Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 2, 303–310

# On global smoothness preservation by Bernstein-type operators

Maria Talpău Dimitriu

**Abstract.** We study global smoothness preservation of a function $f$ by sequences of Bernstein-type operators with respect to a certain modulus of continuity of order two.

**Mathematics Subject Classification (2010):** 41A36, 41A17.

**Keywords:** Global smoothness preservation, second order modulus of continuity, simultaneous approximation, Bernstein-type operators.

## 1. Introduction

Let $(L_n)_{n\in\mathbb{N}}$ a sequence of linear positive operators of approximation on $C[0,1]$-the space of continuous real-valued function on $[0,1]$. If global smoothness of a continuous function $f$ is expressed by a Lipschitz condition with some modulus of continuity, it is of interest if $L_n f$ verify same condition. Further, if the sequence of operators present simultaneous approximation property, namely for $f \in C^r[0,1]$ the sequence $(L_n f)^{(r)}$ converges to the $f^{(r)}$ uniformly on $[0,1]$, it is also important to study if is preserved global smoothness of the derivatives.

The preservation of global smoothness properties by the Bernstein operators

$$B_n(f,x) = \sum_{j=0}^{n} f\left(\frac{j}{n}\right) P_{n,j}(x),\ f \in C[0,1],\ x \in [0,1],$$

$$P_{n,j}(x) = \binom{n}{j} x^j (1-x)^{n-j},$$

were studied in [6], [7], [4], [2], [5], [3]. In [10], D.-X. Zhou showed that the Lipschitz classes with respect to the second order modulus

$$\omega_2(f,t) = \sup\left\{|f(x-h) - 2f(x) + f(x+h)| :\ x \pm h \in [0,1],\ 0 < h \leq t\right\}$$

This paper was presented at the third edition of the International Conference on Numerical Analysis and Approximation Theory (NAAT 2014), Cluj-Napoca, Romania, September 17-20, 2014.

are not preserved by the Bernstein operators. He introduced the following modulus of smoothness of order two

$$\begin{aligned}\widetilde{\omega}_2(f,t) &= \sup\{|f(x+h_1+h_2)-f(x+h_1)-f(x+h_2)+f(x)| : \\ &\quad x, x+h_1+h_2 \in [0,1]\,,\, h_1,h_2>0,\ h_1+h_2 \le 2t\}\end{aligned} \tag{1.1}$$

and proved:

**Theorem A.** *Let $f \in C[0,1]$, $n \in \mathbb{N}$, $M > 0$ and $0 < \alpha \le 1$.*
*If*

$$\widetilde{\omega}_2(f,t) \le Mt^\alpha,\ 0 < t \le \frac{1}{2},$$

*then*

$$\widetilde{\omega}_2(B_n f,t) \le Mt^\alpha,\ 0 < t \le \frac{1}{2}.$$

We consider the Bernstein-type operators

$$L_n(f,x) = \sum_{j=0}^{n} P_{n,j}(x)F_{n,j}(f),\ f \in C[0,1],\ x \in [0,1], \tag{1.2}$$

where $P_{n,j}(x) = \binom{n}{j}x^j(1-x)^{n-j}$ and $F_{n,j} : C[0,1] \longrightarrow \mathbb{R}$, $j = \overline{0,n}$, are linear positive functionals.

In the next section we study simultaneous global smoothness preservation in terms of modulus of continuity $\omega_2^*$ introduced by Păltănea [8], [1], [9], defined for $f \in C[0,1]$ and $t > 0$ by

$$\begin{aligned}\omega_2^*(f,t) &= \sup\{|\Delta(f;u,y,v)| : u,v \in [0,1]\,,\, u \neq v, \\ &\quad u \le y \le v,\ v-u \le 2t\},\end{aligned} \tag{1.3}$$

where

$$\Delta(f;u,y,v) = \frac{v-y}{v-u}f(u) + \frac{y-u}{v-u}f(v) - f(y).$$

## 2. Main result

Firstly, we present two auxiliary results.

