Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 4, 493–507

# $\aleph_1$-$A$-coseperable groups

Ulrich Albrecht

**Abstract.** Let $A$ be a countable self-small Abelian group with a right Noetherian right hereditary endomorphism ring. We show that the question whether strongly-$\aleph_1$-$A$-generated groups are $\aleph_1$-$A$-coseparable is undecidable in ZFC. Our main focus is on the algebraic aspect of the proof, not on the underlying set-theory.

**Mathematics Subject Classification (2010):** 20K20, 20K40.

**Keywords:** $\aleph_1$-a-coseparable groups, Martin's axiom, endomorphism rings.

## 1. Introduction

Let $A$ be an Abelian group with endomorphism ring $E = E(A)$. Associated with $A$ are the functors $H_A(.) = \mathrm{Hom}(A, .)$ and $T_A(.) = . \otimes_E A$ which induce natural maps $\theta_G : T_A H_A(G) \to G$ and $\phi_M : M \to H_A T_A(M)$ defined by $\theta_G(\alpha \otimes a) = \alpha(a)$ and $[\phi_M(x)](a) = x \otimes a$ for all $\alpha \in H_A(G)$, $x \in M$ and $a \in A$. The *A-solvable groups* are the Abelian groups $G$ such that $\theta_G$ is an isomorphism. Finally, a sequence $0 \to G \to H \to L \to 0$ of Abelian group is *A-balanced* if the induced sequence $0 \to H_A(G) \to H_A(H) \to H_A(L) \to 0$ of right $E$-modules is exact

An important class of $A$-solvable groups are the *(finitely) A-projective groups*, i.e. groups which are isomorphic to a direct summand of $\oplus_I A$ for some (finite) index-set $I$. Finitely $A$-projective groups are always $A$-solvable [8], and the same holds for arbitrary $A$-projective groups [9] if $A$ is *self-small*, i.e. if $H_A$ preserves direct sums of copies of $A$. Arnold and Murley showed in [9, Corollary 2.3] that a countable Abelian group is self-small if and only if $E$ is countable.

Epimorphic images of $A$-projective groups are called *A-generated*, but need not be $A$-solvable. It is easy to see that a group $G$ is $A$-generated if and only if $\theta_G$ is onto. Moreover, if $A$ is self-small, then a group $G$ is $A$-solvable if and only if there is an $A$-balanced exact sequence $0 \to U \to F \to G \to 0$ in which $F$ is $A$-projective and $U$ is $A$-generated [3]. Finally, $G$ is *A-torsion-free* if every finitely $A$-generated subgroup of $G$ is isomorphic to a subgroup of a finitely $A$-projective group, and an $A$-generated subgroup $U$ of an $A$-torsion-free group $G$ is *A-pure* if $(U + P)/U$ is $A$-torsion-free for all finitely $A$-generated subgroups $P$ of $G$. If $A$ is flat as an $E$-module, then $A$-torsion-free groups are $A$-solvable [4]. We want to remind the reader that a right $E$-module $M$

is *non-singular* if $xI \neq 0$ for all non-zero $x$ in $M$ and all essential right ideals $I$ of $E$. The ring $R$ is *right non-singular* if $R_R$ is a non-singular module. If $U$ is a submodule of a non-singular right $E$-module $M$, then the $S$-closure of $U$ in $M$ consists of all $x \in M$ such that $xI \subseteq M$ for some essential right ideal $I$ of $E$ [14]. Non-singularity is closely related to $A$-torsion-freeness whenever $A$ is a self-small Abelian group whose endomorphism ring is right non-singular [5]:

a) If an $A$-generated group $G$ is $A$-torsion-free, then $H_A(G)$ is non-singular.
b) An $A$-generated subgroup $U$ of an $A$-torsion-free group $G$ is contained in a smallest $A$-pure subgroup $V$ of $G$ which is obtained as $\theta_G(T_A(W))$ where $W$ is the $S$-closure of $H_A(U)$ in $H_A(G)$.

The focus of this paper are $A$-torsion-free groups $G$ such that all $A$-generated subgroups $U$ of $G$ with $|U| < |G|$ are $A$-projective. Since $A$-generated subgroups of $A$-projective groups need not be $A$-projective in general ([4] and [8]), some immediate restrictions on $A$ are needed to guarantee the existence of non-trivial groups with the above property.

## 2. Hereditary Endomorphism Rings and $\kappa$-$A$-projective groups

An Abelian group is *$\kappa$-$A$-generated*, where $\kappa$ is an infinite cardinal, if it is an epimorphic image of $\oplus_I A$ for some index-set $I$ with $|I| < \kappa$. The $\aleph_0$-$A$-generated groups are referred to as *finitely $A$-generated groups*. An $A$-generated group $G$ is *$\kappa$-$A$-projective* if every $\kappa$-$A$-generated subgroup $U$ of $G$ is $A$-projective. If $|A| < \kappa$, then this is equivalent to the condition that all $A$-generated subgroups $U$ with $|U| < \kappa$ are $A$-projective. Since every finitely $A$-generated subgroup of a $\kappa$-$A$-projective group $G$ is $A$-projective, $G$ is $A$-solvable. In particular, an $A$-generated group $G$ is $\aleph_0$-$A$-projective if every finitely $A$-generated subgroup is $A$-projective. If $A$ is faithfully flat as a left $E$-module, then finitely $A$-generated $A$-projective groups are finitely $A$-projective [4].

**Theorem 2.1.** *The following conditions are equivalent for a self-small torsion-free Abelian group $A$:*

a) i) *$A$-projective groups are $\kappa$-$A$-projective for all infinite cardinals $\kappa$.*
   ii) *Every exact sequence $0 \to U \to G \to H \to 0$, in which $G$ and $H$ is $\kappa$-$A$-projective for some infinite cardinal $\kappa$, is $A$-balanced.*

b) *$E$ is a right hereditary ring.*

*In this case, $A$ is faithfully flat as an $E$-module.*

*Proof.* *a)* $\Rightarrow$ *b)*: To see that $A$ is flat as an $E$-module, observe that $A^n$ is $\aleph_0$-$A$-projective for all $n < \omega$, from which we obtain that $G = \alpha(A^n)$ is $A$-projective for all $\alpha : A^n \to A$. By a.ii), the exact sequence $0 \to U \to A^n \to G \to 0$ with $U = \ker\,\alpha$ is $A$-balanced which yields the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_AH_A(U) & \longrightarrow & T_AH_A(A^n) & \longrightarrow & T_AH_A(G) & \longrightarrow & 0 \\ & & \downarrow \theta_U & & \wr\downarrow \theta_{A^n} & & \wr\downarrow \theta_G & & \\ 0 & \longrightarrow & U & \longrightarrow & A^n & \longrightarrow & G & \longrightarrow & 0. \end{array}$$

Thus, $\theta_U$ is an isomorphism. By Ulmer's Theorem [17], $A$ is $E$-flat.

