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On the univalence of an integral operator

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Abstract. In this paper we introduce a new general integral operator for analytic functions in the open unit disk and we derive some criteria for univalence of this integral operator.

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1. Introduction

Let $\mathcal{P}$ be the class of functions p of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} b_k z^k,$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathcal{C} : |z| < 1\}$, with positive real part in $\mathcal{U}$. We denote by $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $\mathcal{U}$ and we consider $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f \in \mathcal{A}$, which are univalent in $\mathcal{U}$.

In this work we introduce a new integral operator, which is defined by

$$K_{\gamma_1,\ldots,\gamma_n}(z) = \int_0^z \prod_{j=1}^{n} (p_j(u))^{\gamma_j}\, du, \tag{1.1}$$

for functions $p_j \in \mathcal{P}$ and γ_j be complex numbers, $j = \overline{1,n}$.

2. Preliminary results

In order to prove our main results we will use the lemmas.

Lemma 2.1. [1]. *If the function f is analytic in $\mathcal{U}$ and*

$$(1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \le 1, \tag{2.1}$$

for all $z \in \mathcal{U}$, then the function f is univalent in $\mathcal{U}$.

Lemma 2.2. [4]. *Let α be a complex number, $Re\ \alpha > 0$ and $f \in \mathcal{A}$. If*

$$\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le 1, \tag{2.2}$$

for all $z \in \mathcal{U}$, then for any complex number β, $Re\beta \ge Re\alpha$, the function

$$F_\beta(z) = \left[\beta\int_0^z u^{\beta-1} f'(u)du\right]^{\frac{1}{\beta}} \tag{2.3}$$

is regular and univalent in $\mathcal{U}$.

Lemma 2.3. (Schwarz [2]). *Let f be the function regular in the disk*

$$\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$$

with $|f(z)| < M$, M fixed. If $f(z)$ has in $z = 0$ one zero with multiply $\ge m$, then

$$|f(z)| \le \frac{M}{R^m}|z|^m,\ (z \in \mathcal{U}_R), \tag{2.4}$$

the equality (in the inequality (2.4) for $z \ne 0$) can hold if

$$f(z) = e^{i\theta}\frac{M}{R^m}z^m,$$

where θ is constant.

Lemma 2.4. [3]. *If the function f is regular in $\mathcal{U}$ and $|f(z)| < 1$ in $\mathcal{U}$, then for all $\xi \in \mathcal{U}$ and $z \in \mathcal{U}$ the following inequalities hold*

$$\left|\frac{f(\xi)-f(z)}{1-\overline{f(z)}f(\xi)}\right| \le \frac{|\xi - z|}{|1-\overline{z}\xi|}, \tag{2.5}$$

$$|f'(z)| \le \frac{1-|f(z)|^2}{1-|z|^2}, \tag{2.6}$$

the equalities hold only in the case $f(z) = \frac{\epsilon(z+u)}{1+\overline{u}z}$, where $|\epsilon| = 1$ and $|u| < 1$.

Remark 2.5. [3]. For $z = 0$, from inequality (2.5)

$$\left|\frac{f(\xi)-f(0)}{1-\overline{f(0)}f(\xi)}\right| \le |\xi| \tag{2.7}$$

and, hence

$$|f(\xi)| \le \frac{|\xi| + |f(0)|}{1+|f(0)||\xi|}. \tag{2.8}$$

Considering $f(0) = a$ and $\xi = z$, we have

$$|f(z)| \leq \frac{|z| + |a|}{1 + |a||z|}, \tag{2.9}$$

for all $z \in \mathcal{U}$.

3. Main results

Theorem 3.1. *Let γ_j be complex numbers, M_j positive real numbers, $p_j \in \mathcal{P}$,*

$$p_j(z) = 1 + b_{1j}z + b_{2j}z^2 + \ldots, \ \ j = \overline{1, n}.$$

If

$$\left| \frac{zp_j'(z)}{p_j(z)} \right| \leq M_j, \quad (j = \overline{1, n}; z \in \mathcal{U}) \tag{3.1}$$

and

$$|\gamma_1|M_1 + |\gamma_2|M_2 + \ldots + |\gamma_n|M_n \leq \frac{3\sqrt{3}}{2}, \tag{3.2}$$

then the integral operator $K_{\gamma_1,\ldots,\gamma_n}$ defined by (1.1), is in the class $\mathcal{S}$.

