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Discrete operators associated with the Durrmeyer operator

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Abstract. In [3] the author constructed discrete operators associated with certain integral operators using a probabilistic approach. In this article we obtain positive linear operators of discrete type associated with the classical Durrmeyer operator with the aid of some quadrature formulas with positive coefficients. Using Gaussian quadratures we get operators which preserve the moments of the classical Durrmeyer operator up to a given order. Another class of discrete operators is obtained by using the quadratures generated by some positive linear operators. We study the convergence of the new operators and compare them with the Durrmeyer operator. Also, we present some problems of optimality and give numerical examples.

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1. Introduction

If f is an integrable function on $[0,1]$, then the classical Durrmeyer operator is defined by

$$M_n f(x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t) dt, \ x \in [0,1], \tag{1.1}$$

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

This operator was introduced by Durrmeyer in [1].

We observe that the Durrmeyer operator is an integral operator. It uses some integrals as information about the approximated function. In practice it is hard to obtain these integrals. It is known that the Riemann integrals can be approximated by quadrature formulas.

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Using the quadratures

$$(n+1)\int_0^1 p_{n,k}(t)f(t)dt \approx \sum_{j=0}^{m} A_{j,m}^{n,k} f(t_{j,m}^{n,k}),\ k=0,1,...,n \tag{1.2}$$

with positive coefficients, i.e. $A_{j,m}^{n,k} \geq 0$, $j=0,...,m$, $k=0,...,n$, we get the associated discrete operator

$$D_{n,m}f(x) = \sum_{k=0}^{n} p_{n,k}(x) \sum_{j=0}^{m} A_{j,m}^{n,k} f(t_{j,m}^{n,k}),\ x\in[0,1]. \tag{1.3}$$

The operator $D_{n,m}$ uses the values of the function f at the nodes $t_{j,m}^{n,k}$ as information about the approximated function.

Discrete operators associated with some positive operators using a probabilistic approach were obtained in [3].

The remainder terms of the quadratures (1.2) are given by

$$R_{n,m}^{k}(f) = (n+1)\int_0^1 p_{n,k}(t)f(t)dt - \sum_{j=0}^{m} A_{j,m}^{n,k} f(t_{j,m}^{n,k}),\ k=0,1,...,n.$$

If the quadrature formulas have the degree of exactness r, i.e.

$$\begin{aligned} R_{n,m}^{k}(e_i) &= 0,\ i=0,...,r \\ R_{n,m}^{k}(e_{r+1}) &\neq 0 \end{aligned}$$

where $e_i(x) = x^i$, then the associated discrete operator has the same images of the monomials e_i, $i=0,...,r$ as the original operator, i.e.

$$M_n e_i = D_{n,m} e_i,\ i=0,...,r.$$

The next result follows by using the well known Korovkin theorem and taking into account the uniform convergence of the Durrmeyer operators on the test functions e_i, $i=0,1,2$.

Theorem 1.1. *If the quadratures (1.2) have the degree of exactness at least two, then the sequence $(D_{n,m}f)_{n\geq 1}$ converges uniformly to the function f, for every $f\in C[0,1]$.*

Using the Shisha-Mond result from [2] we have the following estimations of the errors for the Durrmeyer operator and the associated discrete operator respectively:

$$|M_n f(x) - f(x)| \leq \left(1 + \frac{2(n-3)x(1-x)+2}{(n+2)(n+3)\delta^2}\right)\omega(f,\delta),\ x\in[0,1], \tag{1.4}$$

$$\begin{aligned} |D_{n,m}f(x) - f(x)| \leq |f(x)|\,|D_{n,m}e_0(x) - e_0(x)| + \\ \left(D_{n,m}e_0(x) + \frac{1}{\delta^2} D_{n,m}(e_2 - 2xe_1 + x^2 e_0)(x)\right)\omega(f,\delta),\ x\in[0,1], \end{aligned} \tag{1.5}$$

where $\delta > 0$ and $\omega(f,\cdot)$ is the modulus of continuity i.e.,

$$\omega(f,\delta) = \sup\{|f(x+h) - f(x)| : x, x+h \in [0,1], 0 \leq h \leq \delta\}.$$

We observe that if the quadrature formulas have the degree of exactness at least two then the associated discrete operator has the same approximation order as the Durrmeyer operator.

Also, we get the estimation

$$|M_n f(x) - D_{n,m} f(x)| \leq \max_{k\in\{0,\ldots,n\}} |R^k_{n,m}(f)|, \; x \in [0,1]. \tag{1.6}$$

In this article we obtain discrete operators associated with the Durrmeyer operator generated by quadratures of Gauss type and by quadratures obtained using positive linear operators.