Consider a right ideal $I$ of $E$. Because $A$ is $E$-flat, $T_A(I) \cong IA \subseteq A$. Since $IA$ is an $A$-generated subgroup of $A$, and $A$ is $|IA|^+$-$A$-projective by a.i), $IA$ is $A$-projective. Thus, $H_AT_A(I)$ is a projective module fitting into the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & H_AT_A(I) & \longrightarrow & H_AT_A(E) \\ & & \uparrow \phi_I & & \wr\uparrow \phi_E \\ 0 & \longrightarrow & I & \longrightarrow & E \end{array}$$

from which we obtain that $\phi_I$ is one-to-one.

On the other hand, consider an exact sequence $0 \to V \to F \to I \to 0$ where $F$ is a free right $E$-module. It induces the exact sequence

$$0 \to T_A(V) \to T_A(F) \to T_A(I) \to 0.$$

The latter sequence is $A$-balanced by a.ii). Hence, the top-row in the commutative diagram

$$\begin{array}{ccccc} H_AT_A(F) & \longrightarrow & H_AT_A(I) & \longrightarrow & 0 \\ \wr\uparrow \phi_F & & \uparrow \phi_I & & \\ F & \longrightarrow & I & \longrightarrow & 0 \end{array}$$

is exact, which yields that $\phi_I$ is onto. Consequently, $I$ is projective, and $E$ is right hereditary.

*b)* $\Rightarrow$ *a)*: Let $M$ be a right $E$-module. Since $E$ is right hereditary, we can find an exact sequence $0 \to P \to F \to M \to 0$ in which $P$ and $F$ are projective. It induces exact sequence

$$0 = \mathrm{Tor}_1^R(F, A) \to \mathrm{Tor}_1^R(M, A) \to T_A(P) \to T_A(F) \to T_A(M) \to 0.$$

We obtain the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_A(\mathrm{Tor}_1^R(M, A)) & \longrightarrow & H_AT_A(P) & \longrightarrow & H_AT_A(F) \\ & & & & \wr\uparrow \phi_P & & \wr\uparrow \phi_F \\ & & 0 & \longrightarrow & P & \longrightarrow & F. \end{array}$$

Therefore, $H_A(\mathrm{Tor}_1^R(M, A)) = 0$ for all right $R$-modules $M$.
If $M^+$ is torsion-free, then it is isomorphic to a submodule of $\mathbb{Q}M = \mathbb{Q} \otimes_{\mathbb{Z}} M$. Since $\mathrm{Tor}_1^R(\mathbb{Q}M, A)$ is torsion-free and divisible, $H_A(\mathrm{Tor}_1^R(\mathbb{Q}M, A)) = 0$ is only possible if $\mathrm{Tor}_1^R(\mathbb{Q}M, A) = 0$. However, because $E$ is right hereditary, we have the exact sequence $0 \to \mathrm{Tor}_1^R(M, A) \to \mathrm{Tor}_1^R(\mathbb{Q}M, A) = 0$, and $\mathrm{Tor}_1^R(M, A) = 0$.

If $M^+$ is torsion, then we select an exact sequence $0 \to U \to F_1 \to A \to 0$ in which $F_1$ is a free left $E$-module. It induces

$$0 = \mathrm{Tor}_1^R(M, F_1) \to \mathrm{Tor}_1^R(M, A) \to M \otimes_E V.$$

Since $M \otimes_E V$ is torsion, the same holds for $\mathrm{Tor}_1^R(M, A)$. But, the latter also is isomorphic to a subgroup of the torsion-free group $T_A(P)$. Thus, $\mathrm{Tor}_1^R(M, A) = 0$.

For an arbitrary $M$, we consider the exact sequence

$$0 = \operatorname{Tor}_1^R(tM, A) \to \operatorname{Tor}_1^R(M, A) \to \operatorname{Tor}_1^R(M/tM, A) = 0$$

where the first and the last term vanish but what has already been shown. Thus, $A$ is $E$-flat.

To show that $A$ is faithful as a left $E$-module, suppose that $T_A(M) = 0$. The sequence $0 \to P \to F \to M \to 0$ yields the exact sequence

$$0 \to T_A(P) \to T_A(F) \to T_A(M) = 0$$

since $A$ is flat as an $E$-module. Hence, the top-row of the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_AT_A(P) & \longrightarrow & H_AT_A(F) & \longrightarrow & 0 & & \\ & & \wr\big\uparrow \phi_P & & \wr\big\uparrow \phi_F & & & & \\ 0 & \longrightarrow & P & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

is exact. A simple diagram chase shows $M = 0$.

Finally, $A$-generated subgroups of $A$-projective groups are $A$-projective if $A$ is faithfully flat and $E$ is right hereditary [4], and a.i) holds. Finally, a.ii) is a direct consequence of [6] since $\kappa$-$A$-projective groups are $A$-solvable. □

In particular, the last result shows that $A$-generated subgroups of self-small groups with right hereditary endomorphism ring are $A$-projective. Our next results summarizes other properties of such groups which we use frequently in this paper:

**Corollary 2.2.** *Let $A$ be a self-small torsion-free Abelian group whose endomorphism ring is right hereditary:*

a) *Every exact sequence $G \to P \to 0$ such that $G$ is $A$-generated and $P$ is $A$-projective splits.*
b) *An $A$-generated group is $A$-torsion-free if and only if it is $\aleph_0$-$A$-projective.*
c) *An $A$-generated subgroup of an $A$-torsion-free group is $A$-pure if and only if $U$ is a direct summand of $U + V$ for all finitely $A$-generated subgroups $V$ of $G$.*

*Proof.* a) follows directly from the fact that $A$ is faithfully flat which was established in Theorem 2.1.

b) It remains to show that $A$-torsion-free groups are $\aleph_0$-$A$-projective. Suppose that $G$ is $A$-torsion-free, and let $U$ be a finitely $A$-generated subgroup of $G$. Then $U$ can be embedded into an $A$-projective group, and thus is $A$-projective by Theorem 2.1.

c) Let $U$ be an $A$-pure subgroup of an $A$-torsion-free group $G$. If $V$ is a finitely $A$-generated subgroup of $G$, then $(U + V)/U$ can be embedded into an $A$-projective group by Theorem 2.1. Thus, $(U+V)/U$ is $A$-projective. By a), $U$ is a direct summand of $U + V$. □

However, the $S$-closure of a countable submodule of a non-singular module does not need to be countable even if $R$ is countable. For instance, if $Q = \mathbb{Q}^\omega$ and $R = \mathbb{Z}1_S + \mathbb{Z}^{(\omega)}$, then $Q$ is the maximal ring of quotients of $R$ and $|Q| > |R|$ although $Q$ is an essential extension of $R$. We want to remind the reader that a right $E$-module $M$ has *Goldie-dimension* $m < \infty$ if it contains an essential submodule which is the direct sum of $m$ non-zero uniform submodules where a module $X \neq 0$ is uniform if all its non-zero submodules are essential.

