Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 3, 413–420

Second Hankel determinant for the class of Bazilevic functions

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Abstract. The objective of this paper is to obtain a sharp upper bound to the second Hankel determinant $H_2(2)$ for the function f when it belongs to the class of Bazilevic functions, using Toeplitz determinants. The result presented here include two known results as their special cases.

Mathematics Subject Classification (2010): 30C45, 30C50.

Keywords: Analytic function, Bazilevic function, upper bound, second Hankel functional, positive real function, Toeplitz determinants.

1. Introduction

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disc $E = \{z : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions.

The Hankel determinant of f for $q \geq 1$ and $n \geq 1$ was defined by Pommerenke ([15]) as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}, (a_1 = 1). \tag{1.2}$$

This determinant has been considered by many authors in the literature. Noonan and Thomas ([13]) studied about the second Hankel determinant of areally mean p-valent functions. Ehrenborg ([5]) studied the Hankel determinant of exponential polynomials. One can easily observe that the Fekete-Szegö functional is $H_2(1)$. Fekete-Szegö then further generalized the estimate $|a_3 - \mu a_2^2|$ with μ real and $f \in S$. Ali ([2]) found sharp bounds for the first four coefficients and sharp estimate for the Fekete-Szegö functional $|\gamma_3 - t\gamma_2^2|$, where t is real, for the inverse function of f defined as

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n$$

when it belongs to the class of strongly starlike functions of order α $(0 < \alpha \le 1)$ denoted by $\widetilde{ST}(\alpha)$. In this paper, we consider the Hankel determinant in the case of $q = 2$ and $n = 2$, known as the second Hankel determinant, given by

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2. \tag{1.3}$$

Janteng, Halim and Darus ([8]) have considered the functional $|a_2a_4 - a_3^2|$ and found sharp upper bound for the function f in the subclass RT of S, consisting of functions whose derivative has a positive real part studied by Mac Gregor ([11]). In their work, they have shown that if $f \in RT$ then $|a_2a_4 - a_3^2| \le \frac{4}{9}$. Janteng, Halim and Darus ([7]) also obtained the second Hankel determinant and sharp bounds for the familiar subclasses of S, namely, starlike and convex functions denoted by ST and CV and have shown that $|a_2a_4 - a_3^2| \le 1$ and $|a_2a_4 - a_3^2| \le \frac{1}{8}$ respectively. Similarly, the same coefficient inequality was calculated for certain subclasses of analytic functions by many authors ([1], [3], [9], [12], [18]).

Motivated by the results obtained by different authors in this direction mentioned above, in this paper, we seek an upper bound to the functional $|a_2a_4 - a_3^2|$ for the function f when it belongs to the class of Bazilevic functions denoted by B_γ $(0 \le \gamma \le 1)$, defined as follows.

Definition 1.1. *A function $f(z) \in A$ is said to be Bazilevic function, if it satisfies the condition*

$$Re\left\{ z^{1-\gamma} \frac{f'(z)}{f^{1-\gamma}(z)} \right\} > 0, \ \forall z \in E \tag{1.4}$$

where the powers are meant for principal values. This class of functions was denoted by B_γ, studied by Ram Singh ([16]). It is observed that for $\gamma = 0$ and $\gamma = 1$ respectively, we get $B_0 = ST$ and $B_1 = RT$.

Some preliminary Lemmas required for proving our result are as follows:

2. Preliminary results

Let $\mathcal{P}$ denote the class of functions consisting of p, such that

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{2.1}$$

which are regular in the open unit disc E and satisfy $\mathrm{Re}\,p(z) > 0$ for any $z \in E$. Here $p(z)$ is called Carathéodory function [4].

