define an iteration for the system (2.4) as follows:

$$\begin{aligned} u_0^{(N+1)}(t) &= u_0^{(0)}(t) + \int_0^t \int_0^1 f(\xi,\tau,Au^{(N)}(\xi,\tau))\xi d\xi d\tau, \\ u_{2k}^{(N+1)}(t) &= u_{2k}^{(0)}(t) + \int_0^t \int_0^1 e^{-(2\pi k)^2(t-\tau)} f(\xi,\tau,Au^{(N)}(\xi,\tau)) \sin 2\pi k\xi d\xi d\tau, \\ u_{2k-1}^{(N+1)}(t) &= u_{2k-1}^{(0)}(t) + \int_0^t \int_0^1 e^{-(2\pi k)^2(t-\tau)} f(\xi,\tau,Au^{(N)}(\xi,\tau))\xi \cos 2\pi k\xi d\xi d\tau \\ &\quad -4\pi k \int_0^t \int_0^1 e^{-(2\pi k)^2(t-\tau)} f(\xi,\tau,Au^{(N)}(\xi,\tau))(t-\tau) \sin 2\pi k\xi d\xi d\tau, \end{aligned} \tag{2.5}$$

where, for simplicity, let

$$Au^{(N)}(\xi,\tau) = 2u_0^{(N)}(\tau) + 4\sum_{k=1}^{\infty} \left(u_{2k}^{(N)}(\tau)(1-\xi) \sin 2\pi k\xi + u_{2k-1}^{(N)}(\tau) \cos 2\pi k\xi \right).$$

where,

$$u_0^{(0)}(t) = \varphi_0 \,, u_{2k}^{(0)}(t) = \varphi_{2k} \, e^{-(2\pi k)^2 t}, u_{2k-1}^{(0)}(t) = (\varphi_{2k-1} \; - 4\pi k \; \varphi_{2k} \;) \; e^{-(2\pi k)^2 t}.$$

From the condition of the theorem we have $u^{(0)}(t) \in B$. We will prove that the other sequentially approximations satisfy this condition.

Let us write $N = 0$ in (2.5).

$$u_0^{(1)}(t) = u_0^{(0)}(t) + \int_0^t \int_0^1 f(\xi,\tau,Au^{(0)}(\xi,\tau)) d\xi d\tau.$$

Adding and subtracting $\int_0^t \int_0^1 f(\xi,\tau,0) d\xi d\tau$, applying Cauchy inequality, Lipschitz condition, taking the maximum of both sides of the last inequality yields the following:

$$\max_{0\le t\le T} \left| u_0^{(1)}(t) \right| \le |\varphi_0| + \sqrt{T} \left\| b(x,t) \right\|_{L_2(D)} \left\| u^{(0)}(t) \right\| + \sqrt{T} \left\| f(x,t,0) \right\|_{L_2(D)}.$$

$$u_{2k}^{(1)}(t) = \varphi_{2k} e^{-(2\pi k)^2 t} + \int_0^t \int_0^1 e^{-(2\pi k)^2(t-\tau)} f(\xi,\tau,Au^{(N)}(\xi,\tau)) \sin 2\pi k\xi d\xi d\tau.$$

Adding and subtracting $\int_0^t \int_0^1 e^{-(2\pi k)^2(t-\tau)} f(\xi,\tau,0) \sin 2\pi k\xi d\xi d\tau$, applying Cauchy inequality, taking the summation of both sides respect to k and using Hölder inequality, Bessel inequality, Lipschitz condition and taking maximum of both sides of the last inequality yields the following:

$$\sum_{k=1}^{\infty}\max_{0\le t\le T}\left|u_{2k}^{(1)}(t)\right|\le\sum_{k=1}^{\infty}|\varphi_{2k}|+\frac{1}{4\sqrt{3}}\|b(x,t)\|_{L_2(D)}\left\|u^{(0)}(t)\right\|+\frac{1}{4\sqrt{3}}\|f(x,t,0)\|_{L_2(D)}.$$

In the same way, we obtain:

$$\begin{aligned}\sum_{k=1}^{\infty}\max_{0\le t\le T}\left|u_{2k-1}^{(1)}(t)\right|&\le\sum_{k=1}^{\infty}|\varphi_{2k-1}|+\frac{1}{\sqrt{6}}\sum_{k=1}^{\infty}\left|\varphi_{2k}^{''}\right|\\&+\frac{1}{4\sqrt{3}}\|b(x,t)\|_{L_2(D)}\left\|u^{(0)}(t)\right\|+\frac{1}{4\sqrt{3}}\|f(x,t,0)\|_{L_2(D)}\\&+\sqrt{2}\,|T|\,\|b(x,t)\|_{L_2(D)}\left\|u^{(0)}(t)\right\|+\sqrt{2}\,|T|\,\|f(x,t,0)\|_{L_2(D)}.\end{aligned}$$