**Proposition 2.3.** *Let $R$ be a countable right non-singular ring which has finite right Goldie-dimension. The $S$-closure of a countable submodule $U$ of a non-singular right $R$-module $M$ is countable.*

*Proof.* Let $V$ be $S$-closure of $U$, and assume that $V$ is uncountable. Let

$$\mathcal{F} = \{X \subseteq R \mid |X| < \infty \text{ and } \sum_{x \in X} xR \text{ is an essential right ideal}\}.$$

Then, $V = \{y \in M \mid yX \subseteq U$ for some $x \in \mathcal{F}\}$ since $R$ has finite right Goldie-dimension. Since $V$ is uncountable and $\mathcal{F}$ is countable, we can find $X_0 \in \mathcal{F}$ such that $yX_0 \subseteq U$ for uncountably many $y \in V$. Let $Y_0 = \{y \in V \mid yX_0 \subseteq U\}$. Write $X_0 = \{x_1, \ldots, x_n\}$, and consider $Y_0 x_1 \subseteq U$. There is an uncountable subset $Y_1$ of $Y_0$ such that $yx_1 = y'x_1$ for all $y, y' \in Y_1$ since $U$ is countable. Repeating this argument with $x_2$ and $Y_1$ yields an uncountable subset $Y_2$ of $Y_1$ such that $yx_2 = y'x_2$ for all $y, y' \in Y_2$. By induction, we can find an uncountable subset $Y_n$ of $Y_0$ such that $yx_i = y'x_i$ for all $i = 1, \ldots, n$ and all $y, y' \in Y_n$. Thus, $(y-y')(x_1R+\ldots+x_nR) = 0$ for all $y, y' \in Y_n$ which contradicts the fact that $M$ is non-singular because $x_1R+\ldots+x_nR$ is essential. Thus $V$ has to be countable. $\square$

By Sandomierski's Theorem [11], a right finite dimensional, right hereditary ring is right Noetherian.

**Corollary 2.4.** *The following conditions are equivalent for a self-small torsion-free Abelian group $A$ whose endomorphism ring is right hereditary:*

a) *$E$ is right Noetherian.*
b) *An $A$-generated subgroup $U$ of a finitely $A$-projective group $G$ is finitely $A$-projective.*

*Proof. a) $\Rightarrow$ b)*: Suppose that $U$ is an $A$-generated subgroup of a finitely $A$-projective group $P$. Then $H_A(U)$ is a submodule of $H_A(P)$, and hence a finitely generated projective module. By Theorem 2.1, $U$ is $A$-solvable, and $U \cong T_A H_A(U)$ is finitely $A$-projective.

*b) $\Rightarrow$ a)*: Let $I$ be a right ideal of $E$. Arguing as in the proof of Theorem 2.1, $\phi_I$ is an isomorphism. Moreover $T_A(I) \cong IA$ since $A$ is flat as an $E$-module. By b), $IA$ is finitely $A$-projective, from which we obtain that $I$ is finitely generated. $\square$

In view of the results of this section, we assume from this point on that $A$ is a self-small torsion-free group with a right Noetherian right hereditary endomorphism ring. Huber and Warfield showed in [16] that $E$ is a right and left Noetherian ring whenever $A$ is a torsion-free reduced group of finite rank with a right hereditary endomorphism ring. Moreover, no generality is lost if we restrict our discussion to the case that $\kappa$ is a regular cardinal because Shelah's singular compactness theorem applies to $A$-projective groups [2].

## 3. $\aleph_1$-$A$-Coseparable Groups

Let $\kappa > \aleph_0$ be a regular cardinal, and $A$ be a torsion-free Abelian group with $|A| < \kappa$. An $A$-projective subgroup $U$ of an $\aleph_0$-$A$-projective group $G$ is *$\kappa$-$A$-closed* provided that $(U+V)/U$ is $A$-projective for all $\kappa$-$A$-generated subgroups $V$ of $G$. If $|U| < \kappa$, then this is equivalent to saying that $G/U$ is $\kappa$-$A$-projective. The group $G$ is *strongly $\kappa$-$A$-projective* if it is $\kappa$-$A$-projective and every $\kappa$-$A$-generated subgroup $U$ of $G$ is contained in an $\kappa$-$A$-generated, $\kappa$-$A$-closed subgroup $V$ of $G$. By [1], $S_A(A^I)$ is $\aleph_1$-$A$-projective, but not strongly $\aleph_1$-$A$-projective since $\oplus_I A$ is not an $\aleph_1$-$A$-closed subgroup.

In the following we focus on the case $\kappa = \aleph_1$ since we are mainly interested in the algebraic aspects instead of the underlying set-theory. However, most results of this section carry over to the general case. In order to avoid immediate difficulties, we restrict our discussion to the case that $A$ is countable.

**Lemma 3.1.** *Let $A$ be a self-small countable torsion-free group with a right Noetherian right hereditary endomorphism ring.*

a) *If $G$ is $\aleph_1$-$A$-projective, then $G/U$ is $\aleph_1$-$A$-projective for all $\aleph_1$-$A$-closed subgroups $U$ of $G$.*
b) *If $G$ is strongly $\aleph_1$-$A$-projective, then $G/U$ is strongly $\aleph_1$-$A$-projective for all countable $\aleph_1$-$A$-closed subgroups $U$ of $G$.*

*Proof.* a) Let $\{\phi_n | n < \omega\} \subseteq H_A(G/U)$. Since $\Sigma_{n<\omega}\phi_n(A)$ is countable, there is a countable subgroup $K$ of $G$ such that $\Sigma_{n<\omega}\phi_n(A) \subseteq (K+U)/U$. Because $A$ is countable, we can choose $K$ to be $A$-generated. Since $U$ is $\aleph_1$-$A$-closed in $G$, the group $(K+U)/U$ is $U$-projective, and the same holds $\Sigma_{n<\omega}\phi_n(A)$. Therefore, $G/U$ is $\aleph_1$-$A$-projective.

b) Let $V/U$ be a countable $A$-generated subgroup of $G/U$. Without loss of generality, we may assume that $V$ is $A$-generated. Then, $V$ is contained in an $\aleph_1$-$A$-closed subgroup $W$ is a $\aleph_1$-$A$-closed subgroup of $G$. Since $U$ is countable this means that $G/W$ is $\aleph_1$-$A$-projective. Since $G/W \cong (G/U)/(W/U)$ and $G/U$ is $\aleph_1$-$A$-projective, we obtain that $G/U$ is strongly $\aleph_1$-$A$-projective. □

An $A$-generated group $G$ $\aleph_1$-*$A$-coseparable* if it is $\aleph_1$-$A$-projective and every $A$-generated subgroup $U$ of $G$ such that $G/U$ is countable contains a direct summand $V$ of $G$ such that $G/V$ is countable. Our next results describes $\aleph_1$-$A$-coseparable group. Although our arguments follow the general outline of [13], significant modifications are necessary in our setting.

