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A convergence result for a contact problem with adhesion

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Abstract. We prove a convergence result for a system coupling two integral equations with a history-dependent variational inequality. More exactly, we consider the variational formulation of a quasistatic contact problem with adhesion. Then we prove the dependence of the weak solution with respect to the data. The proof is based on arguments of variational inequalities, Fréchet spaces and Gronwall inequalities.

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1. Introduction

The aim of this paper is to present a convergence result associated to a contact problem with adhesion. It is known that the processes of contact involving adhesion overcome in many industrial settings, when different parts are glued together. For this reason a lot of studies have been developed so the literature concerning this area is in a continuous expansion. According to [2] if we want to model a process in which bonding is not present and debonding may take place, an adhesion process is needed in order to describe the contact. Such models containing adhesion can be found in [1, 3, 5, 6, 9, 10].

The present paper represents a continuation of the paper [14] which covers the modelling and the variational analysis of a contact problem with adhesion and surface memory effects within the infinitesimal strain theory. Taking note of that, the present paper aims to prove a convergence result associated to the problem approached in [14].

The paper is structured as follows. Second Section presents the notations we have made and some short preliminary material. In Section 3 we describe the model

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and we list the assumptions on the data as well as the variational formulation of the problem as it was given in [14]. Finally in Section 4 we state and prove our main convergence result.

2. Notation and Preliminaries

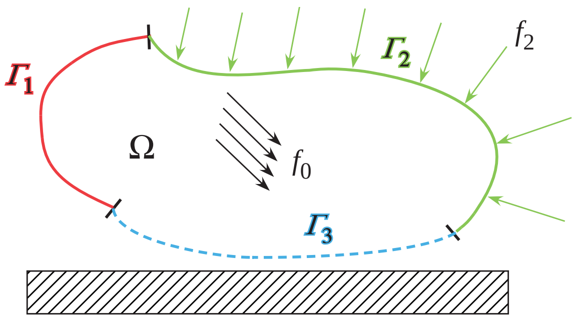

FIGURE 1. The physical setting; Γ_3 is the contact surface

We start this section by presenting the physical setting of the contact process we analyzed throughout the paper. We continue then with some important notation we also shall use throughout this paper. For further details we refer the reader to [2, 4, 7, 8]. Everywhere in this paper we use the notation $\mathbb{N}$ for the set of positive integers and $\mathbb{R}_+$ to denote the set of nonnegative real numbers, i.e. $\mathbb{R}_+ = [0, +\infty)$. For a given $r \in \mathbb{R}$ we denote by r^+ its positive part, i.e. $r^+ = max\,\{r, 0\}$. Also Ω is a bounded domain with a Lipschitz continuous boundary Γ, divided into three measurable parts Γ_1, Γ_2 and Γ_3, such that $meas\,(\Gamma_1) > 0$. Standard notation are used for the Lebesgue and Sobolev spaces associated to Ω and Γ and moreover we use the spaces

$$V = \{\, \boldsymbol{v} = (v_i) \in H^1(\Omega)^d : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1 \,\}$$

and

$$Q = \{\, \boldsymbol{\tau} = (\tau_{ij}) \in L^2(\Omega)^{d\times d} : \tau_{ij} = \tau_{ji} \,\}.$$

These are real Hilbert spaces endowed with the inner products

$$(\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx, \qquad (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\, dx,$$

and the associated norms $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively. Here $\boldsymbol{\varepsilon}$ represents the deformation operator and it is given by

$$\boldsymbol{\varepsilon}(\boldsymbol{v}) = (\varepsilon_{ij}(\boldsymbol{v})), \quad \varepsilon_{ij}(\boldsymbol{v}) = \frac{1}{2}\,(v_{i,j} + v_{j,i}) \qquad \forall\, \boldsymbol{v} \in H^1(\Omega)^d.$$

