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A new class of (j, k)-symmetric harmonic starlike functions

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Abstract. Using the concepts of (j, k)-symmetrical functions we define the class of sense-preserving harmonic univalent functions $\mathcal{SH}_s^{j,k}(\beta)$, and prove certain interesting results.

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1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain $\mathbb{C}$ if both u and v are real harmonic in $\mathbb{C}$. In any simply connected domain $\mathcal{D} \in \mathbb{C}$ we can write $f(z) = h + \overline{g}$, where h and g are analytic in $\mathcal{D}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathcal{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathcal{D}$, [see 3].

Denote by $\mathcal{SH}$ the class of functions $f(z) = h + \overline{g}$ that are harmonic univalent and orientation preserving in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \; and \; |z| < 1\}$, for which $f(0) = f_z(0) - 1 = 0$. Then for $f(z) = h + \overline{g} \in \mathcal{SH}$, we may express the analytic functions f and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \tag{1.1}$$

Note that $\mathcal{SH}$ reduces to the class $\mathcal{S}$ of normalized analytic univalent functions if the coanalytic part of its members is zero. For this class the function $f(z)$ may be expressed as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.2}$$

A function $f(z) = h + \overline{g}$ with h and g given by (1.1) is said to be harmonic starlike of order β for $0 \leq \beta < 1$, for $|z| = r < 1$ if

$$\frac{\partial}{\partial\theta}(\arg f(re^{i\theta})) = \Im\left\{\frac{\frac{\partial}{\partial\theta}f(re^{i\theta})}{f(re^{i\theta})}\right\} = \Re\left\{\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right\} \geq \beta.$$

The class of all harmonic starlike functions of order β is denoted by $\mathcal{S}^*_{\mathcal{H}}(\beta)$ and extensively studied by Jahangiri [4]. The cases $\beta = 0$ and $b_1 = 1$ were studied by Silverman and Silvia [8] and Silverman [7].

Definition 1.1. *Let k be a positive integer. A domain $\mathcal{D}$ is said to be k-fold symmetric if a rotation of $\mathcal{D}$ about the origin through an angle $\frac{2\pi}{k}$ carries $\mathcal{D}$ onto itself. A function f is said to be k-fold symmetric in $\mathcal{U}$ if for every z in $\mathcal{U}$*

$$f(e^{\frac{2\pi i}{k}}z) = e^{\frac{2\pi i}{k}}f(z).$$

The family of all k-fold symmetric functions is denoted by $\mathcal{S}^k$ and for $k = 2$ we get class of odd univalent functions.

The notion of (j,k)-symmetrical functions ($k = 2, 3, ...\ ; j = 0, 1, 2, ..., k-1$) is a generalization of the notion of even, odd, k-symmetrical functions and also generalize the well-known result that each function defined on a symmetrical subset can be uniquely expressed as the sum of an even function and an odd function.
The theory of (j,k) symmetrical functions has many interesting applications, for instance in the investigation of the set of fixed points of mappings, for the estimation of the absolute value of some integrals, and for obtaining some results of the type of Cartan's uniqueness theorem for holomorphic mappings [5].

Definition 1.2. *Let $\varepsilon = (e^{\frac{2\pi i}{k}})$ and $j = 0, 1, 2, .., k-1$ where $k \geq 2$ is a natural number. A function $f : \mathcal{D} \mapsto \mathbb{C}$ and $\mathcal{D}$ is a k-fold symmetric set, f is called (j,k)-symmetrical if*

$$f(\varepsilon z) = \varepsilon^j f(z),\ \ z \in \mathcal{U}.$$

We note that the family of all (j,k)-symmetric functions is denoted be $\mathcal{S}^{(j,k)}$. Also, $\mathcal{S}^{(0,2)}$, $\mathcal{S}^{(1,2)}$ and $\mathcal{S}^{(1,k)}$ are called even, odd and k-symmetric functions respectively. We have the following decomposition theorem.

Theorem 1.3. *[5] For every mapping $f : \mathcal{D} \mapsto \mathbb{C}$, and $\mathcal{D}$ is a k-fold symmetric set, there exists exactly the sequence of (j,k)- symmetrical functions $f_{j,k}$,*

$$f(z) = \sum_{j=0}^{k-1} f_{j,k}(z)$$

where

$$f_{j,k}(z) = \frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{-vj}f(\varepsilon^v z), \tag{1.3}$$

$$(f \in \mathcal{A};\ \ k = 1, 2, ...; j = 0, 1, 2, ..., k-1)$$

From (1.3) we can get

$$f_{j,k}(z)=\frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{-vj}f(\varepsilon^v z)=\frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{-vj}\left(\sum_{n=1}^{\infty}a_n(\varepsilon^v z)^n\right),$$

then

$$f_{j,k}(z)=\sum_{n=1}^{\infty}\delta_{n,j}a_nz^n,\ a_1=1,\quad \delta_{n,j}=\frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{(n-j)v}=\begin{cases}1, & n=lk+j;\\ 0, & n\neq lk+j;\end{cases}, \tag{1.4}$$

