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Extension of Karamata inequality for generalized inverse trigonometric functions

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Abstract. Discussing Ramanujan's Question 294, Karamata established the inequality

$$\frac{\log x}{x-1} \le \frac{1+\sqrt[3]{x}}{x+\sqrt[3]{x}}, \qquad (x>0,\, x\neq 1)\,, \tag{*}$$

which is the cornerstone of this paper. We generalize the above inequality transforming into terms of arctan and artanh. Moreover, we expand the established result to the class of generalized inverse p–trigonometric arctan_p and to hyperbolic artanh_p functions.

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1. Introduction

The monumental *Analytical Inequalities* monograph by Mitrinović [6] contains several results by the famous Serbian mathematician Jovan Karamata. The first (Serbo–Croatian) edition's page 267 presents two Karamata's inequalities [6, **3.6.15.**, **3.6.16.**]

$$\frac{\log x}{x-1} \le \begin{cases} \dfrac{1}{\sqrt{x}} \\ \dfrac{1+\sqrt[3]{x}}{x+\sqrt[3]{x}} \end{cases}, \tag{1.1}$$

which hold for all $x \in \mathbb{R}_+ \setminus \{1\}$. Both estimates Karamata [4] applied in showing the monotone decreasing behavior of a sequence occurring in the famous Ramanujan's

Question 294 [7, p. 128] *Show that* [*if x is a positive integer*]

$$\frac{1}{2}\,\mathrm{e}^x = \sum_{k=0}^{x-1} \frac{x^k}{k!} + \frac{x^x}{x!}\,\theta\,,$$

where θ lies between $\frac{1}{3}$ and $\frac{1}{2}$.

For further information about Question 294 consult [2, p. 46 *et seq.*], while subsequent results concerning (1.1) belong also to Simić [8], see also the related references therein.

Being $\sqrt{x} \le (x + \sqrt[3]{x})(1 + \sqrt[3]{x})^{-1}$, the second Karamata's upper bound is more accurate on the whole range of their validity, therefore we concentrate to (*). In Mitrinović's monograph the proofs of inequalities (1.1) belong to B. Mesihović; we present the sketch of the proof's idea for the cubic–root–bound. By putting

$$(1+x)^3(1-x)^{-3} \mapsto x,$$

the radicals disappear in (*), and it transforms into

$$\frac{3}{2x}\log\frac{1+x}{1-x} - \frac{x^2+3}{1-x^4} < 0, \qquad (0 < |x| < 1)\ . \tag{1.2}$$

Expanding this expression into a power series, we get

$$K_{3,1}^{(2)}(4;x) := 3\sum_{k\ge 0}\left(1 - \frac{1}{4k+1}\right)x^{4k} + \sum_{k\ge 0}\left(1 - \frac{3}{4k+3}\right)x^{4k+2} > 0\,,$$

which finishes in an elegant way the proof.

However, summing up $K_{3,1}^{(2)}(4;x)$, we can write

$$K_{3,1}^{(2)}(4;x) = \frac{x^2+3}{1-x^4} - 3\cdot {}_2F_1\left[\begin{matrix}1, \frac{1}{4}\\ \frac{5}{4}\end{matrix};x^4\right] - x^2\,{}_2F_1\left[\begin{matrix}1, \frac{3}{4}\\ \frac{7}{4}\end{matrix};x^4\right],$$

such that gives the new form of (1.2):

$$3\cdot {}_2F_1\left[\begin{matrix}1, \frac{1}{4}\\ \frac{5}{4}\end{matrix};x^4\right] + x^2\,{}_2F_1\left[\begin{matrix}1, \frac{3}{4}\\ \frac{7}{4}\end{matrix};x^4\right] < \frac{3+x^2}{1-x^4},$$

which simplifies into

$$\frac{3}{x}\operatorname{arctanh} x < \frac{3+x^2}{1-x^4}, \qquad (0 < |x| < 1)\ , \tag{1.3}$$

since

$$\begin{aligned}{}_2F_1\left[\begin{matrix}1, \frac{1}{4}\\ \frac{5}{4}\end{matrix};z\right] &= \frac{1}{\sqrt[4]{z}}\left(\operatorname{arctanh}\sqrt[4]{z} + \arctan\sqrt[4]{z}\right)\\ {}_2F_1\left[\begin{matrix}1, \frac{3}{4}\\ \frac{7}{4}\end{matrix};z\right] &= \frac{3}{2\sqrt[4]{z^3}}\left(\operatorname{arctanh}\sqrt[4]{z} - \arctan\sqrt[4]{z}\right)\ .\end{aligned}$$

Here by using the shifted factorial

$$(a)_n = a(a+1)\dots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$$

for $a > 0$, the power series

$${}_2F_1\left[\begin{matrix}a, b\\ a+b\end{matrix};x\right] = \sum_{n\ge 0}\frac{(a)_n(b)_n}{(a+b)_n}\frac{x^n}{n!},$$

stands for the zero-balanced Gaussian hypergeometric series, which converges for $|x| < 1$.

