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Multiple Stackelberg variational responses

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Abstract. Contrary to the standard literature (where the Stackelberg response function is single-valued), we provide a whole class of functions to show that the Stackelberg variational response set may contain at least three elements.

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1. Introduction

The Stackelberg duopoly is a game in which the leader moves first and the follower moves sequentially. In the usual Nash competition, however, the two players are competing with each other at the same level. The Stackelberg model can be handled by the backward induction method, i.e., we find the best response for the follower (by considering the strategy action of the leader as a parameter) and then choose the best strategy of the leader.

We assume in the sequel that the strategies of both players are some sets $K_1, K_2 \subset \mathbb{R}^m$. Let $f : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ be the payoff function of the *f*ollower player. The first step is to determine for every fixed $x_1 \in K_1$ the *Stackelberg equilibrium response set*, defined by

$$R_{SE}(x_1) = \{x_2 \in K_2 : f(x_1, y) - f(x_1, x_2) \geq 0, \ \forall y \in K_2\}.$$

Now, assuming that $R_{SE}(x_1) \neq \emptyset$ for every $x_1 \in K_1$, the concluding step (for the leader) is to minimize the map $x \mapsto l(x, r(x))$ on K_1 where r is a suitable selection of the set-valued map R_{SE}.

The main objective is to locate the elements of the Stackelberg equilibrium response set. In order to do that, we introduce a larger set by means of variational inequalities. For simplicity, we assume that the follower's payoff function $f : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ has the property that $f(x_1, \cdot)$ is a locally Lipschitz function for every $x_1 \in K_1$. Now, we introduce the so-called *Stackelberg variational response set* defined by

$$R_{SV}(x_1) = \left\{x_2 \in K_2 : f^0_{x_2}((x_1, x_2); y - x_2) \geq 0, \ \forall y \in K_2\right\},$$

where $f^0_{x_2}((x_1,x_2);v)$ is the generalized directional derivative of $f(x_1,\cdot)$ at the point $x_2 \in K_2$ in the direction $v \in \mathbb{R}^m$. It is clear that

$$R_{SE}(x_1) \subseteq R_{SV}(x_1).$$

Usually, the standard literature provides examples where the set $R_{SV}(x_1)$ is a singleton, see A. Kristály and Sz. Nagy [4], Sz. Nagy [7] and the monograph by A. Kristály, V. Rădulescu and Cs. Varga [5] for functions of class C^1. However, as expected, one can happen to have examples where this set contains several elements. In fact, this is precisely the aim of the paper to provide a whole class of functions with the latter property.

We focus our attention to a specific payoff function for the follower player; namely, we assume that $f : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is given by

$$f_\lambda(x_1,x_2) := f(x_1,x_2) = \frac{1}{2}\|x_2\|^2 - \lambda\tilde{f}(x_1,x_2) + \delta_{K_2}(x_2), \tag{1.1}$$

where $K_2 \subset \mathbb{R}^m$ is a non-empty, closed, non-compact set, $\lambda > 0$ is a parameter and $\tilde{f}(x_1,\cdot)$ is locally Lipschitz for every $x_1 \in \mathbb{R}^m$. As usual, δ_{K_2} denotes the indicator function of the set K_2.

Let $x_1 \in \mathbb{R}^m$ be arbitrarily fixed. On the locally Lipschitz function $\tilde{f}(x_1,\cdot)$ we assume:

($H^1_{x_1}$) $\max\{\|z\| : z \in \partial_{x_2}\tilde{f}(x_1,x_2)\} = o(\|x_2\|)$ whenever $\|x_2\| \to 0$;

($H^2_{x_1}$) $\max\{\|z\| : z \in \partial_{x_2}\tilde{f}(x_1,x_2)\} = o(\|x_2\|)$ whenever $\|x_2\| \to +\infty$;

($H^3_{x_1}$) $\tilde{f}(x_1,0) = 0$ and there exists $\tilde{x}_2 \in K_2$ such that $\tilde{f}(x_1,\tilde{x}_2) > 0$.

Here, $o(\cdot)$ is the usual Landau symbol.

Remark 1.1. (a) Hypotheses ($H^1_{x_1}$) and ($H^2_{x_1}$) mean that $\partial_{x_2}\tilde{f}(x_1,\cdot)$ is *superlinear* at the origin and *sublinear* at infinity, respectively. Hypothesis ($H^3_{x_1}$) implies that $\tilde{f}(x_1,\cdot)$ is not identically zero.

(b) According to hypotheses ($H^1_{x_1}$) and ($H^2_{x_1}$), the number

$$\tilde{c} = \max_{x_2 \in \mathbb{R}^m \setminus \{0\}} \frac{\max\{\|z\| : z \in \partial_{x_2}\tilde{f}(x_1,x_2)\}}{\|x_2\|} \tag{1.2}$$

is well-defined, finite, and from the upper semicontinuity of $\partial_{x_2}\tilde{f}(x_1,\cdot)$ and hypothesis ($H^3_{x_1}$), we have $0 < \tilde{c} < \infty$.