Ahuja and Jahangiri [2] discussed the class $\mathcal{SH}(\beta)$ which denotes the class of complex-valued, sense-preserving, harmonic univalent functions f of the form (1.1) and satisfying

$$\Im\left\{\frac{2\frac{\partial}{\partial\theta}f(re^{i\theta})}{f(re^{i\theta})-f(-re^{i\theta})}\right\}\geq\beta,\quad 0\leq\beta<1.$$

Al-Shaqsi and and Maslina Darus [1] discussed the class $\mathcal{SH}_s^k(\beta)$ which denotes the class of complex-valued, sense-preserving, harmonic univalent functions f of the form (1.1) and satisfying

$$\Im\left\{\frac{\frac{\partial}{\partial\theta}f(re^{i\theta})}{f_k(re^{i\theta})}\right\}\geq\beta,\quad 0\leq\beta<1.$$

Now using the concepts of $(j,k)-$symmteric points we define the following

Definition 1.4. *For $0\leq\beta<1$ and $k=1,2,3,...,\ j=0,1,...,k-1$, let $\mathcal{SH}_s^{j,k}(\beta)$ denote the class of sense-preserving, harmonic univalent functions f of the form* (1.1) *which satisfy the condition*

$$\Im\left\{\frac{\frac{\partial}{\partial\theta}f(re^{i\theta})}{f_{j,k}(re^{i\theta})}\right\}\geq\beta. \tag{1.5}$$

Where $z=re^{i\theta},\ 0\leq r<1,\ 0\leq\theta<2\pi$ and $f_{j,k}=h_{j,k}+\overline{g_{j,k}}$, where $h_{j,k},\ g_{j,k}$ given by

$$h_{j,k}(z)=\frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{-vj}h(\varepsilon^v z),\quad g_{j,k}(z)=\frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{-vj}g(\varepsilon^v z). \tag{1.6}$$

The following special cases are of interest
(i) $\mathcal{SH}_s^{1,k}(\beta)=\mathcal{SH}_s^k(\beta)$ the class introduced by Al-Shaqsi and Darus [1].
(ii) $\mathcal{SH}_s^{1,2}(\beta)=\mathcal{SH}(\beta)$ the class introduced by Ahuja and Jahangiri [2].
(iii) $\mathcal{SH}_s^{1,1}(\beta)=\mathcal{S_H}^*(\beta)$ the class introduced by Jahangiri [4].
(iv) $\mathcal{SH}_s^{1,1}(0)=\mathcal{S_H}^*$ the class introduced by Silverman and Silvia [8].
We need the following lemma to prove our main results.

Lemma 1.5. [4] *Let $f=f+\overline{g}$ with h and g are given by* (1.1). *If*

$$\sum_{n=1}^{\infty}\left\{\frac{n-\beta}{1-\beta}|a_n|+\frac{n+\beta}{1-\beta}|b_n|\right\}\leq 2,\ a_1=1,\ 0\leq\beta<1.$$

Then f is sense-preserving, harmonic univalent and starlike of order β in $\mathcal{U}$.

2. Main result

Theorem 2.1. *Let* $f \in \mathcal{SH}_s^{(j,k)}(\beta)$ *where* f *given by* (1.1)*, then* $f_{j,k}(z)$ *is in* $\mathcal{S_H}^*(\beta)$*, where* $f_{j,k}$ *given by* (1.6)*.*

Proof. Suppose that $f \in \mathcal{SH}_s^{(j,k)}(\beta)$. Then we get

$$\Im\left\{\frac{\frac{\partial}{\partial\theta} f(re^{i\theta})}{f_{j,k}(re^{i\theta})}\right\} \geq \beta. \tag{2.1}$$

replacing $re^{i\theta}$ by $\varepsilon^v re^{i\theta}$ in (2.1), we get

$$\Im\left\{\frac{\frac{\partial}{\partial\theta} f(\varepsilon^v re^{i\theta})}{f_{j,k}(\varepsilon^v re^{i\theta})}\right\} \geq \beta.$$