Finally we have the following inequality:

$$\begin{aligned}\left\|u^{(1)}(t)\right\|_B&=2\max_{0\le t\le T}\left|u_0^{(1)}(t)\right|+4\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}\left|u_{2k}^{(1)}(t)\right|+\max_{0\le t\le T}\left|u_{2k-1}^{(1)}(t)\right|\right)\\&\le2|\varphi_0|+4\sum_{k=1}^{\infty}(|\varphi_{2k}|+|\varphi_{2k-1}|)+\frac{2\sqrt{6}\,|T|}{3}\sum_{k=1}^{\infty}\left|\varphi_{2k}^{''}\right|\\&+\left(2\sqrt{T}+\frac{2\sqrt{3}}{3}+4\sqrt{2}\,|T|\right)\left(\|b(x,t)\|_{L_2(D)}\left\|u^{(0)}(t)\right\|_B\right)\\&+\left(2\sqrt{T}+\frac{2\sqrt{3}}{3}+4\sqrt{2}\,|T|\right)\|f(x,t,0)\|_{L_2(D)}.\end{aligned}$$

Hence $u^{(1)}(t)\in B$. In the same way, for a general value of N we have

$$\begin{aligned}\left\|u^{(N)}(t)\right\|_B&=2\max_{0\le t\le T}\left|u_0^{(N)}(t)\right|+4\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}\left|u_{2k}^{(N)}(t)\right|+\max_{0\le t\le T}\left|u_{2k-1}^{(N)}(t)\right|\right)\\&\le2|\varphi_0|+4\sum_{k=1}^{\infty}(|\varphi_{2k}|+|\varphi_{2k-1}|)+\frac{2\sqrt{6}\,|T|}{3}\sum_{k=1}^{\infty}\left|\varphi_{2k}^{''}\right|\\&+\left(2\sqrt{T}+\frac{2\sqrt{3}}{3}+4\sqrt{2}\,|T|\right)\left(\|b(x,t)\|_{L_2(D)}\left\|u^{(N-1)}(t)\right\|_B\right)\\&+\left(2\sqrt{T}+\frac{2\sqrt{3}}{3}+4\sqrt{2}\,|T|\right)\|f(x,t,0)\|_{L_2(D)},\end{aligned}$$

$u^{(N-1)}(t)\in B$, we deduce that $u^{(N)}(t)\in B$, we obtain

$$\{u(t)\}=\{u_0(t),u_{2k}(t),u_{2k-1}(t),\ k=1,2,\ldots\}\in B.$$

Now we prove that the iterations $u^{(N+1)}(t)$ converge in B, as $N\to\infty$.

$$u^{(1)}(t)-u^{(0)}(t)=2(u_0^{(1)}(t)-u_0^{(0)}(t))+4\sum_{k=1}^{\infty}[(u_{2k}^{(1)}(t)-u_{2k}^{(0)}(t))+(u_{2k-1}^{(1)}(t)-u_{2k-1}^{(0)}(t))]$$

$$= 2\int_0^t\int_0^1 \left[f(\xi,\tau,Au^{(0)}(\xi,\tau)) - f(\xi,\tau,0)\right]\xi d\xi d\tau$$

$$+4\int_0^t\int_0^1 \left[f(\xi,\tau,Au^{(0)}(\xi,\tau)) - f(\xi,\tau,0)\right]e^{-(2\pi k)^2(t-\tau)}\sin 2\pi k\xi d\xi d\tau$$

$$+4\int_0^t\int_0^1 \left[f(\xi,\tau,Au^{(0)}(\xi,\tau)) - f(\xi,\tau,0)\right]e^{-(2\pi k)^2(t-\tau)}\xi\cos 2\pi k\xi d\xi d\tau$$

$$-16\pi k\int_0^t\int_0^1 (t-\tau)\left[f(\xi,\tau,Au^{(0)}(\xi,\tau)) - f(\xi,\tau,0)\right]e^{-(2\pi k)^2(t-\tau)}\sin 2\pi k\xi d\xi d\tau.$$