Completeness of the space $(V, \|\cdot\|_V)$ follows from the assumption $meas\,(\Gamma_1) > 0$, which allows the use of Korn's inequality. Moreover, the below mentioned sets are used in the proof of our result.

$$U = \{\, \boldsymbol{v} \in V : v_\nu \le g \quad \text{a.e. on} \quad \Gamma_3 \,\},$$

$$Z = \left\{\, \omega \in L^2(\Gamma_3) \,:\, 0 \le \omega \le 1 \quad \text{a.e. on} \;\; \Gamma_3 \,\right\},$$

where g is a positive constant. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$. Their corresponding inner product and norm are defined by

$$\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i \ , \qquad \|\boldsymbol{v}\| = (\boldsymbol{v}\cdot\boldsymbol{v})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{u},\boldsymbol{v} \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \sigma_{ij}\tau_{ij} \ , \qquad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{\sigma},\boldsymbol{\tau} \in \mathbb{S}^d.$$

Here and below the indices i, j, k, l run between 1 and d and, unless stated otherwise, the summation convention over repeated indices is used.

In this paper we assume that the material's behavior follows a viscoelastic constitutive law with long memory of the form

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds \quad \text{in } \Omega, \tag{2.1}$$

where, here and below, $\boldsymbol{u}$ denotes the displacement field, $\boldsymbol{\sigma}$ represents the stress field, $\boldsymbol{\varepsilon}(\boldsymbol{u})$ is the linearized strain tensor and $t \in \mathbb{R}_+$ represents the time variable. Also, $\mathcal{A}$ and $\mathcal{B}$ represent the elasticity operator and the relaxation tensor, respectively, and are assumed to verify the following conditions.

$$\left\{\begin{array}{l}
\text{(a) } \mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d.\\
\text{(b) There exists } L_{\mathcal{A}} > 0 \text{ such that}\\
\qquad \|\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{A}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|\\
\qquad\quad \forall\, \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega.\\
\text{(c) There exists } m_{\mathcal{A}} > 0 \text{ such that}\\
\qquad (\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2))\cdot(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{A}}\,\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2\\
\qquad\quad \forall\, \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega.\\
\text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}) \text{ is measurable on } \Omega,\\
\qquad \text{for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d.\\
\text{(e) The mapping } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x},\boldsymbol{0}) \text{ belongs to } Q.
\end{array}\right. \tag{2.2}$$

$$\mathcal{B} \in C(\mathbb{R}_+;\mathbf{Q}_\infty). \tag{2.3}$$

The contribution of the bonding to the normal traction, $\sigma_\nu^A(t)$, satisfies

$$\sigma_\nu^A(t) = \gamma_\nu \beta^2(t)\widetilde{R}(u_\nu(t)) \quad \text{on } \Gamma_3, \tag{2.4}$$

where $\widetilde{R}$ is the truncation function given by

$$\widetilde{R}(s) = \left\{\begin{array}{ll} L & \text{if } s < -L\\ -s & \text{if } -L \le s \le 0\\ 0 & \text{if } s > 0 \end{array}\right. \tag{2.5}$$

We follow [5, 6, 11] and assume that the bonding field satisfies the unilateral constraint

$$0 \le \beta(t) \le 1 \qquad \text{on} \;\; \Gamma_3. \tag{2.6}$$

Moreover, its evolution is governed by the differential equation

$$\dot{\beta}(t) = -\big(\gamma_\nu \beta(t)[R(u_\nu(t))]^2 - \varepsilon_a\big)^+ \quad \text{on} \;\; \Gamma_3 \tag{2.7}$$

in which ε_a represents the Dupré energy and R is the truncation operator given by

$$R(s) = \begin{cases} -L & \text{if } s < -L, \\ s & \text{if } -L \le s \le L, \\ L & \text{if } s > L. \end{cases} \tag{2.8}$$