**Theorem 3.2.** *Let $A$ be a self-small countable torsion-free group with a right Noetherian right hereditary endomorphism ring. A group $G$ is $\aleph_1$-$A$-coseparable if and only if $G$ is $A$-solvable and every exact sequence*

$$0 \to P \to X \to G \to 0$$

*with $P$ a direct summand of $\oplus_\omega A$ and $X$ $A$-generated splits.*

*Proof.* Suppose that $G$ is $\aleph_1$-$A$-coseparable, and consider an exact sequence

$$0 \to P \stackrel{\alpha}{\to} X \stackrel{\beta}{\to} G \to 0$$

with $P$ a direct summand of $\oplus_\omega A$ and $X$ $A$-generated. Since $A$ is faithfully flat, $X$ is $A$-generated and $G$ is $A$-solvable, the induced sequence

$$0 \to H_A(P) \xrightarrow{\alpha} H_A(X) \xrightarrow{H_A(\beta)} H_A(G) \to 0$$

of right $E$-modules is exact by Theorem 2.1. Since $H_A(G)$ is non-singular by the remarks in the introduction, the same holds for $H_A(X)$. Observe that $H_A(P)$ is countable since it is a direct summand of $H_A(\oplus_\omega A)$, and the latter is countable because $A$ is self-small. We choose a complement $W$ of $im(H_A(\alpha))$ in $H_A(X)$, and observe that $H_A(X)/W$ is nonsingular. Since

$$M = H_A(X)/(im(H_A(\alpha) \oplus W) \cong [H_A(X)/W][(im(H_A(\alpha) \oplus W)/W]$$

is singular and $(im(H_A(\alpha) \oplus W)/W$ is countable, $H_A(X)/W$ is countable as the $S$-closure of a countable submodule by Proposition 2.3 because $E$ is right Noetherian and countable. Applying $T_A$ yields the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_AH_A(P) & \xrightarrow{T_AH_A(\alpha)} & T_AH_A(X) & \xrightarrow{T_AH_A(\beta)} & T_AH_A(G) & \longrightarrow & 0 \\ & & \wr\downarrow \theta_P & & \downarrow \theta_X & & \wr\downarrow \theta_G & & \\ 0 & \longrightarrow & P & \xrightarrow{\alpha} & X & \xrightarrow{\beta} & G & \longrightarrow & 0. \end{array}$$

Therefore, $X$ is $A$-solvable, and $U = \theta_X(T_A(W))$ is an $A$-generated subgroup of $X$ such that $\alpha(P) \cap X = 0$ and

$$X/[\alpha(P) \oplus U] \cong T_A(M)$$

is countable. If $H = \beta(U)$, then $\beta|U$ is one-to-one. Since $\beta(U) \cong U \cong T_A(W)$ is $A$-generated and $G/\beta(U)$ is countable, there is a subgroup $K$ of $U$ such that $G = \beta(K) \oplus B$ for some countable subgroup $B$ of $G$ using the fact that $G$ is $\aleph_1$-$A$-coseparable. Select a subgroup $V$ of $X$ containing $\alpha(P)$ such that $\beta(V) = B$. Clearly, $V$ is countable.

To show $X = K \oplus V$, take $x \in X$ and write $\beta(x) = \beta(k) + \beta(v)$ for some $k \in K$ and $v \in V$. Then $x - k - b \in \alpha(P) \subseteq V$. On the other hand, suppose that $y \in K \cap V$. Then $\beta(y) \in \beta(K) \cap B = 0$, from which we obtain

$$y \in \alpha(P) \cap K \subseteq \alpha(P) \cap U = 0.$$

Moreover, $V$ is $A$-generated since it is a direct summand of $X$, while $\beta(V) \cong V/\alpha(P)$ is $A$-projective as a countable subgroup of $G$. Therefore, $\alpha(P)$ is a direct summand of $V$.

Conversely, assume that $G$ is an $A$-solvable group such that every exact sequence $0 \to P \to X \to G \to 0$ with $P$ a direct summand of $\oplus_\omega A$ and $X$ $A$-generated splits. Suppose that $G$ contains a countable $A$-generated subgroup $U$ which is not $A$-projective. Since $U$ is $A$-solvable because $A$ is $E$-flat by Theorem 2.1, $H_A(U)$ is not projective. Looking at projective resolutions of $H_A(U)$, we can find a countable projective module $P$ with $\mathrm{Ext}^1_E(H_A(U), P) \neq 0$. Since $E$ is right hereditary, we have an exact sequence

$$\mathrm{Ext}^1_E(H_A(G), P) \to \mathrm{Ext}^1_E(H_A(U), P) \to 0.$$

Thus, there is a non-splitting sequence $0 \to P \to M \to H_A(G) \to 0$ which induces $0 \to T_A(P) \to T_A(M) \to T_A H_A(G) \to 0$ which splits since $G \cong T_A H_A(G)$. We therefore obtain the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_A T_A(P) & \longrightarrow & H_A T_A(M) & \longrightarrow & H_A T_A H_A(G) & \longrightarrow & 0 \\ & & \wr\uparrow \phi_P & & \uparrow \phi_M & & \wr\uparrow \phi_{H_A(G)} & & \\ 0 & \longrightarrow & P & \longrightarrow & M & \longrightarrow & H_A(G) & \longrightarrow & 0 \end{array}$$

in which $\phi_M$ is an isomorphism by the 3-Lemma. Since the top-row splits, the same has to hold for the bottom, which contradicts its choice. Therefore, $G$ is $\aleph_1$-$A$-projective.

Consider an $A$-generated subgroup $C$ of $G$ such that $G/C$ is countable. We can find a countable subgroup $B$ such that $G = C + B$, and no generality is lost if we assume in addition that $B$ is $A$-generated. By what was shown in the last paragraph, $B$ is $A$-projective. We consider the exact sequence $0 \to K \to B \oplus C \xrightarrow{\pi} G \to 0$ with $\pi(b, c) = b + c$. Since $G$ is $A$-solvable, and $C$ is an $A$-generated subgroup of $G$, the group $B \oplus C$ is $A$-solvable. By Theorem 2.1, $K = \{(b, -b) \mid b \in B \cap C\}$ is $A$-generated, and hence $A$-solvable since $A$ is $E$-flat. Since $K$ is isomorphic to a subgroup of the countable $A$-projective group $B$, another application of Theorem 2.1 yields that $K$ is a countable $A$-projective group. By our hypotheses, the map $\pi$ splits, say $\pi\delta = 1_G$ for some homomorphism $\delta : G \to B \oplus C$. Let $\rho : B \oplus C \to B$ be the projection onto $B$ with kernel $C$, and consider $D = \ker(\rho\delta)$. Since $G/D$ is $A$-generated and isomorphic to a subgroup of the countable $A$-projective group $B$, it is $A$-projective itself. By Theorem 2.1, $D$ is a direct summand of $G$. Moreover, every $d \in D$ satisfies $\delta(d) = (0, c)$ for some $c \in C$ since $\rho\delta(d) = 0$ yields $\delta(d) \in \ker\ \rho = C$. Then $d = \pi\delta(d) = \pi(0, c) = c$, and $D \subseteq C$. □

A group $W$ is an *$A$-Whitehead group* if it admits an $A$-balanced exact sequence $0 \to U \to F \to W \to 0$ in which $F$ is $A$-projective and $U$ is $A$-generated with the property that

$$0 \to \mathrm{Hom}(W, A) \to \mathrm{Hom}(F, A) \to Hom(U, A) \to 0$$

is exact.