Applying Cauchy inequality, Hölder inequality, Lipschitz condition and Bessel inequality to the right side of $u^{(1)}(t) - u^{(0)}(t)$ respectively, we obtain:

$$\left|u^{(1)}(t) - u^{(0)}(t)\right| \le 2\left|u_0^{(1)}(t) - u_0^{(0)}(t)\right|$$

$$+4\sum_{k=1}^{\infty}\left(\left|u_{2k}^{(1)}(t) - u_{2k}^{(0)}(t)\right| + \left|u_{2k-1}^{(1)}(t) - u_{2k-1}^{(0)}(t)\right|\right)$$

$$\le \left(2\sqrt{T} + 4\sqrt{2}\,|T| + \frac{2\sqrt{3}}{3}\right)\left(\int_0^t\int_0^1 b^2(\xi,\tau)d\xi d\tau\right)^{\frac{1}{2}}\left|u^{(0)}(t)\right|$$

$$+\left(2\sqrt{T} + 4\sqrt{2}\,|T| + \frac{2\sqrt{3}}{3}\right)\left(\int_0^t\int_0^1 f^2(\xi,\tau,0)d\xi d\tau\right)^{\frac{1}{2}},$$

$$A_T = \left[\left(2\sqrt{T} + 4\sqrt{2}\,|T| + \frac{2\sqrt{3}}{3}\right)\left(\int_0^t\int_0^1 b^2(\xi,\tau)d\xi d\tau\right)^{\frac{1}{2}}\left|u^{(0)}(t)\right|\right.$$

$$\left.+\left(2\sqrt{T} + 4\sqrt{2}\,|T| + \frac{2\sqrt{3}}{3}\right)\left(\int_0^t\int_0^1 f^2(\xi,\tau,0)d\xi d\tau\right)^{\frac{1}{2}}\right].$$

Applying Cauchy unequality, Hölder inequality, Lipschitz condition and Bessel inequality to the right hand side of $u^{(2)}(t) - u^{(1)}(t)$ respectively, we obtain:

$$\left|u^{(2)}(t) - u^{(1)}(t)\right| \le 2\left|u_0^{(2)}(t) - u_0^{(1)}(t)\right|$$

$$+4\sum_{k=1}^{\infty}\left(\left|u_{2k}^{(2)}(t) - u_{2k}^{(1)}(t)\right| + \left|u_{2k-1}^{(2)}(t) - u_{2k-1}^{(1)}(t)\right|\right)$$

$$\le \left(2\sqrt{T} + 4\sqrt{2}\,|T| + \frac{2\sqrt{3}}{3}\right)\left(\int_0^t\int_0^1 b^2(\xi,\tau)d\xi d\tau\right)^{\frac{1}{2}} A_T.$$

In the same way, for a general value of N we have

$$\left|u^{(N+1)}(t)-u^{(N)}(t)\right| \leq 2\left|u_0^{(N+1)}(t)-u_0^{(N)}(t)\right|$$
$$+4\sum_{k=1}^{\infty}\left(\left|u_{2k}^{(N+1)}(t)-u_{2k}^{(N)}(t)\right|+\left|u_{2k-1}^{(N+1)}(t)-u_{2k-1}^{(N)}(t)\right|\right)$$
$$\leq (2\sqrt{T}+4\sqrt{2}\,|T|+\frac{2\sqrt{3}}{3})^N\frac{A_T}{\sqrt{N!}}\left[\left(\int_0^t\int_0^1 b^2(\xi,\tau)d\xi d\tau\right)^2\right]^{\frac{N}{2}}$$
$$\leq (2\sqrt{T}+4\sqrt{2}\,|T|+\frac{2\sqrt{3}}{3})^N A_T\frac{1}{\sqrt{N!}}\left\|b(x,t)\right\|_{L_2(D)}^N . \qquad (2.6)$$

Then the last inequality shows that the $u^{(N+1)}(t)$ convergence in B.

Now let us show $\lim\limits_{N\to\infty} u^{(N+1)}(t)=u(t)$. It follows that if we prove

$$\lim_{N\to\infty}\left\|u(\tau)-u^{(N)}(\tau)\right\|_B=0,$$

then we may deduce that $u(t)$ satisfies (2.4). For this aim we estimate the difference $\left\|u(t)-u^{(N+1)}(t)\right\|_B$, after some transformation we obtain:

$$\left|u(t)-u^{(N+1)}(t)\right| = 2\left|u_0(t)-u_0^{(N+1)}(t)\right|$$
$$+4\sum_{k=1}^{\infty}\left(\left|u_{2k}(t)-u_{2k}^{(N+1)}(t)\right|+\left|u_{2k-1}(t)-u_{2k-1}^{(N+1)}(t)\right|\right)$$
$$\leq 2\left|\int_0^t\int_0^1\left\{f[\xi,\tau,Au(\xi,\tau)]-f[\xi,\tau,Au^{(N)}(\xi,\tau)]\right\}\xi d\xi d\tau\right|$$
$$+4\left|\sum_{k=1}^{\infty}\int_0^t\int_0^1 e^{-(2\pi k)^2(t-\tau)}\left\{f[\xi,\tau,Au(\xi,\tau)]-f[\xi,\tau,Au^{(N)}(\xi,\tau)]\right\}\sin 2\pi k\xi d\xi d\tau\right|$$
$$+4\left|\sum_{k=1}^{\infty}\int_0^t\int_0^1 e^{-(2\pi k)^2(t-\tau)}\left\{f[\xi,\tau,Au(\xi,\tau)]-f[\xi,\tau,Au^{(N)}(\xi,\tau)]\right\}\xi\cos 2\pi k\xi d\xi d\tau\right|$$
$$+16\pi k\left|\int_0^t\int_0^1(t-\tau)\left[f(\xi,\tau,Au(\xi,\tau))-f(\xi,\tau,Au^{(N)}(\xi,\tau))\right]\right.$$
$$\left.\cdot e^{-(2\pi k)^2(t-\tau)}\sin 2\pi k\xi d\xi d\tau\right|.$$

Adding and subtracting $f(\xi,\tau,Au^{(N+1)}(\xi,\tau))$ under appropriate integrals to the right hand side of the inequality we obtain

$$\left|u(t)-u^{(N+1)}(t)\right|\leq\left(2\sqrt{T}+4\sqrt{2}|T|+\frac{2\sqrt{3}}{3}\right)\left\{\int_0^t\int_0^1 b^2(\xi,\tau)\left|u(\tau)-u^{(N+1)}(\tau)\right|^2 d\xi d\tau\right\}^{\frac{1}{2}}$$

$$+\left(2\sqrt{T}+4\sqrt{2}\,|T|+\frac{2\sqrt{3}}{3}\right)\left\{\int_0^t\int_0^1 b^2(\xi,\tau)d\xi d\tau\right\}^{\frac{1}{2}}\left\|u^{(N+1)}(t)-u^{(N)}(t)\right\|_B.$$

Applying Gronwall's inequality to the last inequality and using inequality (2.6), we have

$$\left\|u(t)-u^{(N+1)}(t)\right\|_B\leq\sqrt{\frac{2}{N!}}A_T\left(2\sqrt{T}+4\sqrt{2}\,|T|+\frac{2\sqrt{3}}{3}\right)^{(N+1)}\|b(x,t)\|_{L_2(D)}^{(N+1)}$$

$$\times\exp\left(2\sqrt{T}+4\sqrt{2}\,|T|+\frac{2\sqrt{3}}{3}\right)^2\|b(x,t))\|_{L_2(D)}^2. \tag{2.7}$$

For the uniqueness, we assume that the problem (1.1)-(1.4) has two solutions u,v. Applying Cauchy inequality, Hölder inequality, Lipschitz condition and Bessel inequality to the right hand side of $|u(t)-v(t)|$ respectively, we obtain:

$$|u(t)-v(t)|^2\leq\left(2\sqrt{T}+4\sqrt{2}\,|T|+\frac{2\sqrt{3}}{3}\right)^2\int_0^t\int_0^1 b^2(\xi,\tau)\,|u(\tau)-v(\tau)|^2\,d\xi d\tau,$$

applying Gronwall's inequality to the last inequality we have $u(t)=v(t)$.
The theorem is proved. □

3. Solution of the problem (1.1)-(1.4)

Using the solution of the system (2.4) we compose the series

$$2u_0(t)+4\sum_{k=1}^{\infty}[u_{2k}(t)(1-x)sin2\pi kx+u_{2k-1}(t)cos2\pi kx].$$

It is evident that these series convergence uniformly on D. Therefore the sum

$$u(\xi,\tau)=2u_0(\tau)+4\sum_{k=1}^{\infty}[u_{2k}(\tau)(1-\xi)sin2\pi k\xi+u_{2k-1}(\tau)cos2\pi k\xi],$$

continuous on D.

$$u_l(\xi,\tau)=2u_0(\tau)+4\sum_{k=1}^{l}[u_{2k}(\tau)(1-\xi)sin2\pi k\xi+u_{2k-1}(\tau)cos2\pi k\xi]. \tag{3.1}$$