**Lemma 2.1.** *The derivative of $r$-th order of the Bernstein-type polynomial $L_n f$, $r \in \mathbb{N}$, $n \ge r$, has the expression*

$$(L_n f)^{(r)}(x) = \sum_{j=0}^{n-r} P_{n-r,j}(x)G_{n,j,r}(f), \tag{2.1}$$

*with*

$$G_{n,j,r}(f) = (n)_r \sum_{i=0}^{r} (-1)^{r-i}\binom{r}{i} F_{n,j+i}(f), \tag{2.2}$$

*where $(n)_r = n(n-1)\cdots(n-r+1)$ is the Pochhammer symbol.*

*Proof.* We prove the formula by induction with regard to $r$. For $r=1$ we have

$$\begin{aligned}
(L_n f)'(x) &= -nF_{n,0}(f)(1-x)^{n-1} \\
&\quad + \sum_{j=1}^{n-1} \binom{n}{j} [x^j(1-x)^{n-j}]' F_{n,j}(f) + nF_{n,n}(f)x^{n-1} \\
&= n\sum_{j=1}^{n} \binom{n-1}{j-1} x^{j-1}(1-x)^{n-j} F_{n,j}(f) \\
&\quad - n\sum_{j=0}^{n-1} \binom{n-1}{j} x^{j}(1-x)^{n-1-j} F_{n,j}(f) \\
&= n\sum_{j=0}^{n-1} P_{n-1,j}(x) \left[F_{n,j+1}(f) - F_{n,j}(f)\right].
\end{aligned}$$

Suppose now that the formula (2.1) is true for $r \in \mathbb{N}$, $r \geq 2$ and prove it for $r+1$, $n \geq r+1$.

$$\begin{aligned}
(L_n f)^{(r+1)}(x) &= (n-r)\sum_{j=0}^{n-r-1} \binom{n-r-1}{j} x^j(1-x)^{n-r-1-j} \left[G_{n,j+1,r}(f) - G_{n,j,r}(f)\right] \\
&= \sum_{j=0}^{n-r-1} P_{n-r-1,j}(x) G_{n,j,r+1}(f).
\end{aligned}$$

□

In [10] is given the following representation of the Bernstein operators

$$B_n(f, x+ty) = \sum_{k+l=0}^{n} P_{n,k,l}(x,y) \sum_{m=0}^{l} P_{l,m}(t) f\left(\frac{k+m}{n}\right),$$

$0 \leq x \leq 1$, $y > 0$, $x+y \leq 1$ and $0 \leq t \leq 1$, where

$$P_{n,k,l}(x,y) = \frac{n!}{k!\,l!\,(n-k-l)!} x^k y^l (1-x-y)^{n-k-l}.$$

Similarly is obtained:

**Lemma 2.2.** *Let $f \in C[0,1]$, $r \in \mathbb{N} \cup \{0\}$, $0 \leq u < v \leq 1$, $\lambda \in [0,1]$. Then for $n \geq r+1$ we have*

$$\begin{aligned}
&(L_n f)^{(r)}((1-\lambda)u + \lambda v) \qquad (2.3) \\
&= \sum_{k+l=0}^{n-r} P_{n-r,k,l}(u, v-u) \sum_{m=0}^{l} P_{l,m}(\lambda) G_{n,k+m,r}(f),
\end{aligned}$$

*where $G_{n,j,r}(f)$, $0 \leq j \leq n-r$, $r \in \mathbb{N}$ is defined in (2.2) and $G_{n,j,0}(f) = F_{n,j}(f)$, $0 \leq j \leq n$. We agree that $(L_n f)^{(0)} = L_n f$.*

*Sketch of proof.* We have

$$\begin{aligned}
&(L_nf)^{(r)}\left((1-\lambda)u+\lambda v\right)=(L_nf)^{(r)}\left(u+\lambda(v-u)\right)\\
&=\sum_{j=0}^{n-r}P_{n-r,j}(u+\lambda(v-u))G_{n,j,r}(f)\\
&=\sum_{j=0}^{n-r}\binom{n-r}{j}\sum_{k=0}^{j}\binom{j}{k}u^k\lambda^{j-k}(v-u)^{j-k}\cdot\\
&\cdot\sum_{p=0}^{n-r-j}\binom{n-r-j}{p}(1-v)^{n-r-j-p}(1-\lambda)^p(v-u)^pG_{n,j,r}(f)\\
&=\sum_{j=0}^{n-r}\sum_{k=0}^{j}\sum_{p=0}^{n-r-j}P_{n-r,k,j-k+p}(u,v-u)P_{j-k+p,j-k}(\lambda)G_{n,j,r}(f)
\end{aligned}$$

It makes the change of index $j-k+p=l$ and reverses the order of summations, afterwards it changes the index $j-k=m$ and is obtained (2.3).