(c) We also introduce the number

$$\tilde{\lambda} = \frac{1}{2} \inf_{\substack{\tilde{f}(x_1,x_2)>0 \\ x_2 \in K_2}} \frac{\|x_2\|^2}{\tilde{f}(x_1,x_2)}, \tag{1.3}$$

which is well-defined, finite and $0 < \tilde{\lambda} < \infty$. The discussion on this number is postponed to Proposition 4.2.

Note that the Stackelberg variational response set for the function f_λ in (1.1) is given by

$$R^\lambda_{SV}(x_1) = \left\{x_2 \in K_2 : \langle x_2, y - x_2\rangle + \lambda\tilde{f}^0_{x_2}((x_1,x_2); -y + x_2) \geq 0,\ \forall y \in K_2\right\}.$$

The main theorem of our paper is the following.

Theorem 1.2. *Let $K_i \subset \mathbb{R}^m$ be two convex sets ($i = 1, 2$), and let $f_\lambda : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ be the follower payoff function of the form* (1.1) *such that $\tilde{f}(x_1, \cdot)$ is locally Lipschitz for every $x_1 \in K_1$. Assume that K_2 is closed and non-compact such that $0 \in K_2$. Fix $x_1 \in K_1$ and assume that the hypotheses ($H^i_{x_1}$) hold true, $i \in \{1, 2, 3\}$. Then the following statements hold:*

(a) $0 \in R^\lambda_{SV}(x_1)$ *for every $\lambda > 0$;*
(b) $R^\lambda_{SV}(x_1) = \{0\}$ *for every $\lambda \in (0, \tilde{c}^{-1})$, where $\tilde{c}$ is from* (1.2)*;*
(c) $\mathrm{card}(R^\lambda_{SV}(x_1)) \geq 3$ *for every $\lambda > \tilde{\lambda} > 0$, where $\tilde{\lambda}$ is from* (1.3)*.*

Remark 1.3. By the conclusions of Theorem 1.2 (b) and (c) it is clear that

$$\tilde{c}^{-1} < \tilde{\lambda}.$$

At this moment, we have no precise information what can be said about Stackelberg responses in the gap-interval $[\tilde{c}^{-1}, \tilde{\lambda}]$; in fact, this will be the subject of Section 5.

In the sequel we provide an application.

Example 1.4. Let $K_2 = [0, \infty)$ and $\tilde{f} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$\tilde{f}(x_1, x_2) = (1 + |x_1|) \left(\min \left(8x_2^3, (x_2 + 3)^{\frac{3}{2}} \right) \right)_+,$$

where $s_+ = \max(s, 0)$. A simple calculation shows that

$$\partial_{x_2} \tilde{f}(x_1, x_2) = \begin{cases} \{0\}, & \text{if} \quad x_2 < 0; \\ \left\{24(1 + |x_1|)x_2^2\right\}, & \text{if} \quad x_2 \in [0, 1); \\ [3(1 + |x_1|), 24(1 + |x_1|)], & \text{if} \quad x_2 = 1; \\ \left\{ \frac{3}{2}(1 + |x_1|)(x_2 + 3)^{\frac{1}{2}} \right\}, & \text{if} \quad x_2 > 1. \end{cases}$$

Now, hypotheses ($H^1_{x_1}$) and ($H^2_{x_1}$) hold since

$$\lim_{x_2 \searrow 0} \frac{24(1 + |x_1|)x_2^2}{x_2} = \lim_{x_2 \to \infty} \frac{\frac{3}{2}(1 + |x_1|)(x_2 + 3)^{\frac{1}{2}}}{x_2} = 0.$$

Hypothesis ($H^3_{x_1}$) holds since

$$\tilde{f}(x_1, 0) = 0 < \tilde{f}(x_1, 1) = 8(1 + |x_1|).$$

Let $x_1 \in \mathbb{R}$ be fixed. We notice that $\tilde{c} = 24(1+|x_1|)$ and $\tilde{\lambda} = \frac{1}{16(1+|x_1|)}$. According to Theorem 1.2, only the zero solution is given for $\lambda \in (0, \frac{1}{24(1+|x_1|)})$, while for $\lambda > \frac{1}{16(1+|x_1|)}$ there are three solutions for the inclusion

$$x_2 \in \lambda \partial_{x_2} \tilde{f}(x_1, x_2), \; x_2 \geq 0, \tag{1.4}$$

which is equivalent to $x_2 \in R^\lambda_{SV}(x_1)$.

For λ large enough we solve the inclusion (1.4), obtaining that $R^\lambda_{SV}(x_1)$ contains exactly three elements; namely, $R^\lambda_{SV}(x_1) = \{0, x_2^\lambda, y_2^\lambda\}$ where

$$x_2^\lambda = \frac{9\lambda^2(1 + |x_1|)^2 + 3\lambda(1 + |x_1|)\sqrt{9\lambda^2(1 + |x_1|)^2 + 48}}{8}$$

and

$$y_2^\lambda = \frac{1}{24\lambda(1+|x_1|)}.$$

After a simple computation we conclude that the Stackelberg equilibrium response set is $R_{SE}^\lambda(x_1) = \{x_2^\lambda\}$ whenever λ is large. $\square$

The paper has the following structure. In the next section we recall some notions and results from non-smooth analysis for Lipschitz functions and critical point theory for Motreanu-Panagiotopoulos functionals. In Section 3 the proof of Theorem 1.2 (a) and (b) is provided while Section 4 is devoted the proof of Theorem 1.2 (c). Finally, the last section is devoted to the study of the gap-interval.