From the conditions of Theorem 2.2 and from

$$\lim_{l\to\infty}u_l(\xi,\tau)=u(\xi,\tau),$$

it follows

$$\lim_{l\to\infty} f(\xi,\tau,u_l(\tau,\xi)) = f(\xi,\tau,u(\xi,\tau)).$$

Using $u_l(\xi,\tau)$ and

$$\varphi_l(x) = 2\varphi_0 + 4\sum_{k=1}^{l}[\varphi_{2k}(1-x)sin2\pi kx + \varphi_{2k-1}cos2\pi kx],\ x\in[0,1]$$

on the left hand side of (1.5) we denote the obtained expression by J_l:

$$J_l = \int_0^T\int_0^1 \left[\left(\frac{\partial v}{\partial t} + \frac{\partial^2 v}{\partial x^2}\right) u_{(l)}(x,t) + f(x,t,u_{(l)}(x,t))v(x,t)\right] dxdt$$
$$+ \int_0^1 \varphi_{(l)}(x)v(x,0)dx. \tag{3.2}$$

Applying the integration by part formula to the right hand side the last equation and using the conditions of Theorem 2.2 , we can show that

$$\lim_{l\to\infty} J_l = 0.$$

This shows that the function $u(x,t)$ is a generalized(weak) solution of the problem (1.1)-(1.4).
The following theorem shows the existence and uniqueness results for the generalized solutions of problem (1.1)-(1.4).

Theorem 3.1. *Under the assumptions of Theorem 2.2, Problem (1.1)-(1.4) possesses a unique generalized solution* $u = u(x,t) \in C(\overline{D})$ *in the following form*

$$u(x,t) = 2u_0(t) + 4\sum_{k=1}^{\infty}[u_{2k}(t)(1-x)sin2\pi kx + u_{2k-1}(t)cos2\pi kx].$$

4. Continuous dependence upon the data

In this section, we shall prove the continuous dependence of the solution

$$u = u(x,t)$$

using an iteration method.

Theorem 4.1. *Under the conditions of Theorem 2.2, the solution* $u = u(x,t)$ *depends contiunously upon the data.*

Proof. Let $\phi = \{\varphi, f\}$ and $\overline{\phi} = \{\overline{\varphi}, \overline{f}\}$ be two sets of data which satisfy the conditions of Theorem 1.Let $u = u(x,t)$ and $v = v(x,t)$ be the solutions of the problem (1.1)-(1.4) corresponding to the data ϕ and $\overline{\phi}$ respectively and

$$\left|f(t,x,0) - \overline{f}(t,x,0)\right| \le \varepsilon, \quad for\ \varepsilon \ge 0.$$

The solution $v = v(x,t)$ is in the following form

$$v_0(t) = \overline{\varphi}_0 + \int_0^t \overline{f}_0(\tau)d\tau,$$

$$v_{2k}(t) = \overline{\varphi}_{2k}\, e^{-(2\pi k)^2 t} + \int_0^t \overline{f}_{2k}(\tau)e^{-(2\pi k)^2(t-\tau)}d\tau,$$

$$v_{2k-1}(t) = (\overline{\varphi}_{2k-1} - 4\pi k t\overline{\varphi}_{2k})e^{-(2\pi k)^2 t}$$
$$+ \int_0^t e^{-(2\pi k)^2(t-\tau)}[\overline{f}_{2k-1}(\tau) - 4\pi k(t-\tau)\overline{f}_{2k}(\tau)]d\tau,$$

where, for simplicity, let

$$Av^{(N)}(\xi,\tau) = 2v_0^{(N)}(\tau) + 4\sum_{k=1}^{\infty}\left(v_{2k}^{(N)}(\tau)(1-\xi)\sin 2\pi k\xi + v_{2k-1}^{(N)}(\tau)\cos 2\pi k\xi\right)$$

$$v_0^{(N+1)}(t) = v_0^{(0)}(t) + \int_0^t\int_0^1 \overline{f}(\xi,\tau,Av^{(N)}(\xi,\tau))\xi d\xi d\tau,$$

$$v_{2k}^{(N+1)}(t) = v_{2k}^{(0)}(t) + \int_0^t\int_0^1 e^{-(2\pi k)^2(t-\tau)}\overline{f}(\xi,\tau,Av^{(N)}(\xi,\tau))\sin 2\pi k\xi d\xi d\tau,$$

$$v_{2k-1}^{(N+1)}(t) = v_{2k-1}^{(0)}(t) + \int_0^t\int_0^1 e^{-(2\pi k)^2(t-\tau)}\overline{f}(\xi,\tau,Av^{(N)}(\xi,\tau))\xi\cos 2\pi k\xi d\xi d\tau$$
$$-4\pi k\int_0^t\int_0^1 e^{-(2\pi k)^2(t-\tau)}\overline{f}(\xi,\tau,Av^{(N)}(\xi,\tau))(t-\tau)\sin 2\pi k\xi d\xi d\tau,$$

where

$$v_0^{(0)}(t) = \overline{\varphi}_0\,,\, v_{2k}^{(0)}(t) = \overline{\varphi}_{2k}\, e^{-(2\pi k)^2 t},\, u_{2k-1}^{(0)}(t) = (\overline{\varphi}_{2k-1} - 4\pi k\overline{\varphi}_{2k}\,)\, e^{-(2\pi k)^2 t}.$$