In order to complete the differential equation we give the initial condition

$$\beta(0) = \beta_0 \quad \text{on} \ \ \Gamma_3 \tag{2.9}$$

and we assume that the adhesion coefficient, γ_ν, the Dupré energy ε_a, and initial bonding field, β_0, satisfy the conditions

$$\gamma_\nu \in L^\infty(\Gamma_3), \quad \gamma_\nu \ge 0, \quad \varepsilon_a \in L^\infty(\Gamma_3), \quad \varepsilon_a \ge 0, \tag{2.10}$$

$$\beta_0 \in L^2(\Gamma_3), \quad 0 \le \beta_0 \le 1 \quad \text{a.e. on} \ \ \Gamma_3. \tag{2.11}$$

Note that here and below $L > 0$ is the characteristic length of the bond, beyond which it stretches without offering any additional resistance (see, e.g., [9]). More details on this condition can be found in [11] and references therein. According to [14] when all the adhesive bonds are inactive, or broken, the motion is frictionless. Thus, the tangential traction depends on the intensity of adhesion and on the tangential displacement, but only up to the bond length L, that is

$$-\boldsymbol{\sigma}_\tau(t) = p_\tau(\beta(t))\boldsymbol{R}^*(\boldsymbol{u}_\tau(t)) \quad \text{on } \Gamma_3. \tag{2.12}$$

The truncation operator $\boldsymbol{R}^*$ is given by

$$\boldsymbol{R}^*(\boldsymbol{v}) = \begin{cases} \boldsymbol{v} & \text{if } \ \|\boldsymbol{v}\| \le L \\ \dfrac{L}{\|\boldsymbol{v}\|}\,\boldsymbol{v} & \text{if } \ \|\boldsymbol{v}\| \ge L. \end{cases} \tag{2.13}$$

The function p_ν will be used later in the paper. It satisfies

$$\begin{cases} \text{(a) } p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+. \\ \text{(b) There exists } L_\nu > 0 \text{ such that} \\ \qquad |p_\nu(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2)| \le L_\nu \, |r_1 - r_2| \\ \qquad \ \forall\, r_1,\, r_2 \in \mathbb{R}, \ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c) } (p_\nu(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2))(r_1 - r_2) \ge 0 \\ \qquad \forall\, r_1,\, r_2 \in \mathbb{R}, \ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto p_\nu(\boldsymbol{x}, r) \text{ is measurable on } \Gamma_3, \\ \qquad \text{for any } r \in \mathbb{R}. \\ \text{(e) } p_\nu(\boldsymbol{x}, r) = 0 \text{ for all } r \le 0, \ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases} \tag{2.14}$$

Next we will briefly present some of the other notation that are used in the paper during the proofs of the main result.

We use the Riesz representation Theorem to define the operator $P : V \to V$ and the function $\boldsymbol{f} : \mathbb{R}_+ \to V$ by equalities

$$(P\boldsymbol{u}, \boldsymbol{v})_V = \int_{\Gamma_3} p_\nu(u_\nu) v_\nu \, da \qquad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V, \tag{2.15}$$

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_{\Omega} \boldsymbol{f}_0(t) \cdot \boldsymbol{v} \, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v} \, da \qquad \forall\, \boldsymbol{v} \in V. \tag{2.16}$$

We assume that the densities of body forces and surface tractions have regularity

$$\boldsymbol{f}_0 \in C(\mathbb{R}_+; L^2(\Omega)^d), \quad \boldsymbol{f}_2 \in C(\mathbb{R}_+; L^2(\Gamma_2)^d). \tag{2.17}$$

We also consider b a surface memory function which verifies

$$b \in C(\mathbb{R}_+; L^\infty(\Gamma_3)), \quad b(t, \boldsymbol{x}) \geq 0 \quad \text{for all } t \in \mathbb{R}_+, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \tag{2.18}$$

Finally, we consider the functional $j : Z \times V \times V \to \mathbb{R}$ defined by

$$j(\beta, \boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} \Big[p_\tau(\beta(t)) \boldsymbol{R}^*(\boldsymbol{u}_\tau(t)) \cdot \boldsymbol{v}_\tau - \gamma_\nu \beta^2(t) \widetilde{R}(u_\nu(t)) v_\nu \Big] \, da \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in V, \beta \in Z. \tag{2.19}$$

3. The model

We start this section by presenting the problem statement as it was given in [14].