Proof. The function $K_{\gamma_1,\ldots,\gamma_n}$ is regular in $\mathcal{U}$ and

$$K_{\gamma_1,\ldots,\gamma_n}(0) = K'_{\gamma_1,\ldots,\gamma_n}(0) - 1 = 0.$$

We have

$$\frac{zK''_{\gamma_1,\ldots,\gamma_n}(z)}{K'_{\gamma_1,\ldots,\gamma_n}(z)} = \sum_{j=1}^{n} \gamma_j \frac{zp_j'(z)}{p_j(z)}, \quad (z \in \mathcal{U}), \tag{3.3}$$

and hence, we obtain

$$(1 - |z|^2) \left| \frac{zK''_{\gamma_1,\ldots,\gamma_n}(z)}{K'_{\gamma_1,\ldots,\gamma_n}(z)} \right| \leq (1 - |z|^2) \sum_{j=1}^{n} |\gamma_j| \left| \frac{zp_j'(z)}{p_j(z)} \right|, \tag{3.4}$$

for all $z \in \mathcal{U}$.

From (3.1) and Lemma 2.3 we get

$$\left| \frac{zp_j'(z)}{p_j(z)} \right| \leq M_j|z|, \quad (j = \overline{1, n}; z \in \mathcal{U}) \tag{3.5}$$

and by (3.4) we have

$$(1 - |z|^2) \left| \frac{zK''_{\gamma_1,\ldots,\gamma_n}(z)}{K'_{\gamma_1,\ldots,\gamma_n}(z)} \right| \leq (1 - |z|^2)|z| \sum_{j=1}^{n} |\gamma_j|M_j, \tag{3.6}$$

for all $z \in \mathcal{U}$.

Since

$$\max_{|z|\leq 1}(1 - |z|^2)|z| = \frac{2}{3\sqrt{3}},$$

from (3.2) and (3.6) we obtain that

$$(1-|z|^2)\left|\frac{zK''_{\gamma_1,\dots,\gamma_n}(z)}{K'_{\gamma_1,\dots,\gamma_n}(z)}\right| \le 1, \tag{3.7}$$

for all $z \in \mathcal{U}$ and by Lemma 2.1, it results that the integral operator $K_{\gamma_1,\dots,\gamma_n}$ belongs to the class $\mathcal{S}$. □

Theorem 3.2. *Let* α, γ_j *be complex numbers,* $j=\overline{1,n}$, $0 < Re\ \alpha \le 1$ *and* $p_j \in \mathcal{P}$, $p_j(z) = 1 + b_{1j}z + b_{2j}z^2 + \dots$, $j = \overline{1,n}$.

If

$$\left|\frac{zp'_j(z)}{p_j(z)}\right| \le \frac{(2Re\ \alpha+1)^{\frac{2Re\ \alpha+1}{2Re\ \alpha}}}{2}, \quad (j=\overline{1,n}; z\in\mathcal{U}) \tag{3.8}$$

and

$$|\gamma_1| + |\gamma_2| + \dots | + \gamma_n| \le 1, \tag{3.9}$$

then the integral operator $K_{\gamma_1,\dots,\gamma_n}$*, defined by (*1.1*), is in the class* $\mathcal{S}$*.*

Proof. From (3.3) we obtain

$$\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}\left|\frac{zK''_{\gamma_1,\dots,\gamma_n}(z)}{K'_{\gamma_1,\dots,\gamma_n}(z)}\right| \le \frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}\sum_{j=1}^{n}|\gamma_j|\left|\frac{zp'_j(z)}{p_j(z)}\right|, \tag{3.10}$$

for all $z \in \mathcal{U}$.

By (3.8) and Lemma 2.3, we get

$$\left|\frac{zp'_j(z)}{p_j(z)}\right| \le \frac{(2Re\ \alpha+1)^{\frac{2Re\ \alpha+1}{2Re\ \alpha}}}{2}|z|, \quad (j=\overline{1,n}; z\in\mathcal{U}) \tag{3.11}$$

and hence, by (3.10) we have

$$\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha} \cdot \left|\frac{zK''_{\gamma_1,\dots,\gamma_n}(z)}{K'_{\gamma_1,\dots,\gamma_n}(z)}\right| \le \\ \le \frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}|z|\frac{(2Re\ \alpha+1)^{\frac{2Re\ \alpha+1}{2Re\ \alpha}}}{2}\sum_{j=1}^{n}|\gamma_j|, \tag{3.12}$$

for all $z \in \mathcal{U}$.