2. Preliminaries

Let X be a real Banach space and $U \subset X$ an open subset.

Definition 2.1. (F.H. Clarke [3]) *A function $f : U \to \mathbb{R}$ is called* locally Lipschitz *if every point $x \in U$ possesses a neighborhood $N_x \subset U$ such that*

$$|f(x_1) - f(x_2)| \le K\|x_1 - x_2\|, \quad \forall x_1, x_2 \in N_x,$$

for a constant $K > 0$ depending on N_x.

Definition 2.2. (F.H. Clarke [3]) *The* generalized directional derivative *of the locally Lipschitz function $f : U \to \mathbb{R}$ at the point $x \in U$ in the direction $v \in X$ is defined by*

$$f^0(x; v) = \limsup_{\substack{w \to x \\ t \searrow 0}} \frac{1}{t}(f(w + tv) - f(w)).$$

The following result presents some important properties of the generalized directional derivative.

Proposition 2.3. (D. Motreanu and P.D. Panagiotopoulos [6]) *Let $f : U \to \mathbb{R}$ be a locally Lipschitz function. Then we have:*

(a) *For every $x \in U$ the function $f^0(x;\ \cdot) : X \to \mathbb{R}$ is positively homogeneous and subadditive (therefore convex) and satisfies*

$$|f^0(x; v)| \le K\|v\|, \quad \forall v \in X. \tag{2.1}$$

Moreover, it is Lipschitz continuous on X with the Lipschitz constant K, where $K > 0$ is a Lipschitz constant of f near x.

(b) $f^0(\cdot;\cdot) : U \times X \to \mathbb{R}$ *is upper semicontinuous.*

(c) $f^0(x; -v) = (-f)^0(x; v), \quad \forall x \in U,\ \forall v \in X.$

Definition 2.4. *The* generalized gradient *of f at the point $x \in X$ is defined by*

$$\partial f(x) = \{x^* \in X^* :\ \langle x^*, v\rangle \le f^0(x; v) \ \ \textit{for each } v \in X\}.$$

By using the Hahn-Banach theorem it follows that the set $\partial f(x) \ne \emptyset$ for every $x \in U$. Some important properties of the generalized gradient are collected below.

Proposition 2.5. (F.H. Clarke [3], D. Motreanu and P.D. Panagiotopoulos [6]) *Let $f : U \to \mathbb{R}$ be a locally Lipschitz function. We have:*

(a) *For every* $x \in U$, $\partial f(x)$ *is a nonempty, weak***-compacts and convex subset of* $X^\star$ *which is bounded by the Lipschitz constant* $K > 0$ *of* f *near* x.
(b) *For every* $\lambda \in \mathbb{R}$ *and* $x \in U$ *one has* $\partial(\lambda f)(x) = \lambda \partial f(x)$.
(c) *If* $g : U \to \mathbb{R}$ *is another locally Lipschitz function then for every* $x \in U$, *one has* $\partial(f + g)(x) \subset \partial f(x) + \partial g(x)$.
(d) *For every* $x \in U$, $f^0(x; \cdot)$ *is the support function of* $\partial f(x)$, *i.e.,* $f^0(x; v) = \max\{\langle z,\ v\rangle :\ z \in \partial f(x)\},\ \forall\, v \in X$.
(e) (Upper semicontinuity) *The set-valued map* $\partial f : U \to 2^{X^\star}$ *is weakly***-closed, that is, if* $\{x_n\} \subset U$ *and* $\{z_n\} \subset X^*$ *are sequences such that* $x_n \to x$ *strongly in* X *and* $z_n \in \partial f(x_n)$ *with* $z_n \rightharpoonup z$ *weakly** *in* X^*, *then* $z \in \partial f(x)$. *In particular, if* X *is finite dimensional, then* ∂f *is upper semicontinuous.*
(f) (Lebourg's mean value theorem) *If* $x, y \in U$ *are two points such that* $[x, y] \subset U$ *then there exists a point* $z \in [x, y] \setminus \{x, y\}$ *such that for some* $z^* \in \partial f(z)$ *the following relation is satisfied:*

$$f(y) - f(x) = \langle z^*, y - x\rangle.$$

Let $E : X \to \mathbb{R}$ be a locally Lipschitz function and let $\zeta : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, lower semicontinuous function. Then, $I = E + \zeta$ is called *Motreanu-Panagiotopoulos-type functional*, see [6]. In particular, if E is of class C^1, the functional I is a *Szulkin-type functional*, see A. Szulkin [8].

Definition 2.6. (D. Motreanu and P.D. Panagiotopoulos [6, p.64]) *An element* $x \in X$ *is called a* critical point of $I = E + \zeta$ *if*

$$E^0(x; v - x) + \zeta(v) - \zeta(x) \geq 0 \quad \text{for all } v \in X. \tag{2.2}$$

The number $I(x)$ *is a* critical value *of* I.