Problem $\mathcal{P}$. *Find a displacement field* $\boldsymbol{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$, *a stress field* $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ *and an adhesion field* $\beta : \Gamma_3 \times \mathbb{R}_+ \to [0, 1]$ *such that*

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{B}(t - s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s)) ds \quad \text{in} \quad \Omega, \tag{3.1}$$

$$\operatorname{Div} \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{in} \quad \Omega, \tag{3.2}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_1, \tag{3.3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on} \quad \Gamma_2, \tag{3.4}$$

$$-\boldsymbol{\sigma}_\tau(t) = p_\tau(\beta(t)) \boldsymbol{R}^*(\boldsymbol{u}_\tau(t)) \quad \text{on} \quad \Gamma_3, \tag{3.5}$$

$$\dot{\beta}(t) = -\big(\gamma_\nu \beta(t) [R(u_\nu(t))]^2 - \varepsilon_a\big)^+ \quad \text{on} \quad \Gamma_3, \tag{3.6}$$

for all $t \in \mathbb{R}_+$, there exists $\xi : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}$ which satisfies

$$\left.\begin{aligned} & u_\nu(t) \le g,\ \sigma_\nu(t) + p_\nu(u_\nu(t)) + \xi(t) - \gamma_\nu \beta^2(t)\widetilde{R}(u_\nu(t)) \le 0, \\ & (u_\nu(t) - g)[\sigma_\nu(t) + p_\nu(u_\nu(t)) + \xi(t) - \gamma_\nu \beta^2(t)\widetilde{R}(u_\nu(t))] = 0, \\ & 0 \le \xi(t) \le \int_0^t b(t-s)\, u_\nu^+(s)\, ds, \\ & \xi(t) = 0 \ \text{ if } u_\nu(t) < 0, \\ & \xi(t) = \int_0^t b(t-s)\, u_\nu^+(s)\, ds \text{ if } u_\nu(t) > 0 \end{aligned}\right\} \quad \text{on} \quad \Gamma_3, \tag{3.7}$$

for all $t \in \mathbb{R}_+$ and, moreover,

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \qquad \text{on} \ \ \Omega. \tag{3.8}$$

$$\beta(0) = \beta_0 \qquad \text{on} \ \ \Gamma_3, \tag{3.9}$$

We recall that (3.7) describes a condition with unilateral constraint. We assume that at a given moment t there is penetration which did not reach the bound g, i.e. $0 < u_\nu(t) < g$. Then, (3.7) yields

$$-\sigma_\nu(t) = p_\nu(u_\nu(t)) + \int_0^t b(t-s)\, u_\nu^+(s)\, ds. \tag{3.10}$$

This equality shows that at the moment t, the reaction of the foundation depends both on the current value of the penetration (represented by the term $p_\nu(u_\nu(t))$) and on the history of the penetration (represented by the integral term in (3.10)). A contact condition with unilateral constraint, normal compliance and surface memory effects was used in [12] and [13]. Assume now that at a given moment t there is separation between the body and the foundation, i.e. $u_\nu(t) < 0$. Then, (3.7) shows that

$$\sigma_\nu(t) = \gamma_\nu \beta^2(t)\widetilde{R}(u_\nu(t)), \tag{3.11}$$

which means that the reaction of the foundation is nonnegative and depends on the adhesion coefficient, on the square of intensity of adhesion and on the normal displacement, but as it does not exceed the bound length L. Once it exceeds it the normal traction remains constant and $|\sigma_\nu(t)| \le \gamma_\nu L$.

The unique weak solvability of this problem was proved in [14]. Further on, we present its variational formulation.

