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Some new subclasses of bi-univalent functions

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Abstract. The purpose of the present paper is to obtain the initial coefficients for normalized analytic functions f in the open unit disk U with its inverse $g = f^{-1}$ belonging to the classes $H_\sigma^n(\phi)$, $ST_\sigma^n(\alpha, \phi)$, $M_\sigma^n(\alpha, \phi)$ and $L_\sigma^n(\alpha, \phi)$. Relevant connections of the results presented here with various known results are briefly indicated. Finally, we give an open problem for the readers.

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1. Introduction

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and satisfy the normalization condition $f(0) = f'(0) - 1 = 0$. Let S be the subclass of A consisting of functions of the form (1.1) which are also univalent in U. The Koebe one-quarter theorem [4] ensures that the image of U under every univalent function $f \in A$ contains a disk of radius $\frac{1}{4}$. Thus every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z$, $(z \in U)$ and $f(f^{-1}(w)) = w$, $(|w| < r_0(f), r_0(f) \geq 1/4)$. A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let σ denote the class of bi-univalent functions defined in the unit disk U.

A domain $U \subset C$ is convex if the line segment joining any two points in U lies entirely in U, while a domain is starlike with respect to a point $w_0 \in U$ if the line segment joining any point of U to w_0 lies inside U. A function $f \in A$ is starlike if $f(U)$ is a starlike domain with respect to origin, and convex if $f(U)$ is convex. Analytically $f \in A$ is starlike if and only if $\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0$, whereas $f \in A$ is convex if and only if $\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0$. The classes consisting of starlike and convex functions are denoted by ST and CV respectively. The classes $ST(\alpha)$ and $CV(\alpha)$ of starlike and convex functions of order α , $0 \leq \alpha < 1$, are respectively characterized by $\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha$ and $\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha$. Ma and Minda [8] unified various subclasses of starlike and convex functions by using subordination. Now we recall the definition of subordination

An analytic function f is subordinate to an analytic function g, written $f(z) \prec g(z)$, provided there is an analytic function w defined on U with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$.

Lewin [7] investigated the class σ of bi-univalent functions and obtained the bound for the second coefficient. Several researchers have subsequently studied similar problems in this direction (see [2], [5], [6], [10], [12], [13]). Brannan and Taha [2] considered certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. They introduced bi-starlike functions and bi-convex functions and obtained estimates on the initial coefficients. Recently, Srivastava *et al.* [12] introduced and investigated subclasses of the bi-univalent functions and obtained bounds for the initial coefficients. The results of [12] were generalized in [5], [6], [10] and [13].

Very recently Ali *et al.* [1] estimates on the initial coefficients for bi-starlike of Ma-Minda type and bi-convex of Ma-Minda type functions are obtained. In this paper, we generalized these results by using Salagean operator and obtain sharp estimates on coefficient for function classes $H_\sigma^n(\phi)$, $ST_\sigma^n(\alpha,\phi)$, $M_\sigma^n(\alpha,\phi)$ and $L_\sigma^n(\alpha,\phi)$.

2. Coefficient estimates

In the sequel, it is assumed that ϕ is an analytic function with positive real part in the disk U, satisfying $\phi(0) = 1$, $\phi'(0) > 0$ and $\phi(U)$ is symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots,\ (B_1 > 0)\,. \tag{2.1}$$

A function $f \in \sigma$ is said to be in the class $H_\sigma^n(\phi)$ if the following subordination hold:

$$\frac{D^n f(z)}{z} \prec \phi(z)$$

and

$$\frac{D^n g(w)}{w} \prec \phi(w)\,,\ g(w) = f^{-1}(w)\,,$$

where D^n stands for the Salagean operator introduced by Salagean [11] for function f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

analytic in the open unit disk U as follows

$$D^0 f(z) = f(z)$$
$$D^1 f(z) = zf'(z)$$
$$\cdots\cdots$$
$$D^n f(z) = D(D^{n-1} f(z))$$

Thus

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k.$$

For functions in the class $H_\sigma^n(\phi)$, we obtain the following result.