Remark 2.7. We notice that an equivalent formulation for (2.2) is

$$0 \in \partial E(x) + \partial_C \zeta(x) \quad \text{in} \quad X^\star, \tag{2.3}$$

where ∂_C denotes the subdifferential in the sense of convex analysis.

Proposition 2.8. *Every local minimum point of* $I = E + \zeta$ *is a critical point of* I *in the sense of* (2.2).

Definition 2.9. (D. Motreanu and P.D. Panagiotopoulos [6, p.64]) *The functional* $I = E + \zeta$ *satisfies the* Palais-Smale condition at level $c \in \mathbb{R}$, (*shortly,* $(PS)_c$*-condition*) *if every sequence* $\{x_n\} \subset X$ *such that* $\lim_{n\to\infty} I(x_n) = c$ *and*

$$E^0(x_n; v - x_n) + \zeta(v) - \zeta(x_n) \geq -\varepsilon_n \|v - x_n\| \quad \textit{for all } v \in X,$$

where $\varepsilon_n \to 0$, *possesses a convergent subsequence.*

Remark 2.10. When $\zeta = 0$, $(PS)_c$-condition is equivalent to the $(PS)_c$-condition introduced by K.-C. Chang [2]. In particular, if E is of class C^1 and $\zeta = 0$, the $(PS)_c$-condition from Definition 2.9 reduces to the standard Palais-Smale condition.

Theorem 2.11. *Let* X *be a Banach space,* $I = E + \zeta : X \to \mathbb{R} \cup \{+\infty\}$ *a Motreanu-Panagiotopoulos-type functional which is bounded from below. If* $I = E + \zeta$ *satisfies the Palais-Smale condition at level* $c = \inf_{x\in X} I(x)$, *then* $c \in \mathbb{R}$ *is a critical value of* I.

We conclude this section by recalling a non-smooth version of the Mountain Pass theorem (initially established by A. Ambrosetti and P. Rabinowitz [1] for C^1 functionals):

Theorem 2.12. (D. Motreanu and P.D. Panagiotopoulos [6, p. 77]) *Let X be a Banach space, $I = E + \zeta : X \to \mathbb{R} \cup \{+\infty\}$ a Motreanu-Panagiotopoulos-type functional and we assume that*

(a) *$I(u) \geq \alpha$ for all $\|u\| = \rho$ with α, $\rho > 0$, and $I(0) = 0$;*
(b) *there is $e \in X$ with $\|e\| > \rho$ and $I(e) \leq 0$.*

If I satisfies the $(PS)_c$-condition (in the sense of Definition 2.9) for

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\},$$

then c is a critical value of I and $c \geq \alpha$.

3. Null Stackelberg response: proof of Theorem 1.2 (a) and (b)

Proof of Theorem 1.2 (a). The claim is equivalent to prove that

$$\tilde{f}^0_{x_2}((x_1, 0); -y) \geq 0, \ \forall y \in K_2. \tag{3.1}$$

By contradiction, we assume that there exists $y_0 \in K_2$ such that $\tilde{f}^0_{x_2}((x_1, 0); -y_0) < 0$. By Proposition 2.5 (d), we have that

$$0 > \tilde{f}^0_{x_2}((x_1, 0); -y_0) = \max\{\langle z, -y_0 \rangle : z \in \partial_{x_2} \tilde{f}(x_1, 0)\},$$

thus, with our assumption, it follows that

$$0 \notin \partial_{x_2} \tilde{f}(x_1, 0).$$

Since the set $\partial_{x_2} \tilde{f}(x_1, 0)$ is compact (see Proposition 2.5), we have that

$$\varepsilon_0 = \operatorname{dist}\left(0, \partial_{x_2} \tilde{f}(x_1, 0)\right) > 0.$$

The upper semicontinuity of $\partial_{x_2} \tilde{f}(x_1, \cdot)$ (see Proposition 2.5 (e)) implies that there exists $\eta_0 > 0$ such that for every $y \in B_{\mathbb{R}^m}(0, \eta_0)$, we have

$$\partial_{x_2} \tilde{f}(x_1, y) \subseteq \partial_{x_2} \tilde{f}(x_1, 0) + B_{\mathbb{R}^m}\left(0, \frac{\varepsilon_0}{2}\right).$$

If $\{x_n\} \subset \mathbb{R}^m$ is a sequence such that $\lim_{n \to \infty} x_n = 0$, for large enough $n \in \mathbb{N}$, we have that

$$z_n \in \partial_{x_2} \tilde{f}(x_1, 0) + B_{\mathbb{R}^m}\left(0, \frac{\varepsilon_0}{2}\right), \ \forall z_n \in \partial_{x_2} \tilde{f}(x_1, x_n).$$

In particular, for every large $n \in \mathbb{N}$, there exists $z^n_0 \in \partial_{x_2} \tilde{f}(x_1, 0)$ such that

$$\|z_n - z^n_0\| \leq \frac{\varepsilon_0}{2}.$$

Consequently,

$$\|z_n\| \geq \|z^n_0\| - \|z_n - z^n_0\| \geq \operatorname{dist}\left(0, \partial_{x_2} \tilde{f}(x_1, 0)\right) - \frac{\varepsilon_0}{2} = \frac{\varepsilon_0}{2}.$$