**Theorem 2.3.** *Let $r\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $L_n$ defined by (1.2), $n\geq r+1$, with $F_{n,j}: C[0,1]\longrightarrow\mathbb{R}$, $j=\overline{0,n}$ linear positive functionals such that*

$$\left|\Delta\left(G_{n,\bullet,r}(f);j_1,j_2,j_3\right)\right|\leq\omega_2^*\left(f^{(r)},\frac{j_3-j_1}{2n}\right) \tag{2.4}$$

*hold $(\forall)f\in C^r[0,1]$, $(\forall)j_1,\,j_2,\,j_3\in\mathbb{N}:\ 0\leq j_1\leq j_2\leq j_3\leq n-r$, $j_1\neq j_3$, where $G_{n,j,r}(f)$, $0\leq j\leq n-r$, $r\in\mathbb{N}$ is defined in (2.2) and $G_{n,j,0}(f)=F_{n,j}(f)$, $0\leq j\leq n$. Let $f\in C^r[0,1]$, $M>0$, $\alpha\in(0,1]$.*
*If*

$$\omega_2^*\left(f^{(r)},t\right)\leq Mt^\alpha,\ 0<t\leq\frac{1}{2},$$

*then*

$$\omega_2^*\left((L_nf)^{(r)},t\right)\leq Mt^\alpha,\ 0<t\leq\frac{1}{2}.$$

*Proof.* Let $t\in\left(0,\frac{1}{2}\right]$. Let $u,v\in[0,1]$, $u\neq v$, $u\leq y\leq v$, $v-u\leq 2t$. We use the representation (2.3). We have:

$$\begin{aligned}
\left|\Delta\left((L_nf)^{(r)};u,y,v\right)\right|&=\left|\frac{v-y}{v-u}(L_nf)^{(r)}(u)+\frac{y-u}{v-u}(L_nf)^{(r)}(v)-(L_nf)^{(r)}(y)\right|\\
&\leq\sum_{k+l=0}^{n-r}P_{n-r,k,l}(u,v-u)\cdot\\
\cdot\left|\frac{v-y}{v-u}G_{n,k,r}(f)\right.&\left.+\frac{y-u}{v-u}G_{n,k+l,r}(f)-\sum_{m=0}^{l}P_{l,m}\left(\frac{y-u}{v-u}\right)G_{n,k+m,r}(f)\right|\\
&=\sum_{\substack{k+l=0\\ l\neq 0}}^{n-r}P_{n-r,k,l}(u,v-u)\cdot
\end{aligned}$$

$$\cdot\left|\frac{v-y}{v-u}G_{n,k,r}(f)+\frac{y-u}{v-u}G_{n,k+l,r}(f)-\sum_{m=0}^{l}P_{l,m}\left(\frac{y-u}{v-u}\right)G_{n,k+m,r}(f)\right|$$

$$\le \sum_{\substack{k+l=0\\ l\neq 0}}^{n-r}P_{n-r,k,l}(u,v-u)\sum_{m=0}^{l}P_{l,m}\left(\frac{y-u}{v-u}\right)\cdot$$

$$\cdot\left|\left(1-\frac{m}{l}\right)G_{n,k,r}(f)+\frac{m}{l}G_{n,k+l,r}(f)-G_{n,k+m,r}(f)\right|$$

$$=\sum_{\substack{k+l=0\\ l\neq 0}}^{n-r}P_{n-r,k,l}(u,v-u)\sum_{m=0}^{l}P_{l,m}\left(\frac{y-u}{v-u}\right)\left|\Delta\left(G_{n,\bullet,r}(f);k,k+m,k+l\right)\right|$$