Theorem 2.1. *If $f \in H_\sigma^n(\phi)$ is given by*

$$f\left(z\right) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{2.2}$$

then

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{|3^n B_1^2 - 2^{2n} B_2 + 2^{2n} B_1|}} \tag{2.3}$$

and

$$|a_3| \leq \left(\frac{1}{3^n} + \frac{B_1}{2^{2n}}\right) B_1. \tag{2.4}$$

Proof. Let $f \in H_\sigma^n(\phi)$ and $g = f^{-1}$. Then there are analytic functions $u, v : U \to U$, with $u(0) = v(0) = 0$, satisfying

$$\frac{D^n f\left(z\right)}{z} = \phi\left(u\left(z\right)\right) \text{and } \frac{D^n g\left(w\right)}{w} = \phi\left(v\left(w\right)\right). \tag{2.5}$$

Define the functions $p_1(z)$ and $p_2(z)$ by

$$p_1\left(z\right) = \frac{1 + u\left(z\right)}{1 - u\left(z\right)} = 1 + c_1 z + c_2 z^2 + \ldots$$

and

$$p_2\left(z\right) = \frac{1 + v\left(z\right)}{1 - v\left(z\right)} = 1 + b_1 z + b_2 z^2 + \ldots$$

or, equivalently,

$$u\left(z\right) = \frac{p_1\left(z\right) - 1}{p_1\left(z\right) + 1} = \frac{1}{2}\left(c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + \ldots\right) \tag{2.6}$$

and

$$v\left(z\right) = \frac{p_2\left(z\right) - 1}{p_2\left(z\right) + 1} = \frac{1}{2}\left(b_1 z + \left(b_2 - \frac{b_1^2}{2}\right) z^2 + \ldots\right). \tag{2.7}$$

Then $p_1(z)$ and $p_2(z)$ are analytic in U with $p_1(0) = 1 = p_2(0)$. Since $u, v : U \to U$, the functions $p_1(z)$ and $p_2(z)$ have a positive real part in U, and $|b_i| \leq 2$ and $|c_i| \leq 2$. In view of (2.5)-(2.7), clearly

$$\frac{D^n f\left(z\right)}{z} = \phi\left(\frac{p_1\left(z\right) - 1}{p_1\left(z\right) + 1}\right) \tag{2.8}$$

and

$$\frac{D^n g(w)}{w} = \phi\left(\frac{p_2(w)-1}{p_2(w)+1}\right). \tag{2.9}$$

Using (2.5) and (2.7) together with (2.1), it is evident that

$$\phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = 1 + \frac{1}{2}B_1c_1z + \left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2\right)z^2 + \ldots \tag{2.10}$$

and

$$\phi\left(\frac{p_2(w)-1}{p_2(w)+1}\right) = 1 + \frac{1}{2}B_1b_1w + \left(\frac{1}{2}B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}B_2b_1^2\right)w^2 + \ldots \tag{2.11}$$

Since $f \in \sigma$ has the Maclaurin series given by (2.2), a computation shows that its inverse $g = f^{-1}$ has the expansion

$$g(w) = f^{-1}(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right)w^3 + \ldots$$

Since

$$\frac{D^n f(z)}{z} = 1 + 2^n a_2 z + 3^n a_3 z^2 + \ldots$$

and

$$\frac{D^n g(w)}{w} = 1 - 2^n a_2 w + \left(2a_2^2 - a_3\right)3^n w^2 + \ldots,$$

it follows from (2.8)-(2.11) that

$$2^n a_2 = \frac{1}{2}B_1c_1, \tag{2.12}$$

$$3^n a_3 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2 \tag{2.13}$$

$$-2^n a_2 = \frac{1}{2}B_1b_1 \tag{2.14}$$

and

$$3^n\left(2a_2^2 - a_3\right) = \frac{1}{2}B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}B_2b_1^2. \tag{2.15}$$