Therefore,

$$\max\left\{\|z_n\| : z_n \in \partial_{x_2}\tilde{f}(x_1, x_n)\right\} \geq \frac{\varepsilon_0}{2}.$$

Since $\lim_{n\to\infty} x_n = 0$, by hypothesis $(H^1_{x_1})$ and the above estimate we have that

$$\begin{aligned} 0 &= \lim_{x_2\to 0} \frac{\max\{\|z\| : z \in \partial_{x_2}\tilde{f}(x_1, x_2)\}}{\|x_2\|} \\ &\geq \lim_{n\to\infty} \frac{\max\{\|z_n\| : z_n \in \partial_{x_2}\tilde{f}(x_1, x_n)\}}{\|x_n\|} \\ &\geq +\infty, \end{aligned}$$

a contradiction. This fact shows that the claim (3.1) holds true, which implies that

$$0 \in R^{\lambda}_{SV}(x_1) \text{ for every } \lambda > 0.$$

Proof of Theorem 1.2 (b). Let us fix $\lambda \in (0, \tilde{c}^{-1})$ where $\tilde{c}$ comes from relation (1.2) and let $x_2 \in R^{\lambda}_{SV}(x_1)$, i.e.,

$$\langle x_2, y - x_2\rangle + \lambda \tilde{f}^0_{x_2}((x_1, x_2); -y + x_2) \geq 0, \ \forall y \in K_2.$$

Since $0 \in K_2$, we may choose $y = 0$ in the above inequality, obtaining that

$$\|x_2\|^2 \leq \lambda \tilde{f}^0_{x_2}((x_1, x_2); x_2). \tag{3.2}$$

By Proposition 2.5 (d) and (1.2), it follows that

$$\begin{aligned} |\tilde{f}^0_{x_2}((x_1, x_2); x_2)| &= |\max\{\langle z, x_2\rangle : z \in \partial_{x_2}\tilde{f}(x_1, x_2)\}| \\ &\leq \max\{\|z\| : z \in \partial_{x_2}\tilde{f}(x_1, x_2)\} \cdot \|x_2\| \\ &\leq \tilde{c}\|x_2\|^2. \end{aligned}$$

The latter estimate and (3.2) gives that

$$\|x_2\|^2 \leq \lambda\tilde{c}\|x_2\|^2.$$

Since $\lambda \in (0, \tilde{c}^{-1})$, we necessarily have that $x_2 = 0$. Therefore, we have

$$R^{\lambda}_{SV}(x_1) = \{0\}, \ \forall \lambda \in (0, \tilde{c}^{-1}).$$

4. Geometry of Stackelberg responses: proof of Theorem 1.2 (c)

Let $x_1 \in K_1$ be fixed.

Lemma 4.1. *Let $\lambda > 0$ be fixed. The functional $f_\lambda(x_1, \cdot)$ defined in* (1.1) *is bounded from below and coercive, i.e., $f_\lambda(x_1, x_2) \to +\infty$ whenever $\|x_2\| \to +\infty$. Moreover, $f_\lambda(x_1, \cdot)$ satisfies the Palais-Smale condition in the sense of Definition* 2.9.

Proof. According to hypotheses $(H^1_{x_1})$ and $(H^2_{x_1})$ and to the upper semicontinuity of $\partial_{x_2}\tilde{f}(x_1, \cdot)$ (see Proposition 2.5 (e)), for every $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that

$$\max\left\{\|z\| : z \in \partial_{x_2}\tilde{f}(x_1, x_2)\right\} \leq \frac{\varepsilon}{2}\|x_2\| + M_\varepsilon.$$

By Lebourg mean value theorem and from the fact that $\tilde{f}(x_1, 0) = 0$, it follows that for every $x_2 \in \mathbb{R}^m$,

$$|\tilde{f}(x_1, x_2)| = |\tilde{f}(x_1, x_2) - \tilde{f}(x_1, 0)| \leq \frac{\varepsilon}{2}\|x_2\|^2 + M_\varepsilon \|x_2\|.$$

Consequently, if $\varepsilon < \lambda^{-1}$ we have that

$$\begin{aligned} f_\lambda(x_1, x_2) &= \frac{1}{2}\|x_2\|^2 - \lambda \tilde{f}(x_1, x_2) + \delta_{K_2}(x_2) \\ &\geq \frac{1}{2}\left(1 - \varepsilon\lambda\right)\|x_2\|^2 - \lambda M_\varepsilon \|x_2\|. \end{aligned}$$

This estimate shows that $f_\lambda(x_1, \cdot)$ is bounded from below and coercive.