It is worth to mention that as $x \to 0$, we have the strong asymptotic relation

$$K_{3,1}^{(2)}(4;x) = \frac{12}{5}\, x^4 + \mathcal{O}(x^6)\,, \tag{1.4}$$

compare [6, p. 267].

In the sequel our aim is to extend Mesihović's method to general weighted sum of zero–balanced Gaussian hypergeometric functions getting appropriate extensions of Karamata's inequality (*).

2. Extending $K_{3,1}^{(2)}(4;x)$

In this section we are going to investigate the sum

$$K_{p,q}^{(\gamma)}(\mu;x) := p\sum_{k\ge 0}\left(1-\frac{q}{\mu k+q}\right)x^{\mu k} + q\sum_{k\ge 0}\left(1-\frac{p}{\mu k+p}\right)x^{\mu k+\gamma}\,,$$

for the widest possible range of the variable x and its representation in a form of a weighted sum of two zero-balanced hypergeometric terms.

Theorem 2.1. *For all $p,q,\mu>0, \gamma\in\mathbb{R}$ and $0<x<1$ we have*

$$K_{p,q}^{(\gamma)}(\mu;x) = \frac{p+qx^\gamma}{1-x^\mu} - p\,{}_2F_1\left[\begin{matrix}1,\ \frac{q}{\mu}\\ \frac{q}{\mu}+1\end{matrix};x^\mu\right] - q\,x^\gamma\,{}_2F_1\left[\begin{matrix}1,\ \frac{p}{\mu}\\ \frac{p}{\mu}+1\end{matrix};x^\mu\right]. \tag{2.1}$$

Also, there holds

$$p\,{}_2F_1\left[\begin{matrix}1,\ \frac{q}{\mu}\\ \frac{q}{\mu}+1\end{matrix};x^\mu\right] + q\,x^\gamma\,{}_2F_1\left[\begin{matrix}1,\ \frac{p}{\mu}\\ \frac{p}{\mu}+1\end{matrix};x^\mu\right] < \frac{p+q\,x^\gamma}{1-x^\mu}\,. \tag{2.2}$$

Proof. The following conclusion–chain lead us to the asserted expression (2.1) for $K_{p,q}^{(\gamma)}(\mu;x)$, assuming that $a,b>0$ and $0<x<1$ (which enables the convergence of the following power series):

$$\begin{aligned}
L_b(\mu;x) &:= \sum_{k\ge 0}\left(1-\frac{b}{\mu\,k+b}\right)x^{\mu\,k} = \frac{1}{1-x^\mu} - A\sum_{k\ge 0}\frac{x^{\mu\,k}}{k+A}\\
&= \frac{1}{1-x^\mu} - A\sum_{k\ge 0}\frac{(1)_k\,\Gamma(k+A)}{\Gamma(k+A+1)}\,\frac{x^{\mu\,k}}{k!} = \frac{1}{1-x^\mu} - \sum_{k\ge 0}\frac{(1)_k\,(A)_k}{(A+1)_k}\,\frac{x^{\mu\,k}}{k!}\\
&= \frac{1}{1-x^\mu} - {}_2F_1\left[\begin{matrix}1,\ A\\ A+1\end{matrix};x^\mu\right] = \frac{1}{1-x^\mu} - {}_2F_1\left[\begin{matrix}1,\ \frac{b}{\mu}\\ \frac{b}{\mu}+1\end{matrix};x^\mu\right],
\end{aligned}$$

where $A := b\,\mu^{-1}$. Thus, for $p,q>0$, because

$$K_{p,q}^{(\gamma)}(\mu;x) = p\,L_q(\mu;x) + q\,x^\gamma\,L_p(\mu;x),$$

relation (2.1) is proved. Finally, since we have $K_{p,q}^{(\gamma)}(\mu;x)>0$, we deduce the inequality (2.2) and this completes the proof. □

Remark 2.2. For even positive integer values of μ and γ, the results achieved in Theorem 2.1 one extends to all $x \in (-1, 1)$. Moreover, it is worth to mention that if $p, q, \mu < 0$, $x \in (0, 1)$ and $\gamma \in \mathbb{R}$, then we get that

$$K_{p,q}^{(\gamma)}(\mu; x) = p \sum_{k \geq 0} \frac{k}{k + q/\mu} x^{\mu k} + q \sum_{k \geq 0} \frac{k}{k + p/\mu} x^{\mu k + \gamma} < 0,$$

that is, the inequality (2.2) is reversed.