**Corollary 3.3.** *Let $A$ be a self-small countable torsion-free group with a right Noetherian right hereditary endomorphism ring.*

a) *Every $\aleph_1$-$A$-coseparable group $W$ is an $A$-Whitehead group.*
b) *It is consistent with ZFC that there exists a strongly $\aleph_1$-$A$-projective group $G$ which is not $\aleph_1$-$A$-coseparable.*

*Proof.* a) By [7], an $A$-solvable group $W$ is an $A$-Whitehead group if every exact sequence $0 \to A \to X \to W \to 0$ with $S_A(X) = X$ splits which is satisfied by $W$ because of Theorem 3.2.

b) If we assume $V = L$, then all $A$-Whitehead groups are $A$-projective. However, there exist strongly $\aleph_1$-$A$-projective group $G$ with $\mathrm{Hom}(G, A) = 0$ [7]. □

## 4. Strongly $\aleph_1$-$A$-Projective Groups and Martin's Axiom

We use the formulation of Martin's Axiom given in [13, Definition VI.4.2]. A partially ordered set $(P, \leq)$ satisfies the *countable chain condition (ccc)* if any antichain in $(P, \leq)$ is countable. An *antichain* is a subset $A$ of $P$ such that any two distinct members of A are *incompatible*, i.e., whenever $p, q \in A$, then there does not exist $r \in P$ such that $r \geq p$ and $r \geq q$. A subset $D$ of $P$ is dense if, for every $p \in P$ there is $q \in D$ such that $p \leq q$. Finally, a subset $\mathcal{F}$ of $P$ is directed, if, for all $p, q \in \mathcal{F}$, there is $r \in \mathcal{F}$ such that $r \geq p$ and $r \geq q$.

For a cardinal $\kappa$, let $\mathrm{MA}(\kappa)$ denote the statement:

Let $(P, \leq)$ be a partially ordered set satisfying the *countable chain condition (ccc)*. For every family $\mathcal{D} = \{D_\alpha \mid \alpha < \kappa\}$ of dense subsets of $P$, there is a directed subset $\mathcal{F}$ of $P$ such that $\mathcal{F} \cap D_\alpha \neq \emptyset$ for all $\alpha$, i.e. $\mathcal{F}$ is *$\mathcal{D}$-generic.*

Martin's axiom (MA) stipulates that $\mathrm{MA}(\kappa)$ holds for every $\kappa < 2^{\aleph_0}$ [13].

**Theorem 4.1.** *($MA + \aleph_1 < 2^{\aleph_0}$) Let $A$ be a self-small countable torsion-free group with a right Noetherian right hereditary endomorphism ring. If $G$ is a strongly $\aleph_1$-$A$-projective group and $0 \to U \to \oplus_I A \to G \to 0$ is an $A$-balanced exact sequence such that $S_A(U) = U$ and $|I| < 2^{\aleph_0}$, then the induced sequence*

$$0 \to \mathrm{Hom}(G, B) \to Hom(\oplus_I A, B) \to \mathrm{Hom}(G, B) \to 0$$

*is exact for all countable $A$-solvable group $B$.*

*Proof.* We consider an $A$-balanced exact sequence $0 \to U \to \oplus_I A \to G \to 0$ where $U \to \oplus_I A$ is the inclusion map. Let $\mathcal{P}(U)$ be the collection of $A$-generated $A$-pure subgroups $V$ of $F = \oplus_I A$ containing $U$ such that $V/U$ is finitely $A$-projective. Since $V$ is $A$-generated and $A$ is faithfully flat, $U$ is a direct summand of $V$ by Corollary 2.2, say $V = U \oplus R_V$ for some finitely $A$-projective group $R_V$.

To show that the sequence $Hom(\oplus_I A, B) \to \mathrm{Hom}(G, B) \to 0$ is exact whenever $B$ is a countable $A$-solvable group, let $\phi \in \mathrm{Hom}(U, B)$, and consider

$$P = \{(V, \psi) \mid V \in \mathcal{P}(U), \psi \in \mathrm{Hom}(V, B), \text{ and } \psi|U = \phi\}.$$

We partially order $P$ by $(V_1, \psi_1) \geq (V_2, \psi_2)$ if and only if $V_2 \subseteq V_1$ and $\psi_1|V_2 = \psi_2$. Once we have shown that $P$ and $\mathcal{D} = \{D(J)|J \subseteq I \text{ finite}\}$ satisfy the hypotheses of Martin's Axiom, then we can find a $\mathcal{D}$-generic directed directed subset $\mathcal{F}$ of $P$. Define a map $\psi : \oplus_I A \to B$ as follows. For $x \in \oplus_I A$, choose a finite subset $J$ of $I$ such that $x \in \oplus_J A$. Since $\mathcal{F}$ is $\mathcal{D}$-generic, there is $(V, \delta) \in D(J) \cap F$ with $x \in V$. Define $\psi(x) = \delta(x)$. Moreover, if $(V_1, \delta_1)$ and $(V_2, \delta_2)$ are two choices, then there is $(V_3, \delta_3) \in \mathcal{F}$ such that $(V_i, \delta_i) \leq (V_3, \delta_3)$ for $i = 1, 2$ since $\mathcal{F}$ is directed. Thus, $\delta_1(x) = \delta_3(x) = \delta_2(x)$. □

The key towards showing that $(P, \leq)$ satisfies the countable chain condition is

**Theorem 4.2.** *Every uncountable subset $P'$ of $P$ contains an uncountable subset $P''$ for which we can find an $A$-pure $A$-projective subgroup $X$ of $F$ containing $U$ as a direct summand such that $V \subseteq X$ whenever $(V, \psi) \in P''$.*

*Proof.* We may assume that $P' = \{(V_\nu, \psi_\nu)|\nu < \omega_1\}$. Since $U$ is a direct summand of $V_\nu$, we obtain that $H_A(V_\nu/U) \cong H_A(V_\nu)/H_A(U)$ is a finitely generated right $E$-module. In particular, it has finite right Goldie dimension since $E$ is right Noetherian. Therefore, no generality is lost if we assume that there is $m < \omega$ such that $H_A(V_\nu/U)$ has Goldie dimension $m$ for all $\nu < \omega_1$.