$$\le \sum_{\substack{k+l=0\\ l\neq 0}}^{n-r}P_{n-r,k,l}(u,v-u)\sum_{m=0}^{l}P_{l,m}\left(\frac{y-u}{v-u}\right)\omega_2^*\left(f^{(r)},\frac{l}{2n}\right)$$

$$\le \sum_{\substack{k+l=0\\ l\neq 0}}^{n-r}P_{n-r,k,l}(u,v-u)\sum_{m=0}^{l}P_{l,m}\left(\frac{y-u}{v-u}\right)M\left(\frac{l}{2n}\right)^{\alpha}$$

$$=\frac{M}{2^{\alpha}}\left(\frac{n-r}{n}\right)^{\alpha}\sum_{k+l=0}^{n-r}P_{n-r,k,l}(u,v-u)\left(\frac{l}{n-r}\right)^{\alpha}$$

$$\le \frac{M}{2^{\alpha}}\left(\sum_{k+l=0}^{n-r}P_{n-r,k,l}(u,v-u)\frac{l}{n-r}\right)^{\alpha}$$

$$=\frac{M}{2^{\alpha}}(v-u)^{\alpha}\le Mt^{\alpha}.$$

Hence $\omega_2^*\left((L_nf)^{(r)},t\right)\le Mt^{\alpha}$. □

# 3. Applications

## 3.1. Global smoothness preservation by the Stancu operators

For $0\le a\le b$, the Stancu-Bernstein operators are given by

$$S_n^{(a,b)}(f,x)=\sum_{j=0}^{n}f\left(\frac{j+a}{n+b}\right)P_{n,j}(x),\ f\in C[0,1],\ x\in[0,1].$$

So

$$G_{n,j,0}(f)=F_{n,j}(f)=f\left(\frac{j+a}{n+b}\right),\ 1\le j\le n$$

and for $r\in\mathbb{N}$, $n\ge r+1$, $1\le j\le n-r$

$$\begin{aligned}G_{n,j,r}(f)&=(n)_r\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}f\left(\frac{j+a}{n+b}+\frac{i}{n+b}\right)\\&=(n)_r\Delta^r_{\frac{1}{n+b}}f\left(\frac{j+a}{n+b}\right),\end{aligned}$$

where $\Delta_h^r f(x)$ is the forward difference of order $r$ with step $h$ of $f$ at $x$.

Let $r \in \mathbb{N}_0$, $j_1, j_2, j_3 \in \mathbb{N} : 0 \le j_1 \le j_2 \le j_3 \le n-r$, $j_1 \ne j_3$. For $r = 0$, $f \in C[0,1]$ we have:

$$\begin{aligned}
|\Delta(F_{n,\cdot}(f); j_1, j_2, j_3)| &= \left| \frac{j_3 - j_2}{j_3 - j_1} F_{n,j_1}(f) + \frac{j_2 - j_1}{j_3 - j_1} F_{n,j_3}(f) - F_{n,j_2}(f) \right| \\
&= \left| \frac{j_3 - j_2}{j_3 - j_1} f\left(\frac{j_1 + a}{n+b}\right) + \frac{j_2 - j_1}{j_3 - j_1} f\left(\frac{j_3 + a}{n+b}\right) - f\left(\frac{j_2 + a}{n+b}\right) \right| \\
&= \left| \Delta\left(f; \frac{j_1 + a}{n+b}, \frac{j_2 + a}{n+b}, \frac{j_3 + a}{n+b}\right) \right| \\
&\le \omega_2^*\left(f, \frac{j_3 - j_1}{2(n+b)}\right) \le \omega_2^*\left(f, \frac{j_3 - j_1}{2n}\right).
\end{aligned}$$