We have

$$\max_{|z|\le 1}\left[\frac{(1-|z|)^{2Re\ \alpha}|z|}{Re\ \alpha}\right] = \frac{2}{(Re\ \alpha+1)^{\frac{2Re\ \alpha+1}{2Re\ \alpha}}}$$

and from (3.9) and (3.12) we get

$$\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}\left|\frac{zK''_{\gamma_1,\dots,\gamma_n}(z)}{K'_{\gamma_1,\dots,\gamma_n}(z)}\right| \le 1, \tag{3.13}$$

for all $z \in \mathcal{U}$. By (3.13) and Lemma 2.2, for $\beta = 1$, $f = K_{\gamma_1,\dots,\gamma_n}$, it results that the integral operator $K_{\gamma_1,\dots,\gamma_n}$ is in the class $\mathcal{S}$. □

Theorem 3.3. *Let γ_j be complex numbers,*

$$p_j \in \mathcal{P}, p_j(z) = 1 + b_{1j}z + b_{2j}z^2 + \ldots, j = \overline{1,n}.$$

If

$$|\gamma_1| + |\gamma_2| + \ldots | + \gamma_n| \leq \frac{1}{2}, \tag{3.14}$$

then the integral operator $K_{\gamma_1,\ldots,\gamma_n}$ defined by (1.1) belongs to the class $\mathcal{S}$.

Proof. Since $p_j \in \mathcal{P}$, $j = \overline{1,n}$ we have

$$\left|\frac{zp_j'(z)}{p_j(z)}\right| \leq \frac{2|z|}{1-|z|^2}, \quad (z \in \mathcal{U};\ j = \overline{1,n}), \tag{3.15}$$

by (3.3) we obtain

$$(1-|z|^2)\left|\frac{zK''_{\gamma_1,\ldots,\gamma_n}(z)}{K'_{\gamma_1,\ldots,\gamma_n}(z)}\right| \leq 2\sum_{j=1}^{n}|\gamma_j|, \quad (z \in \mathcal{U}). \tag{3.16}$$

From (3.14) and (3.16) we get

$$(1-|z|^2)\left|\frac{zK''_{\gamma_1,\ldots,\gamma_n}(z)}{K'_{\gamma_1,\ldots,\gamma_n}(z)}\right| \leq 1, \tag{3.17}$$

for all $z \in \mathcal{U}$.

By (3.17) and Lemma 2.1 we obtain that the integral operator $K_{\gamma_1,\ldots,\gamma_n}$ belongs to the class $\mathcal{S}$. □

Theorem 3.4. *Let α, γ_j be complex numbers, $j = \overline{1,n}$, $0 < Re\ \alpha \leq 1$, M_j positive real numbers and $p_j \in \mathcal{P}$, $p_j(z) = 1 + b_{1j}z + b_{2j}z^2 + \ldots$, $j = \overline{1,n}$.*

If

$$\left|\frac{p_j'(z)}{p_j(z)}\right| < M_j, \quad (z \in \mathcal{U};\ j = \overline{1,n}), \tag{3.18}$$

$$M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n| \leq \frac{1}{max_{|z|\leq 1}\left[\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}|z|\frac{|z|+|c|}{1+|c||z|}\right]}, \tag{3.19}$$

where

$$c = \frac{b_{11}\gamma_1 + b_{12}\gamma_2 + \ldots + b_{1n}\gamma_n}{M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n|}, \tag{3.20}$$

then the integral operator $K_{\gamma_1,\gamma_2,\ldots,\gamma_n}$ defined by (1.1) is in the class $\mathcal{S}$.

Proof. We have

$$\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}\left|\frac{zK''_{\gamma_1,\ldots,\gamma_n}(z)}{K'_{\gamma_1,\ldots,\gamma_n}(z)}\right| \leq \frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}|z|\sum_{j=1}^{n}|\gamma_j|\left|\frac{p_j'(z)}{p_j(z)}\right|, \tag{3.21}$$

for all $z \in \mathcal{U}$. We consider the function

$$f_n(z) = \frac{1}{M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n|}\frac{K''_{\gamma_1,\ldots,\gamma_n}(z)}{K'_{\gamma_1,\ldots,\gamma_n}(z)}, \quad (z \in \mathcal{U}) \tag{3.22}$$

and from (1.1) we obtain

$$f_n(z) = \frac{\gamma_1}{M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n|} \cdot \frac{p_1'(z)}{p_1(z)} + \ldots +$$

$$+ \frac{\gamma_n}{M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n|} \cdot \frac{p_n'(z)}{p_n(z)}, \tag{3.23}$$

for all $z \in \mathcal{U}$.

From (3.18) and (3.23) we obtain $|f_n(z)| < 1$, $z \in \mathcal{U}$.

We have

$$f_n(0) = \frac{b_{11}\gamma_1 + \ldots + b_{1n}\gamma_n}{M_1|\gamma_1| + \ldots + M_n|\gamma_n|} = c$$

and by Remark 2.5 we get

$$|f_n(z)| \leq \frac{|z| + |c|}{1 + |c||z|}, \quad (z \in \mathcal{U}), \tag{3.24}$$

where

$$|c| = \frac{|b_{11}\gamma_1 + b_{12}\gamma_2 + \ldots + b_{1n}\gamma_n|}{M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n|}.$$

From (3.22) and (3.24) we obtain

$$\frac{1 - |z|^{2Re\ \alpha}}{Re\ \alpha} \left| \frac{zK''_{\gamma_1,\ldots,\gamma_n}(z)}{K'_{\gamma_1,\ldots,\gamma_n}(z)} \right| \leq$$

$$\leq (M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n|) \max_{|z|\leq 1} \left[\frac{1 - |z|^{2Re\ \alpha}}{Re\ \alpha} |z| \frac{|z| + |c|}{1 + |c||z|} \right], \tag{3.25}$$

for all $z \in \mathcal{U}$.