From (2.12) and (2.14), it follows that

$$c_1 = -b_1. \tag{2.16}$$

Now (2.13)-(2.16) yield

$$a_2^2 = \frac{B_1^3(b_2 + c_2)}{4\left(3^n B_1^2 - 2^{2n}B_2 + 2^{2n}B_1\right)}$$

which, in view of the well-known inequalities $|b_2| \leq 2$ and $|c_2| \leq 2$ for functions with positive real part, gives us the desired estimate on $|a_2|$ as asserted in (2.3). By subtracting (2.15) from (2.13), further computations using (2.12) and (2.16) lead to

$$a_3 = \frac{B_1(c_2 - b_2)}{4.3^n} + \frac{B_1^2c_1^2}{4.2^{2n}},$$

and this yields the estimates given in (2.4). □

Remark 2.2. If we put $n = 1$ in Theorem 2.1, then we obtain the corresponding result of Ali *et al.* [1].

Remark 2.3. If we put $n = 1$ with $\phi(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$ in Theorem 2.1, then we obtain the corresponding result of Srivastava *et al.* [12].

Remark 2.4. If we put $n = 1$ with $\phi(z) = \frac{1+(1-2\gamma)z}{1-z}$ in Theorem 2.1, then we obtain the corresponding result of Srivastava *et al.* [12].

A function $f \in \sigma$ is said to be in the class $ST_{\sigma}^{n}(\alpha,\phi)$, $n \in N_0, \alpha \geq 0$, if the following subordinations hold:

$$\frac{(1-\alpha)\,D^{n+1}f(z)+\alpha D^{n+2}f(z)}{D^{n}f(z)} \prec \phi(z)$$

and

$$\frac{(1-\alpha)\,D^{n+1}g(w)+\alpha D^{n+2}g(w)}{D^{n}g(w)} \prec \phi(w)\,;\;\; g(w)=f^{-1}(w).$$

Note that $ST_{\sigma}^{n}(\phi) \equiv ST_{\sigma}^{n}(0,\phi)$. For the functions in the class $ST_{\sigma}^{n}(\alpha,\phi)$, the following coefficient estimates are obtained.

Theorem 2.5. *Let f given by (2.2) be in the class $ST_{\sigma}^{n}(\alpha,\phi)$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{\left|B_1^2\left(3^n 2\,(1+3\alpha)-2^{2n}\,(1+2\alpha)\right)+(B_1-B_2)\,2^{2n}(1+2\alpha)^2\right|}}$$

and

$$|a_3| \leq \frac{B_1+|B_2-B_1|}{3^n\cdot 2\,(1+3\alpha)-2^{2n}\,(1+2\alpha)}.$$

Proof. Let $f \in ST_{\sigma}^{n}(\alpha,\phi)$. Then there are analytic functions $u, v : U \to U$, with $u(0) = v(0) = 0$, satisfying

$$\frac{(1-\alpha)\,D^{n+1}f(z)+\alpha D^{n+2}f(z)}{D^{n}f(z)} = \phi(u(z)) \tag{2.17}$$

and

$$\frac{(1-\alpha)\,D^{n+1}g(w)+\alpha D^{n+2}g(w)}{D^{n}g(w)} = \phi(v(w))\,,\;\; \left(g=f^{-1}\right). \tag{2.18}$$

Since

$$\frac{(1-\alpha)\,D^{n+1}f(z)+\alpha D^{n+2}f(z)}{D^{n}f(z)} = 1+(1+2\alpha)\,2^{n}a_2 z$$
$$+\left(3^n\cdot 2\,(1+3\alpha)\,a_3-2^{2n}\,(1+2\alpha)\,a_2^2\right)z^2+\dots$$

and

$$\frac{(1-\alpha)\,D^{n+1}g(w)+\alpha D^{n+2}f(w)}{D^{n}g(w)} = 1-(1+2\alpha)\,2^{n}a_2 w$$
$$+\left(\left(3^n\cdot 4\,(1+3\alpha)-2^{2n}\,(1+2\alpha)\right)a_2^2-3^n\cdot 2\,(1+3\alpha)\,a_3\right)w^2+\dots,$$