Problem $\mathcal{P}^V$. *Find a displacement field $\boldsymbol{u} : \mathbb{R}_+ \to U$, a stress field $\boldsymbol{\sigma} : \mathbb{R}_+ \to Q$ and a bonding field $\beta : \mathbb{R}_+ \to Z$ such that for all $t \in \mathbb{R}_+$ we have*

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds, \tag{3.12}$$

$$\begin{aligned} & (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + (P\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V + j(\beta(t), \boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t)) \\ & + \Big(\int_0^t b(t-s)\, u_\nu^+(s)\, ds, v_\nu^+ - u_\nu^+(t) \Big)_{L^2(\Gamma_3)} \ge (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \forall \boldsymbol{v} \in U, \end{aligned} \tag{3.13}$$

$$\beta(t) = \beta_0 - \int_0^t \left(\gamma_\nu \beta(s)[R(u_\nu(s))]^2 - \varepsilon_a\right)^+ ds. \tag{3.14}$$

In the next section we will present the main result of this paper namely the continuous dependence of the weak solution with respect to the data.

4. Dependence on the data

For each $\rho > 0$ let $\mathcal{B}_\rho$, b_ρ, $\boldsymbol{f}_{0\rho}$, $\boldsymbol{f}_{2\rho}$, $\beta_{0\rho}$ represent perturbations of $\mathcal{B}$, b, $\boldsymbol{f}_0$, $\boldsymbol{f}_2$ β_0, respectively, which satisfy conditions (2.3), (2.18), (2.17) and (2.9), respectively. In other words, let

$$\mathcal{B}_\rho \to \mathcal{B} \quad \text{in } C(\mathbb{R}_+; \mathbf{Q}_\infty) \quad \text{as} \quad \rho \to 0, \tag{4.1}$$

$$b_\rho \to b \quad \text{in} \quad C(\mathbb{R}_+; L^\infty(\Gamma_3)) \quad \text{as} \quad \rho \to 0, \tag{4.2}$$

$$\boldsymbol{f}_{0\rho} \to \boldsymbol{f}_0 \quad \text{in} \quad C(\mathbb{R}_+; L^2(\Omega)^d) \quad \text{as} \quad \rho \to 0, \tag{4.3}$$

$$\boldsymbol{f}_{2\rho} \to \boldsymbol{f}_2 \quad \text{in} \quad C(\mathbb{R}_+; L^2(\Gamma_2)^d) \quad \text{as} \quad \rho \to 0, \tag{4.4}$$

$$\beta_{0\rho} \to \beta_0 \quad \text{in} \quad C(\mathbb{R}_+; L^2(\Gamma_3)) \quad \text{as} \quad \rho \to 0. \tag{4.5}$$

Moreover, there exists

$$\begin{cases} F : \mathbb{R}_+ \to \mathbb{R}_+ \text{ and } \alpha \in \mathbb{R}_+ \text{ s. t.} \\ \text{(a) } |p_\rho(\boldsymbol{x}, r) - p(\boldsymbol{x}, r)| \le F(\rho)(|r| + \alpha) \\ \qquad \forall\, r \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3, \text{ for each } \rho > 0. \\ \text{(b) } F(\rho) \to 0 \quad \text{as} \quad \rho \to 0. \end{cases} \tag{4.6}$$

So, the perturbed variational problem is as follows.