Lemma 2.1. ([14], [17]) *If $p \in \mathcal{P}$, then $|c_k| \le 2$, for each $k \ge 1$ and the inequality is sharp for the function $\frac{1+z}{1-z}$.*

Lemma 2.2. ([6]) *The power series for p given in (2.1) converges in the open unit disc E to a function in $\mathcal{P}$ if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, n = 1, 2, 3....$$

and $c_{-k} = \overline{c}_k$, are all non-negative. These are strictly positive except for

$$p(z) = \sum_{k=1}^{m} \rho_k p_0(\exp(it_k)z),$$

$\rho_k > 0$, t_k real and $t_k \neq t_j$, for $k \neq j$, where $p_0(z) = \frac{1+z}{1-z}$; in this case $D_n > 0$ for $n < (m-1)$ and $D_n \doteq 0$ for $n \geq m$.

This necessary and sufficient condition found in ([6]) is due to Carathéodory and Toeplitz. We may assume without restriction that $c_1 > 0$. On using Lemma 2.2, for $n = 2$ and $n = 3$ respectively, we obtain

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \overline{c}_1 & 2 & c_1 \\ \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix} = [8 + 2Re\{c_1^2 c_2\} - 2|c_2|^2 - 4|c_1|^2] \geq 0,$$

it is equivalent to

$$2c_2 = \{c_1^2 + x(4 - c_1^2)\}, \text{ for some } x, |x| \leq 1. \tag{2.2}$$

$$\text{and } D_3 = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \overline{c}_1 & 2 & c_1 & c_2 \\ \overline{c}_2 & \overline{c}_1 & 2 & c_1 \\ \overline{c}_3 & \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix}.$$

Then $D_3 \geq 0$ is equivalent to

$$|(4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2| \leq 2(4 - c_1^2)^2 - 2|(2c_2 - c_1^2)|^2. \tag{2.3}$$

Simplifying the relations (2.2) and (2.3), we get

$$4c_3 = \{c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z\}, \text{ with } |z| \leq 1. \tag{2.4}$$

To obtain our result, we refer to the classical method devised by Libera and Zlotkiewicz ([10]).

3. Main result

Theorem 3.1. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in B_\gamma$ $(0 \leq \gamma \leq 1)$ then*

$$|a_2a_4 - a_3^2| \leq \left[\frac{2}{2+\gamma}\right]^2$$

and the inequality is sharp.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in B_\gamma$, by virtue of Definition 1.1, there exists an analytic function $p \in \mathcal{P}$ in the open unit disc E with $p(0) = 1$ and $\text{Re}\{p(z)\} > 0$ such that

$$z^{1-\gamma}\frac{f'(z)}{f^{1-\gamma}(z)} = p(z) \Leftrightarrow z^{1-\gamma}f'(z) = f^{1-\gamma}(z)p(z). \tag{3.1}$$

Replacing the values of $f(z)$, $f'(z)$ and $p(z)$ with their equivalent series expressions in (3.1), we have

$$z^{1-\gamma}\left\{1 + \sum_{n=2}^{\infty} n a_n z^{n-1}\right\} = \left\{z + \sum_{n=2}^{\infty} a_n z^n\right\}^{1-\gamma}\left\{1 + \sum_{n=1}^{\infty} c_n z^n\right\}. \tag{3.2}$$

Using the binomial expansion on the right-hand side of (3.2) subject to the condition

$$|\sum_{n=2}^{\infty} a_n z^n| < 1 - \gamma,$$

upon simplification, we obtain

$$\begin{aligned} 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + ... &= 1 + \{c_1 + (1-\gamma)a_2\}\, z \\ &+ \left[c_2 + (1-\gamma)\left\{c_1 a_2 + a_3 + \frac{(-\gamma)}{2}a_2^2\right\}\right] z^2 \\ &+ \left[c_3 + (1-\gamma)\left\{c_2 a_2 + c_1 a_3 + a_4 + (-\gamma)\left\{\frac{1}{2}c_1 a_2^2 + a_2 a_3 + \frac{(-1-\gamma)}{6}a_2^3\right\}\right\}\right] z^3 + \dots \end{aligned} \tag{3.3}$$