Now, let $\{x_n\} \subset \mathbb{R}^m$ be a Palais-Smale sequence for $f_\lambda(x_1, \cdot)$, i.e.,

$$\lim_{n\to\infty} f_\lambda(x_1, x_n) = c \tag{4.1}$$

and for every $v \in \mathbb{R}^m$,

$$\langle x_n, v - x_n\rangle + \lambda \tilde{f}^0(x_n; -v + x_n) + \delta_{K_2}(v) - \delta_{K_2}(x_n) \geq -\varepsilon_n \|v - x_n\|,$$

where $\varepsilon_n \to 0$ as $n \to \infty$. Since $f_\lambda(x_1, \cdot)$ is coercive, relation (4.1) immediately implies that the sequence $\{x_n\} \subset \mathbb{R}^m$ should be bounded. Consequently, we can extract a convergent subsequence of it, which proves the validity of the Palais-Smale condition. □

Proposition 4.2. *The number* $\tilde{\lambda}$ *in* (1.3) *is well-defined and*

$$0 < \tilde{\lambda} < \infty.$$

Proof. Let $x_1 \in K_1$ be fixed. By Lebourg mean value theorem (see Proposition 2.5 (f)), we have that

$$\tilde{f}(x_1, x_2) = \tilde{f}(x_1, x_2) - \tilde{f}(x_1, 0) = \langle z_\theta, x_2\rangle$$

for some $z_\theta \in \partial_{x_2}\tilde{f}(x_1, \theta x_2)$ with $\theta \in (0, 1)$. Now, by hypothesis $(H^1_{x_1})$ it follows that for arbitrary $\varepsilon > 0$ there exists $\eta > 0$ such that if $x_2 \in K_2$ with $\|x_2\| < \eta$ then

$$|\tilde{f}(x_1, x_2)| \leq \varepsilon \|x_2\|^2.$$

Consequently,

$$\lim_{\substack{x_2 \to 0 \\ x_2 \in K_2}} \frac{\|x_2\|^2}{|\tilde{f}(x_1, x_2)|} = +\infty.$$

A similar reasoning as above shows that

$$\lim_{\substack{\|x_2\| \to \infty \\ x_2 \in K_2}} \frac{\|x_2\|^2}{|\tilde{f}(x_1, x_2)|} = +\infty. \tag{4.2}$$

Indeed, by $(H^2_{x_1})$ we have that for arbitrary $\varepsilon > 0$ there exists $\eta > 0$ such that if $\|x_2\| > \eta$ then

$$\max\{\|z\| : z \in \partial_{x_2}\tilde{f}(x_1, x_2)\} \leq \varepsilon \|x_2\|.$$

Let $x_\eta \in K_2$ be such that $\|x_\eta\| = \eta$. By Lebourg mean value theorem, for every $x_2 \in K_2$ with $\|x_2\| > \eta$, we have that

$$\tilde{f}(x_1, x_2) - \tilde{f}(x_1, x_\eta) = \langle z_\eta, x_2 - x_\eta \rangle$$

for some $z_\eta \in \partial_{x_2} \tilde{f}(x_1, x_2')$ with $x_2' \in K_2$ and $\|x_2'\| > \eta$. Consequently, we obtain for every $x_2 \in K_2$ with $\|x_2\| > \eta$ that

$$|\tilde{f}(x_1, x_2)| \leq |\tilde{f}(x_1, x_\eta)| + \varepsilon \|x_2\| \|x_2 - x_\eta\|,$$

which shows the validity of (4.2). This ends the proof of the fact that $0 < \tilde{\lambda} < \infty$.

We also notice that the above arguments show that instead of "inf" we can write "min" in (1.3). □

Proof of Theorem 1.2 (c). Let us fix $\lambda > \tilde{\lambda}$.

Step 1. (First response) According to property (a), one has $0 \in R^{\lambda}_{SV}(x_1)$, which is the first (trivial) response.

Step 2. (Second response) Combining Lemma 4.1 with Theorem 2.11, it follows that the Motreanu-Panagiotopoulos-type functional $f_\lambda(x_1, \cdot)$ achieves its infimum at a point $x_2^\lambda \in \mathbb{R}^m$ which is a critical point in the sense of Definition 2.6. Therefore,

$$f_\lambda(x_1, x_2^\lambda) = \inf_{x \in \mathbb{R}^m} f_\lambda(x_1, x)$$

and

$$0 \in x_2^\lambda - \lambda \partial_{x_2} \tilde{f}(x_1, x_2^\lambda) + \partial_C \delta_{K_2}(x_2^\lambda) \quad \text{in} \quad \mathbb{R}^m.$$

In fact, the latter relation is nothing but $x_2^\lambda \in R^{\lambda}_{SV}(x_1)$, which is the second response. Note that in fact $x_2^\lambda \in K_2$; otherwise, $f_\lambda(x_1, x_2^\lambda)$ would be $+\infty$, a contradiction.

It remains to prove that $x_2^\lambda \neq 0$. Since $\lambda > \tilde{\lambda}$, by the definition of $\tilde{\lambda}$ it follows the existence of an element $y_0 \in K_2$ such that

$$\lambda > \frac{1}{2} \frac{\|y_0\|^2}{\tilde{f}(x_1, y_0)} > \tilde{\lambda}.$$

Therefore,

$$\begin{aligned} f_\lambda(x_1, x_2^\lambda) &= \inf_{x \in \mathbb{R}^m} f_\lambda(x_1, x) \\ &\leq f_\lambda(x_1, y_0) \\ &= \frac{1}{2}\|y_0\|^2 - \lambda \tilde{f}(x_1, y_0) + \delta_{K_2}(y_0) \\ &= \frac{1}{2}\|y_0\|^2 - \lambda \tilde{f}(x_1, y_0) \\ &< 0. \end{aligned}$$

Since $f_\lambda(x_1, 0) = 0$, we have that $x_2^\lambda \neq 0$.