From the condition of the theorem we have $v^{(0)}(t) \in B$. We will prove that the other sequentially approximations satisfy this condition.

First of all , we consider $u^{(1)}(t) - v^{(1)}(t)$, applying Cauchy inequality, Hölder inequality, Lipschitz condition and Bessel inequality to the $\left|u^{(1)}(t) - v^{(1)}(t)\right|$ respectively, we obtain:

$$\left|u^{(1)}(t) - v^{(1)}(t)\right| \le 2\left|u_0^{(1)}(t) - v_0^{(1)}(t)\right|$$
$$+4\sum_{k=1}^{\infty}\left(\left|u_{2k}^{(1)}(t) - v_{2k}^{(1)}(t)\right| + \left|u_{2k-1}^{(1)}(t) - v_{2k-1}^{(1)}(t)\right|\right) \le 2\max\left|\varphi_0 - \overline{\varphi_0}\right|$$
$$+4\sum_{k=1}^{\infty}\max\left|\varphi_{2k} - \overline{\varphi_{2k}}\right| + \max\left|\varphi_{2k-1} - \overline{\varphi_{2k-1}}\right| + \frac{2\sqrt{6}\,|T|}{3}\sum_{k=1}^{\infty}\max\left|\varphi_{2k}'' - \overline{\varphi_{2k}''}\right|$$

$$\left(2\sqrt{T}+4\sqrt{2}\left|T\right|+\frac{2\sqrt{3}}{3}\right)\left(\int_0^t\int_0^1 b^2(\xi,\tau)d\xi d\tau\right)^{\frac{1}{2}}\left|\overline{u}^{(0)}(t)\right|$$

$$+\left(2\sqrt{T}+4\sqrt{2}\left|T\right|+\frac{2\sqrt{3}}{3}\right)\left(\int_0^t\int_0^1 \overline{b}^2(\xi,\tau)d\xi d\tau\right)^{\frac{1}{2}}\left|\overline{v}^{(0)}(t)\right|$$

$$+\left(2\sqrt{T}+4\sqrt{2}\left|T\right|+\frac{2\sqrt{3}}{3}\right)\left(\int_0^t\int_0^1 f^2(\xi,\tau,0)-\overline{f}^2(\xi,\tau,0)d\xi d\tau\right)^{\frac{1}{2}},$$

$$A_T=\|\varphi-\overline{\varphi}\|+\left[\left(2\sqrt{T}+4\sqrt{2}\left|T\right|+\frac{2\sqrt{3}}{3}\right)\|b(x,t)\|\left|u^{(0)}(t)\right|\right.$$

$$\left.+\left(2\sqrt{T}+4\sqrt{2}\left|T\right|+\frac{2\sqrt{3}}{3}\right)\|b(x,t)\|\left|\overline{v}^{(0)}(t)\right|\right]+\left(2\sqrt{T}+4\sqrt{2}\left|T\right|+\frac{2\sqrt{3}}{3}\right)\left\|f-\overline{f}\right\|.$$

$$\|\varphi-\overline{\varphi}\|=2\max\left|\varphi_0-\overline{\varphi_0}\right|$$

$$+4\sum_{k=1}^{\infty}\max\left|\varphi_{2k}-\overline{\varphi_{2k}}\right|+\max\left|\varphi_{2k-1}-\overline{\varphi_{2k-1}}\right|+\frac{2\sqrt{6}\left|T\right|}{3}\sum_{k=1}^{\infty}\max\left|\varphi_{2k}''-\overline{\varphi_{2k}''}\right|.$$