The generalized trigonometric and generalized inverse trigonometric functions were introduced by Lindqvist [5]. For $p > 0$ the inverse p–trigonometric and p–hyperbolic functions are defined as special zero-balanced hypergeometric series, that is,

$$\operatorname{arctan}_p(x) = \int_0^x (1 + t^p)^{-1}\, \mathrm{d}t = x\, {}_2F_1\left[\begin{matrix} 1, \frac{1}{p} \\ \frac{1}{p} + 1 \end{matrix}; -x^p\right],$$

$$\operatorname{artanh}_p(x) = \int_0^x (1 - t^p)^{-1}\, \mathrm{d}t = x\, {}_2F_1\left[\begin{matrix} 1, \frac{1}{p} \\ \frac{1}{p} + 1 \end{matrix}; x^p\right].$$

Note that these functions were investigated by many authors in the recent years, see for example [1, 3] and the references therein. The following result is a variant of Theorem 2.1 in terms of generalized inverse trigonometric functions.

Theorem 2.3. *For all $p, q, \mu > 0$, $\gamma \in \mathbb{R}$ and $x \in (0, 1)$ we have*

$$p x^{-q} \operatorname{artanh}_{\frac{\mu}{q}}(x^q) + q x^{\gamma - p} \operatorname{artanh}_{\frac{\mu}{p}}(x^p) < \frac{p + q x^\gamma}{1 - x^\mu}. \tag{2.3}$$

Also for all $p > 0$ and $x \in (0, 1)$ it follows

$$\operatorname{artanh}_p(x) < \frac{x}{1 - x^p}. \tag{2.4}$$

Moreover, we have the asymptotic relation as $x \to 0$

$$\frac{p + q x^\gamma}{1 - x^\mu} - \frac{p}{x^q} \operatorname{artanh}_{\frac{\mu}{q}}(x^q) - \frac{q}{x^{p-\gamma}} \operatorname{artanh}_{\frac{\mu}{p}}(x^p) = \frac{p\,\mu}{q + \mu}\, x^\mu + \mathcal{O}\left(x^{\mu + \min(\gamma, \mu)}\right). \tag{2.5}$$

Proof. Transforming

$${}_2F_1\left[\begin{matrix} 1, \frac{p}{\mu} \\ \frac{p}{\mu} + 1 \end{matrix}; x^p\right] = {}_2F_1\left[\begin{matrix} 1, \frac{1}{\mu/p} \\ \frac{1}{\mu/p} + 1 \end{matrix}; (x^p)^{\frac{\mu}{p}}\right],$$

by means of (2.2) we deduce (2.3). Now, taking $p = q$ in (2.3) and then substituting $x \mapsto x^{1/p}$, $\mu = p^2$, we get (2.4). Finally, expanding (2.1), we have for $x \to 0$:

$$K_{p,q}^{(\gamma)}(\mu; x) = \frac{p\,\mu}{q + \mu}\, x^\mu + \mathcal{O}\left(x^{\mu + \min(\gamma, \mu)}\right).$$

Since $K_{p,q}^{(\gamma)}(\mu; x)$ coincides with the left hand side expression in (2.5), the assertion is proved. □

Now, in establishing the companion inequality associated with (1.3), we study the expression

$$\overline{K}^{(2)}_{3,1}(4;x) := 3\sum_{k\geq 0}\left(1-\frac{1}{4k+1}\right)x^{4k} - \sum_{k\geq 0}\left(1-\frac{3}{4k+3}\right)x^{4k+2} > 0\,.$$

To establish the positivity of $\overline{K}^{(2)}_{3,1}(4;x)$ for all $0<|x|<1$, it is enough to observe that

$$\begin{aligned}\overline{K}^{(2)}_{3,1}(4;x) &= 12\sum_{k\geq 0}\frac{k}{4k+1}\,x^{4k} - 4x^2\sum_{k\geq 0}\frac{k}{4k+3}\,x^{4k}\\ &> 4\sum_{k\geq 0}\left(\frac{3k}{4k+1}-\frac{k}{4k+3}\right)x^{4k}\,.\end{aligned}$$