For $r \ge 1$, $f \in C^r[0,1]$ we have

$$\begin{aligned}
&|\Delta(G_{n,\cdot,r}(f); j_1, j_2, j_3)| \\
&= (n)_r \left| \frac{j_3 - j_2}{j_3 - j_1} \Delta^r_{\frac{1}{n+b}} f\left(\frac{j_1 + a}{n+b}\right) + \frac{j_2 - j_1}{j_3 - j_1} \Delta^r_{\frac{1}{n+b}} f\left(\frac{j_3 + a}{n+b}\right) - \Delta^r_{\frac{1}{n+b}} f\left(\frac{j_2 + a}{n+b}\right) \right| \\
&\le (n)_r \int_0^{\frac{1}{n+b}} \cdots \int_0^{\frac{1}{n+b}} |\Delta(f^{(r)}; \frac{j_1 + a}{n+b} + u_1 + \cdots + u_r, \\
&\quad \frac{j_2 + a}{n+b} + u_1 + \cdots + u_r, \frac{j_3 + a}{n+b} + u_1 + \cdots + u_r)| du_r \cdots du_1 \\
&\le \frac{(n)_r}{(n+b)^r} \omega_2^*\left(f^{(r)}, \frac{j_3 - j_1}{2(n+b)}\right) < \omega_2^*\left(f^{(r)}, \frac{j_3 - j_1}{2n}\right).
\end{aligned}$$

We used that $\Delta_h^r f(x) = \int_0^h \cdots \int_0^h f^{(r)}(x + u_1 + \cdots u_r)\, du_r \cdots du_1$.

Thus we obtain:

**Theorem 3.1.** *Let $f \in C^r[0,1]$, $r \in \mathbb{N}_0$, $n \ge r+1$, $M > 0$, $\alpha \in (0,1]$. If*

$$\omega_2^*\left(f^{(r)}, t\right) \le M t^\alpha,\ 0 < t \le \frac{1}{2},$$

*then*

$$\omega_2^*\left((S_n^{(a,b)} f)^{(r)}, t\right) \le M t^\alpha,\ 0 < t \le \frac{1}{2}.$$

*In particular, when $b = 0$ we have the results for the Bernstein operator.*

### 3.2. Global smoothness preservation by the Kantorovich operators

The Kantorovich operators are defined by

$$M_n(f,x) = \sum_{j=0}^{n} \left( (n+1) \int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}} f(u)du \right) P_{n,j}(x),\ f \in L_1[0,1],\ x \in [0,1].$$

So

$$F_{n,j}(f) = (n+1) \int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}} f(u)du = \int_0^1 f\left(\frac{s+j}{n+1}\right) ds$$

and for $r \in \mathbb{N}$, $n \geq r+1$, $1 \leq j \leq n-r$

$$\begin{aligned} G_{n,j,r}(f) &= (n)_r \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} \int_0^1 f\left(\frac{s+j}{n+1} + \frac{i}{n+1}\right) ds \\ &= (n)_r \int_0^1 \Delta^r_{\frac{1}{n+1}} f\left(\frac{s+j}{n+1}\right) ds \end{aligned}$$

Let $r \in \mathbb{N}_0$, $j_1, j_2, j_3 \in \mathbb{N} : 0 \leq j_1 \leq j_2 \leq j_3 \leq n-r$, $j_1 \neq j_3$. For $r = 0$, $f \in C[0,1]$ we have:

$$\begin{aligned} &|\Delta(F_{n,\cdot}(f); j_1, j_2, j_3)| \\ &= \left| \frac{j_3 - j_2}{j_3 - j_1} F_{n,j_1}(f) + \frac{j_2 - j_1}{j_3 - j_1} F_{n,j_3}(f) - F_{n,j_2}(f) \right| \\ &= \left| \int_0^1 \left[ \frac{j_3 - j_2}{j_3 - j_1} f\left(\frac{s+j_1}{n+1}\right) + \frac{j_2 - j_1}{j_3 - j_1} f\left(\frac{s+j_3}{n+1}\right) - f\left(\frac{s+j_2}{n+1}\right) \right] ds \right| \\ &\leq \int_0^1 \left| \Delta\left(f; \frac{s+j_1}{n+1}, \frac{s+j_2}{n+1}, \frac{s+j_3}{n+1}\right) \right| ds \\ &\leq \omega_2^*\left(f, \frac{j_3 - j_1}{2(n+1)}\right) < \omega_2^*\left(f, \frac{j_3 - j_1}{2n}\right). \end{aligned}$$