then (2.10),(2.11), (2.17) and (2.18) yield

$$2^n(1+2\alpha)a_2 = \frac{1}{2}B_1c_1, \tag{2.19}$$

$$3^n2(1+3\alpha)a_3 - 2^{2n}(1+2\alpha)a_2^2 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2, \tag{2.20}$$

$$-2^n(1+2\alpha)a_2 = \frac{1}{2}B_1b_1, \tag{2.21}$$

and

$$\left(3^n \cdot 4\left(1+3\alpha\right) - 2^{2n}\left(1+2\alpha\right)\right)a_2^2 - 3^n \cdot 2\left(1+3\alpha\right)a_3 = \frac{1}{2}B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}B_2b_1^2. \tag{2.22}$$

Now, the required result follows by using the techniques as used in Theorem 2.1. □

Remark 2.6. If we put $n = 0$ in Theorem 2.5, then we obtain the corresponding result of Ali *et al.* [1].
Next, if we put $n = 0$, $\phi(z) = \left(\frac{1+z}{1-z}\right)^\gamma$ with $\alpha = 0$, then we obtain corresponding result of Brannan and Taha [2].

Next, a function $f \in \sigma$ belongs to the class $M_\sigma^n(\alpha, \phi)$, $n \in N_0, \alpha \geq 0$, if the following subordinations hold:

$$(1-\alpha)\frac{D^{n+1}f(z)}{D^nf(z)} + \alpha\frac{D^{n+2}f(z)}{D^{n+1}f(z)} \prec \phi(z)$$

and

$$(1-\alpha)\frac{D^{n+1}g(w)}{D^ng(w)} + \alpha\frac{D^{n+2}g(w)}{D^{n+1}g(w)} \prec \phi(w),\ g(w) = f^{-1}(w).$$

For function in the class $M_\sigma^n(\alpha, \phi)$, the following coefficient estimates hold.

Theorem 2.7. *Let f given by (2.2) be in the class $M_\sigma^n(\alpha, \phi)$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{B_1^2\left(2\cdot 3^n\left(1+2\alpha\right) - 2^{2n}\left(1+3\alpha\right) + 2^{2n}(1+\alpha)^2\left(B_1 - B_2\right)\right)}} \tag{2.23}$$

and

$$|a_3| \leq \frac{B_1 + |B_2 - B_1|}{2\left(1+2\alpha\right)3^n - \left(1+3\alpha\right)2^{2n}}. \tag{2.24}$$

Proof. If $f \in M_\sigma^n(\alpha, \phi)$, then there are analytic functions $u, v : U \to U$, with $u(0) = v(0) = 0$, such that

$$(1-\alpha)\frac{D^{n+1}f(z)}{D^nf(z)} + \alpha\frac{D^{n+2}f(z)}{D^{n+1}f(z)} = \phi(u(z)) \tag{2.25}$$

and

$$(1-\alpha)\frac{D^{n+1}g(w)}{D^ng(w)} + \alpha\frac{D^{n+2}g(w)}{D^{n+1}g(w)} = \phi(v(w)). \tag{2.26}$$

Since

$$(1-\alpha)\frac{D^{n+1}f(z)}{D^{n}f(z)}+\alpha\frac{D^{n+2}f(z)}{D^{n+1}f(z)}=1+(1+\alpha)\,2^{n}a_{2}z$$
$$+\left(2(1+2\alpha)\,3^{n}a_{3}-(1+3\alpha)\,2^{2n}a_{2}^{2}\right)z^{2}+\ldots$$

and

$$(1-\alpha)\frac{D^{n+1}g(w)}{D^{n}g(w)}+\alpha\frac{D^{n+2}g(w)}{D^{n+1}g(w)}=1-(1+\alpha)\,2^{n}a_{2}w$$
$$+\left(\left(4(1+2\alpha)\,3^{n}-(1+3\alpha)\,2^{2n}\right)a_{2}^{2}-2(1+2\alpha)\,3^{n}a_{3}\right)w^{2}+\ldots.$$