According to the definition of $f_{j,k}$ and $\varepsilon^k = 1$, we know $f_{j,k}(\varepsilon^v re^{i\theta}) = \varepsilon^{vj} f_{j,k}(re^{i\theta})$, we get

$$\Im\left\{\frac{\varepsilon^{-vj}\frac{\partial}{\partial\theta} f(\varepsilon^v re^{i\theta})}{f_{j,k}(re^{i\theta})}\right\} \geq \beta, \tag{2.2}$$

letting $(v = 0, 1, 2, ...k-1)$ in (2.2) and summing them we can get

$$\Im\left\{\frac{\frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{-vj}\frac{\partial}{\partial\theta} f(\varepsilon^v re^{i\theta})}{f_{j,k}(re^{i\theta})}\right\} = \Im\left\{\frac{\frac{\partial}{\partial\theta} f_{j,k}(re^{i\theta})}{f_{j,k}(re^{i\theta})}\right\} > \beta, \tag{2.3}$$

that is $f_{j,k}(z) \in \mathcal{S_H}^*(\beta)$. □

Theorem 2.2. *If* $f = h + \overline{g}$ *with* h *and* g *given by* (1.1) *and* $f_{j,k} = h_{j,k} + \overline{g_{j,k}}$ *with* $h_{j,k}$ *and* $g_{j,k}$ *given by* (1.6)*. Let*

$$\sum_{n=1}^{\infty}\left\{\frac{(n-1)k+j-\beta}{1-\delta_{1,j}\beta}|a_{(n-1)k+j}| + \frac{(n-1)k+j+\beta}{1-\delta_{1,j}\beta}|b_{(n-1)k+j}|\right\} \tag{2.4}$$

$$+ \sum_{n\neq lk+j}^{\infty}\frac{n}{1-\delta_{1,j}\beta}\{|a_n| + |b_n|\} \leq 2,$$

where $a_1 = 1$, $0 \leq \beta < 1$, $k = 1, 2, 3, ...$, $j = 0, 1, ..., k-1$, $l \in \mathbb{N}$ *and* $\delta_{1,j}$ *is defined by* (1.4)*. Then* f *is sense-preserving harmonic univalent in* $\mathcal{U}$ *and* $f \in \mathcal{SH}_s^{(j,k)}(\beta)$*.*

Proof. Since

$$\sum_{n=1}^{\infty}\left[\frac{n-\beta}{1-\beta}|a_n| + \frac{n+\beta}{1-\beta}|b_n|\right] \leq \sum_{n=1}^{\infty}\left\{\frac{n-\delta_{n,j}\beta}{1-\delta_{1,j}\beta}|a_n| + \frac{n+\delta_{n,j}\beta}{1-\delta_{1,j}\beta}|b_n|\right\},$$

where $\delta_{n,j}$ is given by (1.4),

$$= \sum_{n=1}^{\infty}\left\{\frac{(n-1)k+j-\beta}{1-\delta_{1,j}\beta}|a_{(n-1)k+j}| + \frac{(n-1)k+j+\beta}{1-\delta_{1,j}\beta}|b_{(n-1)k+j}|\right\}$$

$$+ \sum_{n\neq lk+j}^{\infty}\frac{n}{1-\delta_{1,j}\beta}\{|a_n| + |b_n|\} \leq 2.$$

By Lemma 1.5, we conclude that f is sense-preserving, harmonic univalent and starlike in $\mathcal{U}$. To prove $f \in \mathcal{SH}_s^{(j,k)}(\beta)$, according to condition (1.5), we need to show that

$$\Im\left\{\frac{\frac{\partial}{\partial\theta}f(re^{i\theta})}{f_k(re^{i\theta})}\right\} = \Re\left\{\frac{zh'(z)-\overline{zg'(z)}}{h_{j,k}(z)+g_{j,k}(z)}\right\} = \Re\left\{\frac{A(z)}{B(z)}\right\} \geq \beta.$$

Where $z = re^{i\theta}$, $0 \leq r < 1$, $0 \leq \theta < 2\pi$, $0 \leq \beta < 1$ and $k = 1,2,3,...$, $j = 0,1,...,k-1$.

$$A(z) = zh'(z) - \overline{zg'(z)} = z + \sum_{n=2}^{\infty} na_n z^n - \overline{\sum_{n=1}^{\infty} nb_n z^n} \tag{2.5}$$

and

$$B(z) = f_{j,k}(z) = \sum_{n=1}^{\infty} a_n \delta_{n,j} z^n + \overline{\sum_{n=1}^{\infty} \delta_{n,j} b_n z^n}, \tag{2.6}$$

where $\delta_{n,j}$ is defined by (1.4), and $\varepsilon^k = 1$.
Using the fact that $\Re\{w\} \geq \beta$ if and only if $|1-\beta+w| \geq |1+\beta-w|$, it suffices to show that

$$|A(z) + (1-\beta)B(z)| - |A(z) - (1+\beta)B(z)| \geq 0.$$

For $A(z)$ and $B(z)$ as given in (2.5) and (2.6) respectively, we get

$$|A(z) - (1-\beta)B(z)| - |A(z) - (1-\beta)B(z)|$$

$$= |(1-\beta)h_{j,k} + zh'(z) + \overline{(1-\beta)g_{j,k} - zg'(z)}| - |(1+\beta)h_{j,k} - zh'(z) + \overline{(1+\beta)g_{j,k} + zg'(z)}|$$