Let $0 \leq k \leq m$ be maximal with respect to the property that there exists an $A$-pure $A$-projective subgroup $T$ of $F$ containing $U$ such that $H_A(T/U)$ has Goldie dimension $k$ and $T$ is contained in uncountable many $V_\nu$. This $k$ exists since $U$ is the choice for $T$ in the case $k = 0$. Observe that $T/U$ is $A$-solvable as an $A$-generated subgroup of the $A$-solvable group $G = F/U$. Thus, $0 \to U \to T \to T/U \to 0$ is $A$-balanced, and $H_A(T/U) \cong H_A(T)/H_A(U)$ has finite Goldie-dimension and is non-singular. Thus, it contains a finitely generated essential submodule. Since $E$ is right Noetherian and countable, any essential extension of a non-singular finite dimensional right $E$-module is countable by Proposition 2.3. In particular, $H_A(T/U)$ is countable, and hence $T/U \cong T_A H_A(T/U)$ is countable. Since $G$ is $\aleph_1$-$A$-projective, $T/U$ is $A$-projective, and $T = U \oplus W$ because $A$ is faithfully flat by Corollary 2.2.

Suppose that $T'$ is an $A$-generated subgroup of $F$ containing $T$ such that $T \neq T'$. There exists $\alpha \in H_A(T')$ with $\alpha(A) \not\subseteq T$. Since $T$ is $A$-pure in $F$, we obtain $T + \alpha(A) = T \oplus C$ with $C \neq 0$. Thus, the Goldie-dimension of $H_A(T')$ is at least $k+1$, and $T'$ is contained in only countably many of the $V_\nu$. No generality is lost if we assume that $T$ is contained in $V_\nu$ for all $\nu$. Since $T$ is $A$-pure in $F$ and $V_\nu = U \oplus R_{V_\nu} = T + R_{V_\nu}$ for some finitely $A$-projective subgroup $R_{V_\nu}$ of $F$, we obtain decompositions $V_\nu = T \oplus W_\nu$. Observe that $W_\nu$ is finitely $A$-projective.

We construct $X$ as the union of a smooth ascending chain $\{X_\nu|\nu < \omega_1\}$ of $A$-pure $A$-projective subgroups of $F$ containing $T$ and an ascending chain of ordinals $\{\sigma_\nu|\nu < \omega_1\}$ such that $X_{\nu+1}/X_\nu$ is $A$-projective, $W_{\sigma_{\nu+1}} \subseteq X_{\nu+1}$, and $X_\nu/U$ is an image of $\oplus_\omega A$ for all $\nu < \omega_1$. We set $X_0 = T$, and $X_\alpha = \cup_{\nu<\alpha} X_\nu$ if $\alpha$ is a limit ordinal. Then, $X_\alpha/U$ is a countable subgroup of $F/U$, and hence $A$-projective. Set $\sigma_\alpha = sup(\sigma_\nu|\nu < \alpha)$.

If $\alpha = \nu + 1$, then there exists a subgroup $C_\nu$ of $F$ containing $X_\nu$ such that the group $C_\nu/U$ is an $A$-projective countable $\aleph_1$-$A$-closed subgroup of $F/U$ since $F/U$ is strongly $\aleph_1$-$A$-projective. In particular, $F/C_\nu \cong (F/U)/(C_\nu/U)$ is $A$-solvable. Since $A$ is flat, $C_\nu$ is $A$-generated by Theorem 2.1. If $K$ is a countable $A$-generated subgroup of $F$, then $(K+U)/U$ is a countable subgroup of $F/U$. Hence,

$$(K + C_\nu)/C_\nu \cong [(K + C_\nu)/U]/[C_\nu/U]$$

is $A$-projective.

To construct $\sigma_\alpha$, assume $W_\mu \cap C_\nu \neq 0$ for all $\mu > \sigma_\nu$. Then,

$$W_\mu/(W_\mu \cap C_\nu) \cong (W_\mu + C_\nu)/C_\nu$$

is $A$-projective by the last paragraph. Since $A$ is faithfully flat, $W_\mu \cap C_\nu$ is $A$-generated, and there is a map $0 \neq \alpha_\mu \in H_A(W_\mu \cap C_\nu) \subseteq H_A(C_\nu)$. Since $C_\nu/U$ is a countable subgroup of $F/U$, it is $A$-projective, and $C_\nu = U \oplus P_\nu$ since $A$ is faithfully flat. Observe that $P_\nu$ is countable and $A$-projective. Write $\alpha_\mu = \beta_\mu + \epsilon_\mu$ with $\beta_\mu \in H_A(U)$ and $\epsilon_\mu \in H_A(P_\nu)$. Since $E$ is countable, the same holds for $H_A(P_\nu)$, and there is

$\epsilon \in H_A(P_\nu)$ such that $\epsilon_\mu = \epsilon$ for uncountably many $\mu$. For all these $\mu$, we have $\epsilon(A) \subseteq W_\mu + U \subseteq V_\mu$. Hence, $T + \epsilon(A) \subseteq V_\mu$ for uncountably many $\mu$. However, this is only possible if $\epsilon(A) \subseteq T$. But then, $\alpha_\mu(A) \subseteq T \cap W_\mu = 0$, a contradiction.

Therefore, we can find an ordinal $\sigma_\alpha > \sigma_\nu$ with $C_\nu \cap W_{\sigma_\alpha} = 0$. In particular, $X_\nu \subseteq C_\nu$ yields $X_\nu \cap W_{\sigma_\alpha} = 0$. Let $Y$ be the $S$-closure of

$$H_A(X_\nu \oplus W_{\sigma_\alpha}) = H_A(X_\nu \oplus H_A(W_{\sigma_\alpha}) \supseteq H_A(U)$$

in $H_A(F)$ and let $X_\alpha = \theta_F(Y \otimes A) = YA$. As an $A$-generated subgroup of $F$, $X_\alpha$ is $A$-solvable. Then, the inclusion $Y \subseteq H_A(X_\alpha)$ induces the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_A(Y) & \longrightarrow & T_AH_A(X_\alpha) & \longrightarrow & T_A(H_A(X_\alpha)/Y) & \longrightarrow & 0 \\ & & \wr\big\downarrow \theta_F|T_A(Y) & & \wr\big\downarrow \theta_{X_\alpha} & & & & \\ 0 & \longrightarrow & YA & \xrightarrow{1_{YA}} & YA & \longrightarrow & 0 & & \end{array}$$

from which we get $T_A(H_A(X_\alpha)/Y) = 0$. Since $A$ is faithfully flat, $Y = H_A(X_\alpha)$, and $H_A(X_\alpha)/[H_A(X_\nu) \oplus H_A(W_{\sigma_\alpha})]$ is singular.