Problem ${\mathcal{P}_\rho}^V$. *Find a displacement field* $\boldsymbol{u}_\rho : \mathbb{R}_+ \to U$, *a stress field* $\boldsymbol{\sigma}_\rho : \mathbb{R}_+ \to Q$ *and a bonding field* $\beta_\rho : \mathbb{R}_+ \to Z$ *such that for all* $t \in \mathbb{R}_+$ *we have*

$$\boldsymbol{\sigma}_\rho(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) + \int_0^t \mathcal{B}_\rho(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(s))ds, \tag{4.7}$$

$$\begin{aligned} &(\boldsymbol{\sigma}_\rho(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)))_Q + (P_\rho \boldsymbol{u}_\rho(t), \boldsymbol{v} - \boldsymbol{u}_\rho(t))_V \\ &+ j(\beta_\rho(t), \boldsymbol{u}_\rho(t), \boldsymbol{v} - \boldsymbol{u}_\rho(t)) + \Big(\int_0^t b_\rho(t-s)\, u_{\rho\nu}^+(s)\, ds, v_\nu^+ - u_{\rho\nu}^+(t) \Big)_{L^2(\Gamma_3)} \\ &\ge (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_\rho(t))_V \quad \forall\, \boldsymbol{v} \in U, \end{aligned} \tag{4.8}$$

$$\beta_\rho(t) = \beta_0 - \int_0^t \big(\gamma_\nu \beta_\rho(s)[R(u_\rho \nu(s))]^2 - \varepsilon_a\big)^+ ds. \tag{4.9}$$

Theorem 4.1. *Under the assumptions* (4.1)–(4.5), *the solution* $(\boldsymbol{u}_\rho, \boldsymbol{\sigma}_\rho, \beta_\rho)$ *of Problem* $\mathcal{P}_\rho^V$ *converges to the solution* $(\boldsymbol{u}, \boldsymbol{\sigma}, \beta)$ *of Problem* $\mathcal{P}^V$,

$$\boldsymbol{u}_\rho \to \boldsymbol{u} \quad in \quad C(\mathbb{R}_+; U)$$

$$\boldsymbol{\sigma}_\rho \to \boldsymbol{\sigma} \quad in \quad C(\mathbb{R}_+; Q)$$

$$\beta_\rho \to \beta \quad in \quad C(\mathbb{R}_+; Z)$$

as $\rho \to 0$.

Proof. Let $n \in \mathbb{N}$ and $t \in [0, n]$. We put $\boldsymbol{v} = \boldsymbol{u}_\rho(t)$ in $\mathcal{P}^V$ and $\boldsymbol{v} = \boldsymbol{u}(t)$ in ${\mathcal{P}_\rho}^V$ then we combine the variational problem $\mathcal{P}^V$ with the perturbed variational problem ${\mathcal{P}_\rho}^V$ and we get

$$\begin{aligned}
&(P\boldsymbol{u} - P_\rho \boldsymbol{u}_\rho(t), \boldsymbol{u}_\rho(t) - \boldsymbol{u}(t))_V + j(\beta(t), \boldsymbol{u}(t), \boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)) \\
&\qquad + j(\beta_\rho(t), \boldsymbol{u}_\rho(t), \boldsymbol{u}(t) - \boldsymbol{u}_\rho(t)) \\
&+ \Big(\int_0^t b(t-s)\, u_\nu^+(s)\, ds - \int_0^t b_\rho(t-s)\, u_{\rho\nu}^+(s)\, ds, u_{\rho\nu}^+(t) - u_\nu(t)^+ \Big)_{L^2(\Gamma_3)} \\
&\qquad + (\boldsymbol{f}_\rho(t) - f(t), \boldsymbol{u}_\rho(t) - \boldsymbol{u}(t))_V + \\
&+ \Big(\int_0^t \mathcal{B}_\rho(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(s))ds - \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\, ds, \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \Big)_Q \\
&\geq \Big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \Big)_Q
\end{aligned} \tag{4.10}$$

Using (2.2) we deduce that

$$\Big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \Big)_Q \geq m_\mathcal{A} \|\boldsymbol{u}_\rho - \boldsymbol{u}\|_V^2 \tag{4.11}$$