$$= \left|[1+(1-\beta)\delta_{1,j}]z + \sum_{n=2}^{\infty}[n+(1-\beta)\delta_{n,j}]a_n z^n - \overline{\sum_{n=1}^{\infty}[n-(1-\beta)\delta_{n,j}]b_n z^n}\right|$$

$$- \left|[1-(1+\beta)\delta_{1,j}]z + \sum_{n=2}^{\infty}[n-(1+\beta)\delta_{n,j}]a_n z^n - \overline{\sum_{n=1}^{\infty}[n+(1+\beta)\delta_{n,j}]b_n z^n}\right|$$

$$\geq [1+(1-\beta)\delta_{1,j}]|z| - \sum_{n=2}^{\infty}[n+(1-\beta)\delta_{n,j}]|a_n||z|^n - \sum_{n=1}^{\infty}[n-(1-\beta)\delta_{n,j}]|b_n||z|^n$$

$$-[1-(1+\beta)\delta_{1,j}]|z| - \sum_{n=2}^{\infty}[n-(1+\beta)\delta_{n,j}]|a_n||z|^n - \sum_{n=1}^{\infty}[n+(1+\beta)\delta_{n,j}]|b_n||z|^n$$

$$= 2(1-\beta\delta_{1,j})|z|\left\{1 - \sum_{n=2}^{\infty}\frac{n-\beta\delta_{n,j}}{1-\beta\delta_{1,j}}|a_n||z|^{n-1} - \sum_{n=1}^{\infty}\frac{n+\beta\delta_{n,j}}{1-\beta\delta_{1,j}}|b_n||z|^{n-1}\right\}$$

$$\geq 2(1-\delta_{1,j})|z|\left\{1 - \sum_{n=2}^{\infty}\frac{n-\beta\delta_{n,j}}{1-\beta\delta_{1,j}}|a_n| - \sum_{n=1}^{\infty}\frac{n+\beta\delta_{n,j}}{1-\beta\delta_{1,j}}|b_n|\right\}.$$

From the definition of $\delta_{n,j}$ in (1.4), we have

$$|A(z) + (1-\beta)B(z)| - |A(z) - (1+\beta)B(z)|$$

$$\geq 2(1-\beta\delta_{1,j})|z|\left\{1 - \sum_{n=1}^{\infty}\frac{nk+j-\beta}{1-\beta\delta_{1,j}}|a_{nk+j}| - \sum_{n=1}^{\infty}\frac{nk+j+\beta}{1-\delta_{1,j}\beta}|b_{nk+j}|\right\}$$

$$-(1-\beta\delta_{1,j})|z|\left\{\sum_{n\neq lk+j}^{\infty}\frac{n}{1-\delta_{1,j}\beta}|a_n|+\sum_{n\neq lk+j}^{\infty}\frac{n}{1-\delta_{1,j}\beta}|b_n|+\frac{1+\beta}{1-\delta_{1,j}\beta}|b_1|\right\}$$

$$\geq 2(1-\beta\delta_{1,j})|z|\left\{1-\sum_{n=1}^{\infty}\left[\frac{(n-1)k+j-\beta}{1-\delta_{1,j}\beta}|a_{(n-1)k+j}|-\frac{(n-1)k+j+\beta}{1-\delta_{1,j}\beta}|b_{(n-1)k+j}|\right]\right\}$$

$$-(1-\beta\delta_{1,j})|z|\left\{\sum_{n\neq lk+j}^{\infty}\frac{n}{1-\delta_{1,j}\beta}\left[|a_n|+|b_n|\right]\right\}\geq 0,$$

we note that in (2.4). This concludes the proof of the theorem. □

For $j=1$ we get the result introduced by Al-Shaqsi and Darus in [1].

Corollary 2.3. *If $f=h+\overline{g}$ with h and g given by* (1.1) *and $f_k=h_k+\overline{g_k}$. Let*

$$\sum_{n=1}^{\infty}\left\{\frac{(n-1)k+1-\beta}{1-\beta}|a_{(n-1)k+1}|+\frac{(n-1)k+1+\beta}{1-\beta}|b_{(n-1)k+1}|\right\} \tag{2.7}$$

$$+\sum_{n\neq lk+1}^{\infty}\frac{n}{1-\beta}\left\{|a_n|+|b_n|\right\}\leq 2,$$

where $a_1=1,\ 0\leq\beta<1,\ k=1,2,3,...,\ l\in\mathbb{N}$. Then f is sense-preserving harmonic univalent in $\mathcal{U}$ and $f\in\mathcal{SH}_s^{(k)}(\beta)$.

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