Equating the coefficients of like powers of z, z^2 and z^3 respectively on both sides of (3.3), after simplifying, we get

$$\begin{aligned} a_2 &= \frac{c_1}{(1+\gamma)};\ a_3 = \frac{1}{2(1+\gamma)^2(2+\gamma)}\left\{2(1+\gamma)^2 c_2 + (1-\gamma)(2+\gamma)c_1^2\right\}; \\ a_4 &= \frac{1}{6(1+\gamma)^3(2+\gamma)(3+\gamma)} \times \{6(1+\gamma)^2(2+\gamma)c_3 \\ &\quad + 6(1-\gamma)(1+\gamma)^2(3+\gamma)c_1 c_2 + (\gamma-1)(2+\gamma)(2\gamma^2+5\gamma-3)c_1^3\}. \end{aligned} \tag{3.4}$$

Substituting the values of a_2, a_3 and a_4 from (3.4) in the second Hankel functional $|a_2 a_4 - a_3^2|$ for the function $f \in B_\gamma$, which simplifies to

$$\begin{aligned} |a_2 a_4 - a_3^2| &= \frac{1}{12(1+\gamma)^3(2+\gamma)^2(3+\gamma)}|12(1+\gamma)^2(2+\gamma)^2 c_1 c_3 \\ &\quad - 12(1+\gamma)^3(3+\gamma)c_2^2 + (2+\gamma)^2(3+\gamma)(\gamma-1)c_1^4|. \end{aligned}$$

The above expression is equivalent to

$$|a_2 a_4 - a_3^2| = \frac{1}{12(1+\gamma)^3(2+\gamma)^2(3+\gamma)}\left|d_1 c_1 c_3 + d_2 c_2^2 + d_3 c_1^4\right|, \tag{3.5}$$

where

$$\begin{aligned} d_1 &= 12(1+\gamma)^2(2+\gamma)^2; \quad d_2 = -12(1+\gamma)^3(3+\gamma); \\ d_3 &= (2+\gamma)^2(3+\gamma)(\gamma-1). \end{aligned} \tag{3.6}$$

Substituting the values of c_2 and c_3 from (2.2) and (2.4) respectively from Lemma 2.2 on the right-hand side of (3.5), we have

$$\left|d_1c_1c_3 + d_2c_2^2 + d_3c_1^4\right| = |d_1c_1 \times \frac{1}{4}\{c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z\} + d_2 \times \frac{1}{4}\{c_1^2 + x(4-c_1^2)\}^2 + d_3c_1^4|.$$

Using the facts that $|z| < 1$ and $|xa+yb| \leq |x||a| + |y||b|$, where x, y, a and b are real numbers, after simplifying, we get

$$4\left|d_1c_1c_3 + d_2c_2^2 + d_3c_1^4\right| \leq |(d_1+d_2+4d_3)c_1^4 + 2d_1c_1(4-c_1^2) + 2(d_1+d_2)c_1^2(4-c_1^2)|x| - \left\{(d_1+d_2)c_1^2 + 2d_1c_1 - 4d_2\right\}(4-c_1^2)|x|^2|. \quad (3.7)$$

With the values of $d_1,\ d_2$ and d_3 from (3.6), we can write

$$\begin{aligned} d_1 + d_2 + 4d_3 &= 4(\gamma^4 + 6\gamma^3 + 12\gamma^2 + 2\gamma - 9); \\ d_1 &= 12(1+\gamma)^2(2+\gamma)^2;\ d_1 + d_2 = 12(1+\gamma)^2 \end{aligned} \quad (3.8)$$

and

$$(d_1+d_2)c_1^2 + 2d_1c_1 - 4d_2 = 12(1+\gamma)^2\left\{c_1^2 + 2(2+\gamma)^2c_1 + 4(1+\gamma)(3+\gamma)\right\}. \quad (3.9)$$