Step 3. (Third response) By hypotheses $(H^1_{x_1})$ and $(H^2_{x_1})$ again, for every $\varepsilon \in (0, \frac{1}{\lambda})$ there exists $M_\varepsilon > 0$ such that

$$\max\{\|z\| : z \in \partial_{x_2} \tilde{f}(x_1, x_2)\} \leq \frac{\varepsilon}{2}\|x_2\| + M_\varepsilon \|x_2\|^2, \ \forall x_2 \in \mathbb{R}^m.$$

By Lebourg mean value theorem, one has that

$$\tilde{f}(x_1, x_2) \leq \frac{\varepsilon}{2}\|x_2\|^2 + M_\varepsilon \|x_2\|^3, \ \forall x_2 \in \mathbb{R}^m.$$

Let

$$0 < \rho < \min\left\{\|x_2^\lambda\|, \frac{1}{2M_\varepsilon}\left(\frac{1}{\lambda} - \varepsilon\right)\right\}.$$

Then, for every $x_2 \in \mathbb{R}^m$ with the property $\|x_2\| = \rho$, we have

$$\begin{aligned} f_\lambda(x_1, x_2) &= \frac{1}{2}\|x_2\|^2 - \lambda \tilde{f}(x_1, x_2) + \delta_{K_2}(x_2) \\ &\geq \frac{1}{2}(1 - \varepsilon\lambda)\|x_2\|^2 - \lambda M_\varepsilon \|x_2\|^3 \\ &= \rho^2 \left(\frac{1}{2}(1 - \varepsilon\lambda) - \lambda M_\varepsilon \rho\right) \\ &> 0. \end{aligned}$$

Therefore, by the non-smooth Mountain Pass theorem (see Theorem 2.12), it follows that the number

$$c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f_\lambda(x_1, \gamma(t))$$

is a critical value for $f_\lambda(x_1, \cdot)$, where $\Gamma = \{\gamma \in C([0,1], \mathbb{R}^m) : \gamma(0) = 0, \gamma(1) = x_2^\lambda\}$, and

$$c_\lambda \geq \rho^2 \left(\frac{1}{2}(1 - \varepsilon\lambda) - \lambda M_\varepsilon \rho\right) > 0.$$

Thus, if $y_2^\lambda \in K_2$ is the mountain pass-type critical point of $f_\lambda(x_1, \cdot)$ with $c_\lambda = f_\lambda(x_1, y_2^\lambda) > 0$, we clearly have that $y_2^\lambda \neq 0$ and $y_2^\lambda \neq x_2^\lambda$, which is the third response.

Summing up the above three steps, we conclude that

$$\{0, x_2^\lambda, y_2^\lambda\} \subset R_{SV}^\lambda(x_1), \ \forall \lambda > \tilde{\lambda}.$$

This ends the proof of Theorem 1.2. □

Remark 4.3. As we pointed out before, the Stackelberg variational response set reduces to the null strategy whenever the parameter is small enough. However, when the parameter is beyond a threshold value (see Theorem 1.2 (c)), there are three possible Stackelberg variational responses; in this case, the follower enters actively into the game in order to minimize his loss. More precisely, besides the null strategy (see Step 1), he can choose the global minimum-type solution/response (see Step 2); in this case, his loss function takes a negative value, i.e., he is in a winning position. In the case when the player chooses the mountain pass-type minimax response (see Step 3), his payoff function takes a positive value.

5. Remarks on the gap-interval

The subject of this section is twofold:

(a) to give a direct proof for the inequality $\tilde{c}^{-1} \leq \tilde{\lambda}$ whenever $K_2 = \mathbb{R}^m$ (the strict inequality $\tilde{c}^{-1} < \tilde{\lambda}$ can be proven e.g. when $m = 1$ and the payoff function $\tilde{f}$ is of class C^1);

(b) to provide an example in order to show that the gap-interval $[\tilde{c}^{-1}, \tilde{\lambda}]$ can be arbitrary small.

Proposition 5.1. *When $K_2 = \mathbb{R}^m$, we have $\tilde{c}^{-1} \leq \tilde{\lambda}$.*

Proof. As we already pointed out in the proof of Theorem 1.2, in the definition of $\tilde{\lambda}$ we can write minimum instead of infimum. Accordingly, let $\tilde{x}_2 \in K_2 = \mathbb{R}^m$ be the minimum point of the function $x_2 \mapsto \frac{\|x_2\|^2}{2\tilde{f}(x_1,x_2)}$ in the set

$$S = \{x_2 \in \mathbb{R}^m : \tilde{f}(x_1, x_2) > 0\},$$

i.e.,

$$\tilde{\lambda} = \frac{\|\tilde{x}_2\|^2}{2\tilde{f}(x_1, \tilde{x}_2)}.$$

Since S is open and $0 \notin S$, the element $\tilde{x}_2 \neq 0$ is a local minimum point, thus a critical point of the above locally Lipschitz function. Applying the rules of subdifferentiation, we obtain

$$0 \in \frac{2\tilde{x}_2\tilde{f}(x_1, \tilde{x}_2) - \|\tilde{x}_2\|^2\partial_{x_2}\tilde{f}(x_1, \tilde{x}_2)}{\tilde{f}(x_1, \tilde{x}_2)^2},$$

i.e.,

$$\frac{2\tilde{f}(x_1, \tilde{x}_2)}{\|\tilde{x}_2\|^2}\tilde{x}_2 \in \partial_{x_2}\tilde{f}(x_1, \tilde{x}_2). \tag{5.1}$$