In addition

$$\begin{aligned}
&\Big(\int_0^t \mathcal{B}_\rho(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(s))ds - \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\, ds, \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \Big)_Q \\
&\quad \leq \Big[\Theta_{\rho n} \int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|\, ds + \omega_{\rho n} \int_0^t \|\boldsymbol{u}(s)\|\, ds \Big] \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V,
\end{aligned} \tag{4.12}$$

$$\begin{aligned}
&\Big(\int_0^t b(t-s)\, u_\nu^+(s)\, ds - \int_0^t b_\rho(t-s)\, u_{\rho\nu}^+(s)\, ds, u_{\rho\nu}^+(t) - u_\nu(t)^+ \Big)_{L^2(\Gamma_3)} \\
&\quad \leq \Big[\Theta_{\rho n}^b \int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|\, ds + \omega_{\rho n}^b \int_0^t \|\boldsymbol{u}(s)\|\, ds \Big] \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V,
\end{aligned} \tag{4.13}$$

and

$$(\boldsymbol{f}_\rho(t) - f(t), \boldsymbol{u}_\rho(t) - \boldsymbol{u}(t))_V \leq \delta_{\rho n} \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V, \tag{4.14}$$

where

$$\Theta_{\rho n} = c_0^2 \max_{r\in[0,n]} \|\mathcal{B}_\rho(r)\|_Q, \tag{4.15}$$

$$\omega_{\rho n} = c_0^2 \max_{r\in[0,n]} \|\mathcal{B}_\rho(r) - \mathcal{B}(r)\|_Q, \tag{4.16}$$

$$\Theta_{\rho n}^b = c_0^2 \max_{r\in[0,n]} \|b_\rho(r)\|_{L^\infty(\Gamma_3)}, \tag{4.17}$$

$$\omega_{\rho n}^b = c_0^2 \max_{r\in[0,n]} \|b_\rho(r) - b(r)\|_{L^\infty(\Gamma_3)}, \tag{4.18}$$

$$\delta_{\rho n} = \max_{r\in[0,n]} \|\boldsymbol{f}_\rho(r) - \boldsymbol{f}(r)\|_V. \tag{4.19}$$

Moreover,

$$\begin{aligned}&(P\boldsymbol{u} - P_\rho \boldsymbol{u}_\rho(t), \boldsymbol{u}_\rho(t) - \boldsymbol{u}(t))_V \\ &\qquad \le F(\rho)\big(c_0^2\|\boldsymbol{u}(t)\|_V + c_0\,\alpha\, meas(\Gamma_3)^{1/2}\big)\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V.\end{aligned} \tag{4.20}$$

From [14] we have that

$$j(\beta_1, \boldsymbol{u}_1, \boldsymbol{u}_2 - \boldsymbol{u}_1) + j(\beta_2, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \le c\|\beta_1 - \beta_2\|_{L^2(\Gamma_3)}\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V.$$

Analogous, in our context we have

$$j(\beta, \boldsymbol{u}, \boldsymbol{u}_\rho - \boldsymbol{u}) + j(\beta_\rho, \boldsymbol{u}_\rho, \boldsymbol{u} - \boldsymbol{u}_\rho) \le c\|\beta - \beta_\rho\|_{L^2(\Gamma_3)}\|\boldsymbol{u} - \boldsymbol{u}_\rho\|_V. \tag{4.21}$$

Note that c is a constant which does not depend on t and whose values can change from line to line. From (4.11) – (4.21) we deduce that

$$\begin{aligned}&m_A\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V^2 \le \delta_{\rho n}\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \\ &\quad + F(\rho)\big(c_0^2\|\boldsymbol{u}(t)\|_V + c_0\,\alpha\, meas(\Gamma_3)^{1/2}\big)\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \\ &\quad + \Big[\Theta^b_{\rho n}\int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|\,ds + \omega^b_{\rho n}\int_0^t \|\boldsymbol{u}(s)\|\,ds\Big]\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \\ &\quad + \Big[\Theta_{\rho n}\int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|\,ds + \omega_{\rho n}\int_0^t \|\boldsymbol{u}(s)\|\,ds\Big]\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \\ &\quad + c\|\beta - \beta_\rho\|_{L^2(\Gamma_3)}\|\boldsymbol{u}(t) - \boldsymbol{u}_\rho(t)\|_V\end{aligned} \tag{4.22}$$