Applying Cauchy inequality, Hölder inequality, Lipschitz condition and Bessel inequality to the right hand side of $u^{(2)}(t)-v^{(2)}(t)$ respectively, we obtain:

$$\left|u^{(2)}(t)-v^{(2)}(t)\right|\leq 2\left|u_0^{(2)}(t)-v_0^{(2)}(t)\right|$$

$$+4\sum_{k=1}^{\infty}\left(\left|u_{2k}^{(2)}(t)-v_{2k}^{(2)}(t)\right|+\left|u_{2k-1}^{(2)}(t)-v_{2k-1}^{(2)}(t)\right|\right)$$

$$\leq\left(2\sqrt{T}+4\sqrt{2}\left|T\right|+\frac{2\sqrt{3}}{3}\right)\left(\int_0^t\int_0^1 b^2(\xi,\tau)d\xi d\tau\right)^{\frac{1}{2}}A_T$$

$$+\left(2\sqrt{T}+4\sqrt{2}\left|T\right|+\frac{2\sqrt{3}}{3}\right)\left(\int_0^t\int_0^1 \overline{b}^2(\xi,\tau)d\xi d\tau\right)^{\frac{1}{2}}A_T.$$

In the same way, for a general value of N we have

$$\left|u^{(N+1)}(t)-v^{(N+1)}(t)\right|\leq 2\left|u_0^{(N+1)}(t)-v_0^{(N+1)}(t)\right|$$

$$+4\sum_{k=1}^{\infty}\left(\left|u_{2k}^{(N+1)}(t)-v_{2k}^{(N+1)}(t)\right|+\left|u_{2k-1}^{(N+1)}(t)-v_{2k-1}^{(N+1)}(t)\right|\right)$$

$$\leq\left(2\sqrt{T}+4\sqrt{2}\left|T\right|+\frac{2\sqrt{3}}{3}\right)^N\frac{A_T}{\sqrt{N!}}\left[\left(\int_0^t\int_0^1 b^2(\xi,\tau)d\xi d\tau\right)^2\right]^{\frac{N}{2}}$$

$$+\left(2\sqrt{T}+4\sqrt{2}\,|T|+\frac{2\sqrt{3}}{3}\right)^{N}\frac{A_T}{\sqrt{N!}}\left[\left(\int_0^t\int_0^1\overline{b}^2(\xi,\tau)d\xi d\tau\right)^2\right]^{\frac{N}{2}}$$

$$\leq\left(2\sqrt{T}+4\sqrt{2}\,|T|+\frac{2\sqrt{3}}{3}\right)^{N}A_T\frac{1}{\sqrt{N!}}\left\|b(x,t)\right\|^N_{L_2(D)}$$

$$+\left(2\sqrt{T}+4\sqrt{2}\,|T|+\frac{2\sqrt{3}}{3}\right)^{N}A_T\frac{1}{\sqrt{N!}}\left\|\overline{b}(x,t)\right\|^N_{L_2(D)}$$

$$\leq A_T\cdot a_N=a_N\,\left(\|\varphi-\overline{\varphi}\|+C(t)+M_1\left\|f-\overline{f}\right\|\right)$$

where

$$a_N=\left(2\sqrt{T}+4\sqrt{2}\,|T|+\frac{2\sqrt{3}}{3}\right)^{N}\frac{1}{\sqrt{N!}}\left[\left(\int_0^t\int_0^1 b^2(\xi,\tau)d\xi d\tau\right)^2\right]^{\frac{N}{2}}$$

$$+\left(2\sqrt{T}+4\sqrt{2}\,|T|+\frac{2\sqrt{3}}{3}\right)^{N}\frac{1}{\sqrt{N!}}\left[\left(\int_0^t\int_0^1\overline{b}^2(\xi,\tau)d\xi d\tau\right)^2\right]^{\frac{N}{2}}.$$

and

$$M_1=(2\sqrt{T}+4\sqrt{2}\,|T|+\frac{2\sqrt{3}}{3})^N.$$

(The sequence a_N is convergent then we can write $a_N\leq M$, $\forall N$). It follows from the estimation ([2], page 76-77) that $\lim\limits_{N\to\infty}u^{(N+1)}(t)=u(t)$, then let $N\to\infty$ for last equation

$$|u(t)-v(t)|\leq M\,\|\varphi-\overline{\varphi}\|+M_2\left\|f-\overline{f}\right\|)$$

where $M_2=M.M_1$. If $\left\|f-\overline{f}\right\|\leq\varepsilon$ and $\|\varphi-\overline{\varphi}\|\leq\varepsilon$ then $|u(t)-v(t)|\leq\varepsilon$. □