Therefore,

$$\begin{aligned}
\tilde{c} &= \max_{x_2 \in \mathbb{R}^m \setminus \{0\}} \frac{\max\{\|z\| : z \in \partial_{x_2}\tilde{f}(x_1, x_2)\}}{\|x_2\|} \\
&\geq \frac{1}{\|\tilde{x}_2\|} \cdot \left\| \frac{2\tilde{f}(x_1, \tilde{x}_2)}{\|\tilde{x}_2\|^2}\tilde{x}_2 \right\| = \frac{2\tilde{f}(x_1, \tilde{x}_2)}{\|\tilde{x}_2\|^2} \\
&= \tilde{\lambda}^{-1},
\end{aligned}$$

which concludes the proof. □

Remark 5.2. In general, we have that $\tilde{c}^{-1} < \tilde{\lambda}$. Such a situation occurs e.g. when $m = 1$, $K_2 = [0, \infty)$ and the payoff function $\tilde{f} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is of class C^1 in the second variable.

Indeed, by contradiction, we assume that $\tilde{c}^{-1} = \tilde{\lambda}$. Let

$$\tilde{x}_2^0 = \inf\left\{\tilde{x}_2 > 0 : \tilde{\lambda} = \frac{\tilde{x}_2^2}{2\tilde{f}(x_1, \tilde{x}_2)}\right\},$$

and fix $y_0 \in (0, \tilde{x}_2^0)$. By the latter construction, one clearly has that

$$\tilde{\lambda} < \frac{y_0^2}{2\tilde{f}(x_1, y_0)}.$$

Since $\tilde{f}(x_1,\cdot)$ is of class C^1, it follows that $\partial_{x_2}\tilde{f}(x_1,x_2)=\{\tilde{f}'_{x_2}(x_1,x_2)\}$; thus by the definition of the number $\tilde{c}$ we obtain in particular that

$$\tilde{f}'_{x_2}(x_1,t)\le \tilde{c}t,\ \forall t>0.$$

Thus, the above relations imply that

$$\begin{aligned}
0 &= 2\tilde{f}(x_1,\tilde{x}_2)-\tilde{\lambda}^{-1}\tilde{x}_2^2\\
&= 2\tilde{f}(x_1,y_0)-\tilde{\lambda}^{-1}y_0^2\\
&\quad +2\tilde{f}(x_1,\tilde{x}_2)-\tilde{\lambda}^{-1}\tilde{x}_2^2-(2\tilde{f}(x_1,y_0)-\tilde{\lambda}^{-1}y_0^2)\\
&= 2\tilde{f}(x_1,y_0)-\tilde{\lambda}^{-1}y_0^2+2\int_{y_0}^{\tilde{x}_2}(\tilde{f}'_{x_2}(x_1,t)-\tilde{c}t)dt\\
&< 0,
\end{aligned}$$

a contradiction, which proves the claim.

Proposition 5.3. *The gap-interval* $[\tilde{c}^{-1},\tilde{\lambda}]$ *can be arbitrarily small.*

Proof. For $\eta>1$, let $\tilde{f}:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be defined by

$$\tilde{f}(x_1,x_2)=(1+|x_1|)\int_0^{x_2}\min\{(s-1)_+,\eta-1\}ds,$$

and $K_2=\mathbb{R}$. Note that $\tilde{f}(x_1,\cdot)$ is of class C^1 and

$$\partial_{x_2}\tilde{f}(x_1,x_2)=\{\tilde{f}'_{x_2}(x_1,x_2)\}=\{(1+|x_1|)\min\{(x_2-1)_+,\eta-1\}\}.$$

Consequently, on one hand, we have

$$\tilde{c}=\max_{x_2\in\mathbb{R}\setminus\{0\}}\frac{\max\{|z|:z\in\partial_{x_2}\tilde{f}(x_1,x_2)\}}{|x_2|}=(1+|x_1|)\frac{\eta-1}{\eta}.$$

On the other hand,

$$\tilde{\lambda}=\frac{1}{2}\inf_{\substack{\tilde{f}(x_1,x_2)>0\\ x_2\in\mathbb{R}}}\frac{|x_2|^2}{\tilde{f}(x_1,x_2)}=\frac{1}{1+|x_1|}\cdot\frac{\eta^2}{(\eta-1)^2}.$$

We can see that $\tilde{c}^{-1}<\tilde{\lambda}$ and these numbers can be arbitrary close to each other whenever η is large enough. $\square$

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