Observe that $Y/H_A(U)$ is the $S$-closure of $[H_A(X_\nu) + H_A(W_{\sigma_\alpha})]/H_A(U)$ in $H_A(F)/H_A(U)$ because

$$H_A(F)/Y \cong [H_A(F)/H_A(U)]/[Y/H_A(U)]$$

is non-singular and

$$Y/H_A(X_\nu \oplus W_{\sigma_\alpha}) \cong [Y/H_A(U)]/[H_A((X_\nu \oplus W_{\sigma_\alpha})/H_A(U)]$$

is singular. Since $F/U$ is $A$-solvable, and $X_\nu/U$ is countable, $H_A(X_\nu)/H_A(U)$ is countable. Moreover, $W_\mu$ is finitely $A$-projective. Hence, the $E$-module $H_A(W_{\sigma_\alpha})$ is countable too, and

$$[H_A(X_\nu) + H_A(W_{\sigma_\alpha})]/H_A(U)$$

is countable. Thus, $Y/H_A(U)$ is an essential extension of a countable non-singular right $E$-module. By Proposition 2.3, we obtain that $Y/H_A(U)$ is countable. Thus, there is a countable submodule $Y'$ of $Y$ with $Y = Y' + H_A(U)$. Then $X_\alpha/U$ is countable and $X_\alpha = Y'A + X_\nu$, and. Consequently, $X_\alpha/U$ has to be $A$-projective, and the same holds for $X_\alpha \cong X_\alpha/U \oplus U$.

It remains to show that $X_\alpha/X_\nu$ is $A$-projective. For this, observe that the group

$$X_\alpha/(X_\alpha \cap C_\nu) \cong (X_\alpha + C_\nu)/C_\nu$$

is countable since it is an epimorphic image of $(X_\alpha + C_\nu)/U$ which is countable because $X_\alpha$ and $C_\nu/U$ are countable. Since $C_\nu/U$ is $\aleph_1$-$A$-closed in $F/U$, we have that

$$X_\alpha/(X_\alpha \cap C_\nu) \cong [(X_\alpha + C_\nu)/U]/[C_\nu/U]$$

is $A$-projective. Since $A$ is flat, $X_\alpha \cap C_\nu$ is $A$-generated as in Theorem 2.1. For $\tau$ in $H_A(X_\alpha \cap C_\nu)$, choose a regular element $c \in E$ such that $\tau c \in H_A(X_\nu) \oplus H_A(W_{\sigma_\alpha})$, say $\tau c = \beta + \gamma$ for some $\beta \in H_A(X_\nu)$ and $\gamma \in H_A(W_{\sigma_\alpha})$. Then

$$\gamma = \tau c - \beta \in H_A(W_{\sigma_\alpha}) \cap H_A(C_\nu) = 0.$$

Hence, $\tau c \in H_A(X_\nu)$. Since $X_\nu$ is $A$-pure in $F$, we obtain $\tau \in H_A(X_\nu)$. Therefore, $H_A(X_\alpha \cap C_\nu) \subseteq H_A(X_\nu)$, and $X_\alpha \cap C_\nu \subseteq X_\nu$. Since $X_\nu$ is contained in $X_\alpha$ and in

$C_\nu$, we obtain $X_\alpha \cap C_\nu = X_\nu$. Then $X_\alpha/X_\nu \cong (X_\alpha + C_\nu)/C_\nu$ is $A$-projective by what we have already shown. In particular, $X_\nu$ is a direct summand of $X_\alpha$.

Consequently, $X = \cup_{\nu<\omega_1} X_\nu$ is $A$-pure and $A$-projective. Because

$$X_{\nu+1}/X_\nu \cong [X_{\nu+1}/T]/[X_\nu/T]$$

is $A$-projective for all $\nu$, the group $X/T$ is $A$-projective. This yields $X = T \oplus S$. However, $T = U \oplus W$, so that $X = U \oplus W \oplus S$. Finally,

$$V_{\sigma_{\nu+1}} = T \oplus W_{\sigma_{\nu+}} \subseteq X_{\nu+1} \subseteq X$$

for all $\nu < \omega_1$. Let $P'' = \{(V_{\sigma_{\nu+1}}, \psi_{\sigma_{\nu+1}})|\nu < \omega_1\}$. □

**Corollary 4.3.** *$P$ satisfies the countable chain condition.*

*Proof.* Since $B$ is a countable $A$-solvable group, there is an exact sequence

$$0 \to V \to \oplus_\omega A \to B \to 0$$

which is $A$-balanced by Theorem 2.1. Thus, $H_A(B)$ is countable as an epimorphic image of $H_A(\oplus_\omega A) \cong \oplus_\omega E$ using the self-smallness of $A$.

Let $P'$ be an uncountable subset of $P$. By the previous Lemma, we may assume $P' = \{(V_\nu, \psi_\nu)|\nu < \omega_1\}$ such that there is an $A$-pure $A$-projective subgroup $X$ containing $U$ as a direct summand satisfying $V_\nu \subseteq X$ for all $\nu < \omega_1$. We can write $X = U \oplus Y$ and $Y = \oplus_J Y_j$ where each $Y_j$ is isomorphic to a subgroup of $A$. This is possible since $E$ is hereditary.

For $\nu < \omega_1$, we have $V_\nu = U \oplus (Y \cap V_\nu)$. Since $Y \cap V_\nu$ is finitely $A$-projective, there is a finite subset $J_\nu$ of $J$ such that $H_A(Y \cap V_\nu) \subseteq H_A(\oplus_{J_\nu} Y_j)$, and $Y \cap V_\nu \subseteq \oplus_{J_\nu} Y_j$. Therefore, $V_\nu$ is an $A$-pure subgroup of

$$V_\nu + (\oplus_{J_\nu} Y_j) = U \oplus\ (\oplus_{J_\nu} Y_j).$$

Because $\oplus_{J_\nu} Y_j$ is finitely $A$-generated, $V_\nu$ is a direct summand of $U \oplus (\oplus_{J_\nu} Y_j)$, say $V_\nu + (\oplus_{J_\nu} Y_j) = V_\nu \oplus X_\nu$. Since $V_\nu + (\oplus_{J_\nu} Y_j)$ is $A$-projective, the same holds for $X_\nu$. Thus, $X_\nu$ is isomorphic to a direct summand of $\oplus_{J_\nu} Y_j$. Moreover, $\psi_\nu : V_\nu \to B$ extends to a map $\lambda_\nu : U \oplus (\oplus_{J_\nu} Y_j) \to B$. By the Adjoint-Functor-Theorem,

$$\mathrm{Hom}(\oplus_{J_\nu} Y_j, B) \cong \mathrm{Hom}_E(H_A(\oplus_{J_\nu} Y_j), H_A(B))$$

is countable since $H_A(B)$ is countable as was shown in the first paragraph of the proof and $J_\nu$ is finite. Consequently, there are at most countably many different extensions of $\phi$ to $U \oplus (\oplus_{J_\nu})$.