For $r \geq 1$, $f \in C^r[0,1]$ we have

$$\begin{aligned} &|\Delta(G_{n,\cdot,r}(f); j_1, j_2, j_3)| \\ &\leq (n)_r \int_0^1 \left| \frac{j_3 - j_2}{j_3 - j_1} \Delta^r_{\frac{1}{n+1}} f\left(\frac{s+j_1}{n+1}\right) + \frac{j_2 - j_1}{j_3 - j_1} \Delta^r_{\frac{1}{n+1}} f\left(\frac{s+j_3}{n+1}\right) \right. \\ &\quad \left. - \Delta^r_{\frac{1}{n+1}} f\left(\frac{s+j_2}{n+1}\right) \right| ds \\ &\leq (n)_r \int_0^1 \int_0^{\frac{1}{n+1}} \cdots \int_0^{\frac{1}{n+1}} |\Delta(f^{(r)}; \frac{s+j_1}{n+1} + u_1 + \cdots + u_r, \\ &\quad \frac{s+j_2}{n+1} + u_1 + \cdots + u_r, \frac{s+j_3}{n+1} + u_1 + \cdots + u_r)| du_r \cdots du_1 ds \\ &\leq \frac{(n)_r}{(n+1)^r} \omega_2^*\left(f^{(r)}, \frac{j_3 - j_1}{2(n+1)}\right) < \omega_2^*\left(f^{(r)}, \frac{j_3 - j_1}{2n}\right). \end{aligned}$$

So we obtain:

**Theorem 3.2.** *Let* $f \in C^r[0,1]$, $r \in \mathbb{N}_0$, $M > 0$, $\alpha \in (0,1]$.
*If*

$$\omega_2^*\left(f^{(r)}, t\right) \le Mt^\alpha,\ 0 < t \le \frac{1}{2},$$

*then*

$$\omega_2^*\left((M_n f)^{(r)}, t\right) \le Mt^\alpha,\ 0 < t \le \frac{1}{2}.$$

**Acknowledgement.** This paper is supported by the Sectoral Operational Programme Human Resources Development (SOP HRD), financed from the European Social Fund and by the Romanian Government under the project number POSDRU/159/1.5/S/134378.

## References

[1] Adell, J.A., de la Cal, J., *Preservation of moduli of continuity by Bernstein-type operators*, in Approximation, Probability and Related Fields (G.A. Anastassiou and S.T. Rachev Eds.), Plenum Press, New York, (1994), 1-18.

[2] Anastassiou, G.A., Cottin, C., Gonska, H.H., *Global smoothness of approximating functions*, Analysis, Munich, **11**(1991), 43–57.

[3] Anastassiou, G.A., Gal, S., *Approximation Theory: Moduli of Continuity and Global Smoothness Preservation*, Birkhäuser, Boston, 2000.

[4] Brown, B.M., Elliot, D., Paget, D.F., *Lipschitz constants for the Bernstein polynomials of a Lipschitz continuous function*, J. Approx. Theory, **49**(1987), 196-199.

[5] Cottin, C., Gonska, H.H., *Simultaneous approximation and global smoothness preservation*, Rend. Circ. Mat. Palermo (2) Suppl., **33**(1993), 259–279.

[6] Hajek, D., *Uniform polynomial approximation*, Amer. Math. Monthly, **72**(1965), 681.

[7] Lindvall, T., *Bernstein polynomials and the law of large numbers*, Math. Scientist, **7**(1982), 127-139.

[8] Păltănea, R., *Improved constant in approximation with Bernstein operators*, Research Semin. Fac. Math. Babeş-Bolyai Univ. 6, Univ. Babes-Bolyai, Cluj-Napoca, (1988), 261-268.

[9] Păltănea, R., *Approximation theory using positive linear operators*, Birkhäuser, 2004.

[10] Zhou, D.-X., *On a problem of Gonska*, Results Math., **28**(1995), 169-183.

Maria Talpău Dimitriu
Transilvania University of Braşov
Faculty of Mathematics and Computer Sciences
B-dul Eroilor nr.29,
500036 Braşov, Romania
e-mail: `mdimitriu@unitbv.ro`