Consequently

$$\begin{aligned}&m_A\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V^2 \\ &\le \Big[\delta_{\rho n} + F(\rho)\big(c_0^2\|\boldsymbol{u}(t)\|_V + c_0\,\alpha\, meas(\Gamma_3)^{1/2}\big) + (\omega^b_{\rho n} + \omega_{\rho n})\int_0^t \|\boldsymbol{u}(s)\|\,ds\Big] \\ &\qquad + (\Theta^b_{\rho n} + \Theta_{\rho n})\int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|\,ds + c\|\beta - \beta_\rho\|_{L^2(\Gamma_3)}\end{aligned} \tag{4.23}$$

Next, we denote

$$\xi_{n,n} = \frac{1}{m_A}\max\Big\{1,\ c_0^2\|\boldsymbol{u}(t)\|_V + c_0\,\alpha\, meas(\Gamma_3)^{1/2},\ \int_0^t \|\boldsymbol{u}(s)\|\,ds\Big\}. \tag{4.24}$$

Once again, from [2] we have that

$$\|\beta - \beta_\rho\|_{L^2(\Gamma_3)} \le c\int_0^t \|\boldsymbol{u}(s) - \boldsymbol{u}_\rho(s)\|_V\,ds \tag{4.25}$$

So we have that

$$\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \tag{4.26}$$

$$\leq \Big[\delta_{\rho n} + F(\rho) + \omega^b_{\rho n} + \omega_{\rho n}\Big]\xi_{n,n} + \frac{\Theta^b_{\rho n} + \Theta_{\rho n} + c}{m_A}\int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|_V\, ds.$$

We know that $((\Theta_{\rho n})_\rho,\, (\Theta^b_{\rho n})_\rho)$ are bounded sequences so we can conclude that there exists $\zeta_n > 0$ which only depends on n and it is independent of ρ such that

$$0 \leq \frac{\Theta^b_{\rho n} + \Theta_{\rho n} + c}{m_A} \leq \zeta_n, \text{ for all } \rho \geq 0.$$

We deduce that

$$\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \tag{4.27}$$

$$\leq \Big[\delta_{\rho n} + F(\rho) + \omega^b_{\rho n} + \omega_{\rho n}\Big]\xi_{n,n} + \zeta_n\int_0^n \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V\, ds.$$

Using the Gronwall inequality we get that

$$\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \leq \Big[F(\rho) + \delta_{\rho n} + \omega^b_{\rho n} + \omega_{\rho n}\Big]\xi_{n,n}e^{\xi_n \rho n}. \tag{4.28}$$

Now using the fact that $F(\rho) \to 0, \omega_{\rho n} \to 0, \delta_{\rho n} \to 0, \omega^b_{\rho n} \to 0$ we deduce that

$$\max_{t\in[0,n]} \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \to 0 \text{ for } \rho \to 0. \tag{4.29}$$

In conclusion, we have that

$$\max_{t\in[0,n]} \|\beta(t) - \beta_\rho(t)\|_{L^2(\Gamma_3)} \to 0 \text{ for } \rho \to 0. \tag{4.30}$$

In the same time

$$\|\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}_\rho(t)\|_Q \leq L_{\mathcal{A}}\|\boldsymbol{u}(t) - \boldsymbol{u}_\rho(t)\|_V + \tag{4.31}$$

$$+\Theta_{\rho n}\int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|_V\, ds + \omega_{\rho n}\int_0^n \|\boldsymbol{u}(s)\|_V\, ds.$$

Taking into account (4.29) and the fact that $((\Theta_{\rho n})_\rho)$ is bounded and $\omega_{\rho n} \to 0$ we get

$$\max_{t\in[0,n]} \|\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}_\rho(t)\|_Q \to 0.$$

□

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