By (3.19) and (3.25) we have

$$\frac{1 - |z|^{2Re\ \alpha}}{Re\ \alpha} \left| \frac{zK''_{\gamma_1,\ldots,\gamma_n}(z)}{K'_{\gamma_1,\ldots,\gamma_n}(z)} \right| \leq 1, \quad (z \in \mathcal{U}). \tag{3.26}$$

From (3.26) and Lemma 2.2 for $\beta = 1$, it results that the integral operator $K_{\gamma_1,\ldots,\gamma_n} \in \mathcal{S}$. □

Corollary 3.5. *Let α, γ_j be complex numbers, $j = \overline{1,n}$, $0 < Re\ \alpha \leq 1$, M_j positive real numbers and $p_j \in \mathcal{P}$, $p_j(z) = 1 + b_{1j}z + b_{2j}z^2 + \ldots$, $j = \overline{1,n}$.*

If

$$\left| \frac{p_j'(z)}{p_j(z)} \right| < M_j, \quad (z \in \mathcal{U};\ j = \overline{1,n}), \tag{3.27}$$

$$|b_{11}\gamma_1 + b_{12}\gamma_2 + \ldots + b_{1n}\gamma_n| \leq \frac{(2Re\ \alpha + 1)^{\frac{2Re\ \alpha + 1}{2Re\ \alpha}}}{2}, \tag{3.28}$$

$$|b_{11}\gamma_1 + b_{12}\gamma_2 + \ldots + b_{1n}\gamma_n| = M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n|, \tag{3.29}$$

then the integral operator $K_{\gamma_1,\ldots,\gamma_n} \in \mathcal{S}$.

Proof. From (3.29) and (3.20) we obtain $|c| = 1$. Using the inequality (3.19) we have

$$M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n| \le \frac{1}{max_{|z|\le 1}\left[\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}|z|\right]}, \tag{3.30}$$

Since

$$max_{|z|\le 1}\left[\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}|z|\right] = \frac{2}{(2Re\ \alpha+1)^{\frac{2Re\ \alpha+1}{2Re\ \alpha}}}, \tag{3.31}$$

from (3.30) and (3.29) we obtain (3.28).

The conditions of Theorem 3.4 are satisfied. □

Corollary 3.6. *Let α, γ_j be complex numbers, $j = \overline{1,n}$, $0 < Re\ \alpha \le 1$, M_j positive real numbers and $p_j \in \mathcal{P}$, $p_j(z) = 1 + b_{1j}z + b_{2j}z^2 + \ldots$, $j = \overline{1,n}$, $b_{11}\gamma_1 + b_{12}\gamma_2 + \ldots + b_{1n}\gamma_n = 0$.*

If

$$\left|\frac{p_j'(z)}{p_j(z)}\right| < M_j, \quad (z \in \mathcal{U};\ j = \overline{1,n}), \tag{3.32}$$

$$M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n| \le (Re\ \alpha+1)^{\frac{Re\ \alpha+1}{Re\ \alpha}}, \tag{3.33}$$

then the integral operator $K_{\gamma_1,\ldots,\gamma_n} \in \mathcal{S}$.

Proof. From Theorem 3.4, by (3.20), we obtain $c = 0$ and using the inequality (3.19) we get

$$M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n| \le \frac{1}{max_{|z|\le 1}\left[\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}|z|^2\right]}. \tag{3.34}$$

We have

$$max_{|z|\le 1}\left[\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}|z|^2\right] = \frac{1}{(Re\ \alpha+1)^{\frac{Re\ \alpha+1}{Re\ \alpha}}}$$

and from (3.34) we obtain the inequality (3.33). Since the conditions of Theorem 3.4 are verified it results that $K_{\gamma_1,\ldots,\gamma_n}$ belongs to $\mathcal{S}$. □

References

[1] Becker, J., *Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen*, J. Reine Angew. Math., **255**(1972), 23-43.

[2] Mayer, O., *The Functions Theory of One Variable Complex*, Bucureşti, 1981.

[3] Nehari, Z., *Conformal Mapping*, Mc Graw-Hill Book Comp., New York, 1952 (Dover. Publ. Inc., 1975).

[4] Pascu, N.N., *An improvement of Becker's univalence criterion*, Proceedings of the Commemorative Session Simion Stoilow, Braşov, (1987), 43-48.

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