Consider

$$\begin{aligned} &\left\{c_1^2 + 2(2+\gamma)^2c_1 + 4(1+\gamma)(3+\gamma)\right\} \\ &= \left[\left\{c_1 + (2+\gamma)^2\right\}^2 - (2+\gamma)^4 + 4(1+\gamma)(3+\gamma)\right] \\ &= \left[\left\{c_1 + (2+\gamma)^2\right\}^2 - \left\{\sqrt{\gamma^4 + 8\gamma^3 + 20\gamma^2 + 16\gamma + 4}\right\}^2\right] \\ &= \left[c_1 + \left\{(2+\gamma)^2 + \sqrt{\gamma^4 + 8\gamma^3 + 20\gamma^2 + 16\gamma + 4}\right\}\right] \\ &\times \left[c_1 + \left\{(2+\gamma)^2 - \sqrt{\gamma^4 + 8\gamma^3 + 20\gamma^2 + 16\gamma + 4}\right\}\right] \end{aligned} \quad (3.10)$$

Since $c_1 \in [0,2]$, using the result $(c_1+a)(c_1+b) \geq (c_1-a)(c_1-b)$, where $a, b \geq 0$ on the right-hand side of (3.10), after simplifying, we get

$$\begin{aligned} &\left\{c_1^2 + 2(2+\gamma)^2c_1 + 4(1+\gamma)(3+\gamma)\right\} \\ &\geq \left\{c_1^2 - 2(2+\gamma)^2c_1 + 4(1+\gamma)(3+\gamma)\right\}. \end{aligned} \quad (3.11)$$

From the relations (3.9) and (3.11), we can write

$$\begin{aligned} &-\left\{(d_1+d_2)c_1^2 + 2d_1c_1 - 4d_2\right\} \\ &\leq -12(1+\gamma)^2\left\{c_1^2 - 2(2+\gamma)^2c_1 + 4(1+\gamma)(3+\gamma)\right\}. \end{aligned} \quad (3.12)$$

Substituting the calculated values from (3.8) and (3.12) on the right-hand side of (3.7), we have

$$\begin{aligned} \left|d_1c_1c_3 + d_2c_2^2 + d_3c_1^4\right| \leq |&(\gamma^4 + 6\gamma^3 + 12\gamma^2 + 2\gamma - 9)c_1^4 \\ &+ 6(1+\gamma)^2(2+\gamma)^2c_1(4-c_1^2) + 6(1+\gamma)^2c_1^2(4-c_1^2)|x| \\ &- 3(1+\gamma)^2\left\{c_1^2 - 2(2+\gamma)^2c_1 + 4(1+\gamma)(3+\gamma)\right\}(4-c_1^2)|x|^2|. \end{aligned}$$

Choosing $c_1 = c \in [0,2]$, applying triangle inequality and replacing $|x|$ by μ on the right-hand side of the above inequality, we obtain

$$\begin{aligned}\left|d_1c_1c_3 + d_2c_2^2 + d_3c_1^4\right| \leq & [(-\gamma^4 - 6\gamma^3 + 12\gamma^2 - 2\gamma + 9)c^4 \\ & + 6(1+\gamma)^2(2+\gamma)^2c(4-c^2) + 6(1+\gamma)^2c^2(4-c^2)\mu \\ & + 3(1+\gamma)^2\left\{c^2 - 2(2+\gamma)^2c + 4(1+\gamma)(3+\gamma)\right\}(4-c^2)\mu^2] \\ & = F(c,\mu), \text{ for } 0 \leq \mu = |x| \leq 1, \end{aligned} \tag{3.13}$$

where

$$\begin{aligned}F(c,\mu) = & [(-\gamma^4 - 6\gamma^3 + 12\gamma^2 - 2\gamma + 9)c^4 \\ & + 6(1+\gamma)^2(2+\gamma)^2c(4-c^2) + 6(1+\gamma)^2c^2(4-c^2)\mu \\ & + 3(1+\gamma)^2\left\{c^2 - 2(2+\gamma)^2c + 4(1+\gamma)(3+\gamma)\right\}(4-c^2)\mu^2]. \end{aligned} \tag{3.14}$$