If there are only countably many different $J_\nu$'s, then there is $\nu_0$ such that $J_{\nu_0} = J_\mu$ for uncountable $\mu$. Thus, there are $\mu_1$ and $\mu_2$ with $J_{\nu_0} = J_{\mu_1} = J_{\mu_2}$ and $\lambda_{\mu_1} = \lambda_{\mu_2}$. Thus, $\psi_{\mu_1}$ and $\psi_{\mu_2}$ have a common extension. Therefore, $P' = \{(V_\nu, \psi_\nu)|\nu < \omega_1\}$ cannot be an antichain. On the other hand, if there are uncountably many $J_\nu$'s, then we may assume without loss of generality that $J_\nu \neq J_\mu$ for $\mu \neq \nu$. Finally, we can impose the requirement that all the $J_\nu$ have the same order. Thus, $J_\nu$ cannot be contained in $J_\mu$ for $\mu \neq \nu$. Since $(V_\nu, \psi_\nu) \leq (V_\nu \oplus X_\nu, \lambda_\nu)$, we may assume that $V_\nu = U \oplus (\oplus_{J_\nu} Y_j)$ and $\lambda_\nu = \psi_\nu$.

There is a subset $T$ of $J$ which is maximal with respect to the property that it is contained in uncountably many of the $J_\nu$. We may assume that $T$ is actually contained in all of the $J_\nu$. Observe that $T$ is finite and a proper subset of all the $J_\nu$. Otherwise, all the $J_\nu$ would have to coincide with $T$ since they have the same finite order. Since $\mathrm{Hom}(\oplus_T Y_j, B) \cong \mathrm{Hom}_E(H_A(\oplus_T Y_j), H_A(B))$ is countable by the Adjoint-Functor-Theorem, there are uncountably many $\psi_\nu$ which have the same restriction to $W = U \oplus (\oplus_T Y_j)$. Without loss of generality, we may assume that this happens for all $\nu$.

Let $j \in J_0 \setminus T$. The maximality of $T$ guarantees that $j$ is contained in only countably many of the $J_\nu$. Since $J_0 \setminus T$ is finite, there is $\mu < \omega_1$ with $J_\mu \cap J_0 = T$. The maps $\psi_\mu$ and $\psi_0$ have a common extension $\sigma : U \oplus (\oplus_{J_0 \cup J_\nu} Y_j) \to B$ since they coincide on $W$. Since $U \oplus (\oplus_{J_0 \cup J_\nu} Y_j)$ is a direct summand of $X$, and $X$ is $A$-pure in $F$, we have that $U \oplus (\oplus_{J_0 \cup J_\nu} Y_j)$ is $A$-pure in $F$. Because $J_0 \cup J_\nu$ is finite,

$$(U \oplus (\oplus_{J_0 \cup J_\nu} Y_j), \sigma) \in P.$$

Thus,

$$(U \oplus (\oplus_{J_0 \cup J_\nu} Y_j), \sigma) \geq (V_\mu, \psi_\mu), (V_0, \psi_0).$$

Consequently, $P'$ cannot be an anti-chain. □

For every finite subset $J$ of $I$, let $D(J) = \{(V, \psi) \in P | \oplus_J A \subseteq V\}$.

**Proposition 4.4.** *$P$ and $\mathcal{D} = \{D(J) | J \subseteq I$ finite$\}$ satisfy the hypotheses of Martin's Axiom.*

*Proof.* By Corollary 4.3, it remains to show that $D(J)$ is dense in $P$. For this, let $(V, \psi) \in P$. We have to find $(W, \alpha) \in P$ such that $\oplus_J A$ and $V$ are contained in $W$ and $\alpha | V = \psi$. Since $V/U$ is finitely $A$-projective and $G \cong F/U$ is strongly $\aleph_1$-$A$-projective, there is a subgroup $X$ of $F$ containing $V$ and $\oplus_J A$ such that $X/U$ is a $\aleph_1$-$A$-closed, $A$-projective countable subgroup of $F/U$. Since

$$[F/U]/[X/U] \cong F/X$$

is $\aleph_1$-$A$-projective, it is $A$-solvable by Theorem 2.1. Using the same result once more, we obtain that the sequence $0 \to X \to F \to F/X \to 0$ is $A$-balanced. In particular, $S_A(X) = X$ and $X$ is $A$-projective. Moreover,

$$H_A(F)/H_A(X) \cong H_A(F/X) \cong H_A([F/U]/[X/U]) \cong H_A(F/U)/H_A(X/U)$$

since $X$ in $F$ and $X/U$ in $F/U$ are $A$-balanced by the faithful flatness of $A$. But the latter is non-singular, since $[F/U]/[X/U]$ is $\aleph_1$-$A$projective. Therefore, $X$ is $A$-pure in $F$.

Since the group $X/U$ is $A$-projective, we have a decomposition $X = U \oplus P$. Hence, $V = U \oplus (V \cap P)$ and $V \cap P$ is finitely $A$-projective. In the same way,

$$(\oplus_J A) + U = U \oplus [((\oplus_J A) + U) \cap P]$$

yields that $((\oplus_J A) + U) \cap P$ is $A$-generated and an image of $\oplus_J A$.
Therefore, $((\oplus_J A) + U) \cap P$ and $V \cap P$ are finitely $A$-projective subgroups of $P$. Thus, $H_A(((\oplus_J A) + U) \cap P)$ and $H_A(V \cap P)$ are finitely generated submodule of $H_A(P)$. Since $E$ is right hereditary, $H_A(P)$ is a direct sum of right ideals of $E$, which yields that $H_A(((\oplus_J A) + U) \cap P)$ and $H_A(V \cap P)$ are contained in a finitely generated direct summand of $H_A(P)$. Hence, there is a finitely $A$-projective summand $D$ of $P$

which contains $V \cap P$ and $((\oplus_J A) + U) \cap P$. Since $U \oplus D = V + D$ and $V$ is $A$-pure in $F$, we obtain that $V$ is a direct summand of $U \oplus D$. Thus, $\psi$ extends to a map $\alpha : U \oplus D \to B$. Clearly, $(U \oplus D, \alpha) \in P$ and $(U \oplus D, \alpha) \geq (V, \psi)$. □

An $A$-generated group $G$ $\aleph_1$-*$A$-separable* if every countable subset of $G$ is contained in an $A$-projective direct summand of $G$.

**Corollary 4.5.** $(MA + \aleph_1 < 2^{\aleph_0})$ *If $A$ is a self-small countable torsion-free group with a right Noetherian right hereditary endomorphism ring, then every strongly $\aleph_1$-$A$-projective group is $\aleph_1$-$A$-separable and $\aleph_1$-$A$-coseparable.*

*Proof.* By Theorem 3.2 and Theorem 4.1, a strongly $\aleph_1$-$A$-projective group $G$ is $\aleph_1$-$A$-coseparable. It remains to show that is $\aleph_1$-$A$-separable too. For a countable subset $X$ of $G$ select a countable $\aleph_1$-$A$-closed subgroup $U$ of $G$ containing $X$. Since $G/U$ is strongly $\aleph_1$-$A$-projective, the sequence $0 \to U \to G \to G/U \to 0$ splits by Theorem 4.1. □

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Ulrich Albrecht
Department of Mathematics
Auburn University
Auburn, AL 36849, U.S.A.
e-mail: `albreuf@auburn.edu`