From (2.10), (2.11), (2.25) and (2.26) it follows that

$$(1+\alpha)\,2^{n}a_{2}=\frac{1}{2}B_{1}c_{1}\tag{2.27}$$

$$2(1+2\alpha)\,3^{n}a_{3}-(1+3\alpha)\,2^{2n}a_{2}^{2}=\frac{1}{2}B_{1}\left(c_{2}-\frac{c_{1}^{2}}{2}\right)+\frac{1}{4}B_{2}c_{1}^{2}\tag{2.28}$$

$$-(1+\alpha)\,2^{n}a_{2}=\frac{1}{2}B_{1}b_{1}\tag{2.29}$$

and

$$\left(4(1+2\alpha)\,3^{n}-(1+3\alpha)\,2^{2n}\right)a_{2}^{2}-2(1+2\alpha)\,3^{n}a_{3}=\frac{1}{2}B_{1}\left(b_{2}-\frac{b_{1}^{2}}{2}\right)+\frac{1}{4}B_{2}b_{1}^{2}.\tag{2.30}$$

Equation (2.27) and (2.29) yield

$$c_{1}=-b_{1}.\tag{2.31}$$

From (2.28), (2.30) and (2.31), it follows that

$$a_{2}^{2}=\frac{B_{1}^{3}(b_{2}+c_{2})}{4\left(B_{1}^{2}\left(2\cdot 3^{n}(1+2\alpha)-2^{2n}(1+3\alpha)\right)+2^{2n}(1+\alpha)^{2}(B_{1}-B_{2})\right)}$$

which yields the describe estimate on as describe in (2.23). As in the earlier proofs, use of (2.28)-(2.31) shows that

$$a_{3}=\frac{(B_{1}/2)\left(\left(4(1+2\alpha)\,3^{n}-(1+3\alpha)\,2^{n}\right)c_{2}+(1+3\alpha)\,2^{n}b_{2}\right)+b_{1}^{2}(1+2\alpha)(B_{2}-B_{1})}{4\cdot 3^{n}(1+2\alpha)\left(2(1+2\alpha)\,3^{n}-(1+3\alpha)\,2^{2n}\right)}.$$

Thus the proof of Theorem 2.7 is complete. □

Next, a function $f\in\sigma$ is said to be in the class $L_{\sigma}^{n}(\alpha,\phi)$ $n\in N_{0},\ \alpha\geq 0$, if the following subordinations hold:

$$\left(\frac{D^{n+1}f(z)}{D^{n}f(z)}\right)^{\alpha}\left(\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right)^{1-\alpha}\prec\phi(z)$$

and

$$\left(\frac{D^{n+1}g(w)}{D^{n}g(w)}\right)^{\alpha}\left(\frac{D^{n+2}g(w)}{D^{n+1}g(w)}\right)^{1-\alpha}\prec\phi(w)$$

$$g(w)=f^{-1}(w).$$

For function in this class, the following coefficient estimates are obtained

Theorem 2.8. *Let f given by (2.2) be in the class $L_\sigma^n(\alpha,\phi)$. Then*

$$|a_2| \leq \frac{2B_1\sqrt{B_1}}{\sqrt{\left|2\left(4\left(3-\alpha\right)3^n+\left(\alpha^2+5\alpha-8\right)2^{2n}B_1^2\right)+4\cdot 2^{2n}(\alpha-2)^2\left(B_1-B_2\right)\right|}}, \tag{2.32}$$

and

$$|a_3| \leq \frac{2\left(3-2\alpha\right)3^n\left(B_1+|B_1-B_2|\right)}{|3^n\left(3-2\alpha\right)\left(4\left(3-2\alpha\right)3^n+\left(\alpha^2+5\alpha-8\right)2^{2n}\right)|}. \tag{2.33}$$