5. Numerical procedure for the nonlinear problem (1.1)-(1.4)

We construct an iteration algorithm for the linearization of the problem (1.1)-(1.4):

$$\frac{\partial u^{(n)}}{\partial t}-\frac{\partial^2 u^{(n)}}{\partial x^2}=f(x,t,u^{(n-1)}),\ (x,t)\in D \tag{5.1}$$

$$u^{(n)}(0,t)=u^{(n)}(1,t),\qquad t\in[0,T] \tag{5.2}$$

$$u_x^{(n)}(1,t)=0,\qquad t\in[0,T] \tag{5.3}$$

$$u^{(n)}(x,0)=\varphi(x),\qquad x\in[0,1]. \tag{5.4}$$

Let $u^{(n)}(x,t) = v(x,t)$ and $f(x,t,u^{(n-1)}) = \widetilde{f}(x,t)$. Then the problem (5.1)-(5.4) can be written as a linear problem:

$$\begin{aligned}
\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} &= \widetilde{f}(x,t) \quad (x,t) \in D &(5.5)\\
v(0,t) &= v(1,t), \quad t \in [0,T] &(5.6)\\
v_x(1,t) &= 0, \quad t \in [0,T] &(5.7)\\
v(x,0) &= \varphi(x), \quad x \in [0,1]\ . &(5.8)
\end{aligned}$$

We use the finite difference method to solve (5.5)-(5.8).
We subdivide the intervals $[0,1]$ and $[0,T]$ into M and N subintervals of equal lengths $h = \frac{1}{M}$ and $\tau = \frac{T}{N}$, respectively. Then, we add a line $x = (M+1)\,h$ to generate the fictitious point needed for the second boundary condition.
We choose the implicit scheme, which is absolutely stable and has a second order accuracy in h and a first order accuracy in τ.
The implicit monotone difference scheme for (5.5)-(5.8) is as follows:

$$\frac{v_{i,j+1} - v_{i,j}}{\tau} = \frac{a^2}{h^2}(v_{i-1,j+1} - 2v_{i,j+1} + v_{i+1,j+1}) + \widetilde{f}_{i,j+1}$$
$$v_{i,0} = \varphi_i,\ v_{0,j} = v_{M,j},\ v_{x,M_j} = 0$$

where $0 \leq i \leq M$ and $1 \leq j \leq N$ are the indices for the spatial and time steps, respectively, $v_{i,j}$ is the approximation to $v(x_i,t_j)$, $f_{i,j} = f(x_i,t_j)$, $v_i = v(x_i)$, $x_i = ih$, $t_j = j\tau$. [12]

At the $t = 0$ level, adjustment should be made according to the initial condition and the compatibility requirements.

6. Numerical example

In this section, we will consider an example of numerical solution of the problem (1.1)-(1.4).

These problems were solved by applying the iteration scheme and the finite difference scheme which were explained in the Section 5. The condition

$$error(i,j) := \left\| u_{i,j}^{(n+1)} - u_{i,j}^{(n)} \right\|_\infty$$

with $error(i,j) := 10^{-3}$ was used as a stopping criteria for the iteration process.

Example 6.1. Consider the problem

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \left[1 - (2\pi)^2(\cos 2\pi x + (\sin 2\pi x)^2)\right] u$$
$$u(x,0) = \exp(-\cos 2\pi x),\ x \in [0,1]$$
$$u(0,t) = u(1,t), \quad t \in [0,T]\,, u_x(1,t) = 0, \quad t \in [0,T]\,.$$

It is easy to see that the analytical solution of this problem is

$$u(x,t) = \exp(t - \cos 2\pi x).$$

The comparisons between the analytical solution and the numerical finite difference solution f when $T = 1$ are shown in Figure 1 for the step sizes $h = 0.0025$, $\tau = 0.0025$.

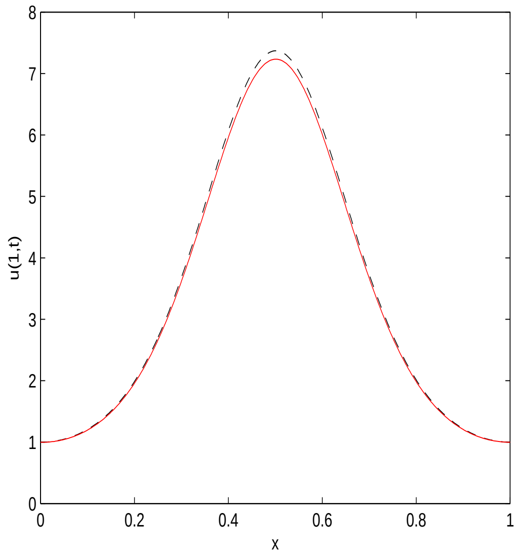

FIGURE 1. The exact and numerical solutions of $u(x,1)$, the exact solution is shown with dashes line.

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