We next maximize the function $F(c,\mu)$ on the closed region $[0,2] \times [0,1]$. Differentiating $F(c,\mu)$ in (3.14) partially with respect to μ, we get

$$\frac{\partial F}{\partial \mu} = 6(1+\gamma)^2[c^2 + \left\{c^2 - 2(2+\gamma)^2c + 4(1+\gamma)(3+\gamma)\right\}\mu] \times (4-c^2). \tag{3.15}$$

For $0 < \mu < 1$, for any fixed c with $0 < c < 2$ and $o \leq \gamma \leq 1$, from (3.15), we observe that $\frac{\partial F}{\partial \mu} > 0$. Therefore, $F(c,\mu)$ is an increasing function of μ and hence it cannot have maximum value any point in the interior of the closed region $[0,2] \times [0,1]$. Moreover, for fixed $c \in [0,2]$, we have

$$\max_{0 \leq \mu \leq 1} F(c,\mu) = F(c,1) = G(c). \tag{3.16}$$

In view of (3.16), replacing μ by 1 in (3.14), upon simplification, we obtain

$$\begin{aligned}G(c) = F(c,1) = & -\gamma(\gamma^3 + 6\gamma^2 - 3\gamma + 20)c^4 - 12\gamma(1+\gamma)^2(4+\gamma)c^2 \\ & + 48(1+\gamma)^3(3+\gamma), \end{aligned} \tag{3.17}$$

$$G'(c) = -4\gamma c\left\{(\gamma^3 + 6\gamma^2 - 3\gamma + 20)c^2 + 6(1+\gamma)^2(4+\gamma)\right\}. \tag{3.18}$$

From the expression (3.18), we observe that $G'(c) \leq 0$, for every $c \in [0,2]$ and for fixed γ with $0 \leq \gamma \leq 1$. Therefore, $G(c)$ is a decreasing function of c in the interval [0,2], whose maximum value occurs at $c = 0$ only. For $c = 0$ in (3.17), the maximum value of $G(c)$ is given by

$$G_{max} = G(0) = 48(1+\gamma)^3(3+\gamma). \tag{3.19}$$

From the expressions (3.13) and (3.19), we have

$$\left|d_1c_1c_3 + d_2c_2^2 + d_3c_1^4\right| \leq 48(1+\gamma)^3(3+\gamma). \tag{3.20}$$

Simplifying the relations (3.5) and (3.20), we obtain

$$|a_2a_4 - a_3^2| \leq \left[\frac{2}{2+\gamma}\right]^2. \tag{3.21}$$

Choosing $c_1 = c = 0$ and selecting $x = 1$ in (2.2) and (2.4), we find that $c_2 = 2$ and $c_3 = 0$. Substituting these values in (3.20), we observe that equality is attained which shows that our result is sharp. For these values, we derive that

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \dots = 1 + 2z^2 + 2z^4 + \dots = \frac{1+z^2}{1-z^2}. \tag{3.22}$$

Therefore, the extremal function in this case is

$$z^{1-\gamma}\frac{f'(z)}{f^{1-\gamma}(z)} = 1 + 2z^2 + 2z^4 + \dots = \frac{1+z^2}{1-z^2}. \tag{3.23}$$

This completes the proof of our Theorem. □

Remark 3.2. Choosing $\gamma = 0$, from (3.21), we get $|a_2a_4 - a_3^2| \le 1$, this inequality is sharp and coincides with that of Janteng, Halim, Darus ([7]).

Remark 3.3. For the choice of $\gamma = 1$ in (3.21), we obtain $|a_2a_4 - a_3^2| \le \frac{4}{9}$ and is sharp, coincides with the result of Janteng, Halim, Darus ([8]) .

Acknowledgement. The authors express their sincere thanks to the esteemed Referee(s) for their careful readings, valuable suggestions and comments, which helped to improve the presentation of the paper.

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