Thus, rewriting $\overline{K}^{(2)}_{3,1}(4;x)$ in terms of hypergeometric series, and then in inverse trigonometric and hyperbolic terms, we conclude that

$$\overline{K}^{(2)}_{3,1}(4;x) = \frac{3-x^2}{1-x^4} - \frac{3}{x}\arctan x\,.$$

Having in mind that $\overline{K}^{(2)}_{3,1}(4;x)>0$, we get

$$\frac{3}{x}\arctan x < \frac{3-x^2}{1-x^4}\,, \qquad \big(0<|x|<1\big).$$

Also, the following asymptotic behavior holds true

$$\overline{K}^{(2)}_{3,1}(4;x) = \frac{12}{5}\,x^4 + \mathcal{O}(x^6), \qquad (x\to 0)$$

which coincides with the one in (1.4).

Now, the counterpart result of Theorem 2.1 reads as follows.

Theorem 2.4. *For all $p,q,\mu,\gamma>0$ such that $p\geq q$ and for all $0<x<1$ we have*

$$px^{-q}\,\mathrm{artanh}_{\frac{\mu}{q}}\,(x^q) - qx^{\gamma-p}\,\mathrm{artanh}_{\frac{\mu}{p}}\,(x^p) < \frac{p-qx^\gamma}{1-x^\mu}\,. \tag{2.6}$$

Proof. Consider the linear combination of power series

$$\overline{K}^{(\gamma)}_{p,q}(\mu;x) := p\sum_{k\geq 0}\left(1-\frac{q}{\mu k+q}\right)x^{\mu k} - q\sum_{k\geq 0}\left(1-\frac{p}{\mu k+p}\right)x^{\mu k+\gamma}\,.$$

For all $x\in(0,1)$ and $\gamma>0$ it follows

$$\begin{aligned}\overline{K}^{(\gamma)}_{p,q}(\mu;x) &> \mu\sum_{k\geq 0}\left(\frac{pk}{\mu k+q}-\frac{qk}{\mu k+p}\right)x^{\mu k}\\ &= \mu(p-q)\sum_{k\geq 0}\frac{k(\mu k+p+q)}{(\mu k+q)(\mu k+p)}\,x^{\mu k}\,;\end{aligned}$$

the last estimate is non–negative for $p\geq q$. Transforming the constituting sums of $\overline{K}^{(\gamma)}_{p,q}(\mu;x)$ into hypergeometric expressions, and following the lines of the proof of Theorem 2.3, we arrive at the desired inequality (2.6). □

We mention that the expression $L_b(\mu;x)$ can be expressed also in another way as

$$\begin{aligned} L_b(\mu;x) &= \sum_{k\geq 0}\frac{\mu k}{\mu k+b}\,x^{\mu k} = x\sum_{k\geq 0}\frac{\mu k}{\mu k+b}\,x^{\mu k-1} = \frac{x}{\mu}\frac{\mathrm{d}}{\mathrm{d}x}\sum_{k\geq 0}\frac{x^{\mu k}}{k+\frac{b}{\mu}} \\ &= \frac{x}{\mu}\frac{\mathrm{d}}{\mathrm{d}x}\sum_{k\geq 0}\frac{\Gamma(k+\frac{b}{\mu})\,\Gamma(k+1)}{(k+\frac{b}{\mu})\Gamma(k+\frac{b}{\mu})}\,\frac{x^{\mu k}}{k!} \\ &= \frac{x\Gamma(\frac{b}{\mu})}{\mu\,\Gamma(1+\frac{b}{\mu})}\frac{\mathrm{d}}{\mathrm{d}x}\sum_{k\geq 0}\frac{(\frac{b}{\mu})_k\,(1)_k}{(1+\frac{b}{\mu})_k}\,\frac{x^{\mu k}}{k!} \\ &= \frac{x}{b}\,\frac{\mathrm{d}}{\mathrm{d}x}\,{}_2F_1\left[\begin{matrix}\frac{b}{\mu},\,1\\ \frac{b}{\mu}+1\end{matrix}\;;x^{\mu}\right] = \frac{\mu}{b+\mu}\,x^{\mu}\,{}_2F_1\left[\begin{matrix}\frac{b}{\mu}+1,\,2\\ \frac{b}{\mu}+2\end{matrix}\;;x^{\mu}\right]. \end{aligned}$$

However, by this expression we cannot reach any rational upper bound for $K_{p,q}^{(\gamma)}(\mu;x)$.

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