Proof. Let $f \in L_\sigma^n(\alpha,\phi)$. Then there are analytic functions $u,v : U \to U$, with $u\left(0\right)=v\left(0\right)=0$, such that

$$\left(\frac{D^{n+1}f\left(z\right)}{D^nf\left(z\right)}\right)^{\alpha}\left(\frac{D^{n+2}f\left(z\right)}{D^{n+1}f\left(z\right)}\right)^{(1-\alpha)} = \phi\left(u\left(z\right)\right) \tag{2.34}$$

and

$$\left(\frac{D^{n+1}g\left(w\right)}{D^ng\left(w\right)}\right)^{\alpha}\left(\frac{D^{n+2}g\left(w\right)}{D^{n+1}g\left(w\right)}\right)^{(1-\alpha)} = \phi\left(v\left(w\right)\right). \tag{2.35}$$

Since

$$\left(\frac{D^{n+1}f\left(z\right)}{D^nf\left(z\right)}\right)^{\alpha}\left(\frac{D^{n+2}f\left(z\right)}{D^{n+1}f\left(z\right)}\right)^{(1-\alpha)} = 1+2^n\left(2-\alpha\right)a_2z$$
$$+\left(3^n\cdot 2\left(3-2\alpha\right)a_3+\left(\frac{\alpha^2-5\alpha+8}{2}\right)2^{2n}a_2^2\right)z^2+\ldots$$

and

$$\left(\frac{D^{n+1}g\left(w\right)}{D^ng\left(w\right)}\right)^{\alpha}\left(\frac{D^{n+2}g\left(w\right)}{D^{n+1}g\left(w\right)}\right)^{(1-\alpha)} = 1-2^n\left(2-\alpha\right)a_2w$$
$$+\left(\left(4\cdot\left(3-2\alpha\right)3^n+\frac{\alpha^2+5\alpha-8}{2}\right)a_2^2-3^n\cdot 2\left(3-2\alpha\right)a_3\right)w^2+\ldots$$

from (2.10), (2.11), (2.34) and (2.35) it follows that

$$2^n\cdot\left(2-\alpha\right)a_2 = \frac{1}{2}B_1c_1 \tag{2.36}$$

$$3^n\ 2\left(3-2\alpha\right)a_3+\left(\alpha^2+5\alpha-8\right)2^{2n}\cdot\frac{a_2^2}{2} = \frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2 \tag{2.37}$$

$$-2^n\left(2-\alpha\right)a_2 = \frac{1}{2}B_1b_1 \tag{2.38}$$

and

$$\left(4\left(3-2\alpha\right)3^n+2^n\frac{\left(\alpha^2+5\alpha-8\right)}{2}\right)a_2^2-3^n\cdot 2\left(3-2\alpha\right)a_3 = \frac{1}{2}B_1\left(b_2-\frac{b_1^2}{2}\right)+\frac{1}{4}B_2b_1^2. \tag{2.39}$$

Now (2.36) and (2.38) clearly yield

$$c_1=-b_1. \tag{2.40}$$

Equation (2.37), (2.39) and (2.40) lead to

$$a_2^2 = \frac{B_1^3\,(b_2+c_2)}{2\left(4\,(3-2\alpha)\,3^n + (\alpha^2+5\alpha-8)\,2^{2n}\right)B_1^2 + 4\cdot 2^{2n}(\alpha-2)^2\,(B_1-B_2)}$$

which yields the desired estimate on $|a_2|$ as asserted in (2.32). Proceeding similarly as in the earlier proof, using (2.37)-(2.40), it following that

$$a_3 = \frac{(B_1/2)((8(3-2\alpha)3^n + 2^{2n}(\alpha^2+5\alpha-8))c_2 - 2^{2n}(\alpha^2+5\alpha-8)b_2) + 3^n 2b_1^2(3-2\alpha)(B_1-B_2)}{4\cdot 3^n(3-2\alpha)(4(3-2\alpha)3^n + (\alpha^2+5\alpha-8)2^{2n})}$$

which yields the estimate (2.33). □

Remark 2.9. If we put $n = 0$ in Theorem 2.7-2.8 then we obtain the corresponding result of Ali *et al.* [1].

Remark 2.10. Sharp estimates for the coefficients $|a_2|$, $|a_3|$ and other coefficients of functions belonging to the classes investigated in this paper are yet open problems. Indeed it would be of interest even to find estimates (not necessarily sharp) for $|a_n|,\ n \geq 4$.

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