2. Discrete operators generated by Gaussian quadratures

We have to approximate the integrals

$$\int_0^1 t^k(1-t)^{n-k} f(t)dt, \; k = 0, 1, ..., n.$$

Using the substitution $t = (1+u)/2$ we get

$$\frac{1}{2^{n+1}} \int_{-1}^1 (1+u)^k (1-u)^{n-k} f\left(\frac{1+u}{2}\right) du, \; k = 0, 1, ..., n. \tag{2.1}$$

To approximate the integrals from (2.1) we can use the Gauss Jacobi quadratures

$$\int_{-1}^1 (1+u)^\alpha (1-u)^\beta g\,(u)\, du \approx \sum_{j=0}^m B_{j,m} g(u_{j,m}). \tag{2.2}$$

We consider two cases.

2.1. The first case

We take

$$\alpha = k, \; \beta = n-k, \; g(u) = f\left(\frac{1+u}{2}\right).$$

The quadrature formulas (2.2) become

$$\int_{-1}^1 (1+u)^k (1-u)^{n-k} g\,(u)\, du \approx \sum_{j=0}^m B^{n,k}_{j,m} g(u^{n,k}_{j,m}), \; k = 0, 1, ..., n. \tag{2.3}$$

The nodes $u^{n,k}_{j,m}$, $j = 0, ..., m$, $k = 0, ..., n$ are the roots of the Jacobi orthogonal polynomial of degree $m+1$:

$$J^{(k,n-k)}_{m+1}(u) = \frac{1}{2^{m+1}(m+1)!} \frac{1}{\rho(u)} \frac{d^{m+1}}{du^{m+1}} \left[\rho(u)(u^2-1)^{m+1}\right], \; u \in [-1,1]$$

where

$$\rho(u) = (1+u)^k (1-u)^{n-k}.$$

Using [5, Th. 11.5.3], we get the coefficients

$$B_{j,m}^{n,k} = \frac{2^n(2m+n+2)(m+k)!(m+n-k)!}{(m+1)!(m+n+1)!J_m^{(k,n-k)}(u_{j,m}^{n,k})\frac{d}{du}\left[J_{m+1}^{(k,n-k)}(u)\right]_{u=u_{j,m}^{n,k}}}$$

for $j = 0, ..., m$ and $k = 0, ..., n$.

The associated discrete operator is

$$D_{n,m}^{GJ} f(x) = \frac{n+1}{2^{n+1}} \sum_{k=0}^{n} p_{n,k}(x) \binom{n}{k} \sum_{j=0}^{m} B_{j,m}^{n,k} f\left(\frac{1+u_{j,m}^{n,k}}{2}\right), \ x \in [0,1].$$

We have

$$D_{n,m}^{GJ} e_i = M_n e_i, \ i = 0, ..., 2m+1.$$

For $m = 0$ we get a Stancu operator [4]

$$D_{n,0}^{GJ} f(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k+1}{n+2}\right), \ x \in [0,1].$$

This operator was associated with the Durrmeyer operator in [3].

Theorem 2.1. *For every $m \in \mathbb{N}$ and for every $f \in C[0,1]$ we have that the sequence $(D_{n,m}^{GJ} f)_{n \geq 1}$ converges uniformly to the function f.*

Proof. If $m = 0$ then the convergence follows from [4].

For $m \geq 1$ the quadrature formulas (2.3) have the degree of exactness at least three. Using the Theorem 1.1 we get the conclusion. □

2.2. The second case

We take

$$\alpha = \beta = 0, \ g_{k,n}(u) = (1+u)^k (1-u)^{n-k} f\left(\frac{1+u}{2}\right)$$

and we use the Gauss Legendre quadratures

$$\int_{-1}^{1} g_{k,n}(u) du \approx \sum_{j=0}^{m} B_{j,m} g_{k,n}(u_{j,m}), \ k = 0, 1, ..., n. \tag{2.4}$$

The discrete operator is

$$D_{n,m}^{GL} f(x) =$$

$$\frac{n+1}{2^{n+1}} \sum_{k=0}^{n} p_{n,k}(x) \binom{n}{k} \sum_{j=0}^{m} B_{j,m} (1+u_{j,m})^k (1-u_{j,m})^{n-k} f\left(\frac{1+u_{j,m}}{2}\right), \ x \in [0,1],$$

where the nodes $u_{j,m}$, $j = 0, ..., m$ are roots of the Legendre orthogonal polynomial $J_{m+1}^{(0,0)}(u)$ and the coefficients are given by (see [5, Th. 11.6.2])

$$B_{j,m} = \frac{2}{(m+1)J_m^{(0,0)}(u_{j,m})\frac{d}{du}\left[J_{m+1}^{(0,0)}(u)\right]_{u=u_{j,m}}}$$

for $j = 0, ..., m$ and $k = 0, ..., n$.

Theorem 2.2. *If* $m \in \mathbb{N}$ *and*

$$m \geq \frac{n+1}{2}, \tag{2.5}$$

then the sequence $(D^{GL}_{n,m} f)_{n\geq 1}$ *converges to the function* f*, for every* $f \in C[0,1]$.

Proof. The quadrature formulas (2.4) have the degree of exactness $2m+1$. From the inequality (2.5) we get that the degree of exactness of the quadrature formulas are at least $n+2$. It follows that exists $r \in \mathbb{N}$, $r \geq 2$ such that

$$D^{GL}_{n,m} e_i = M_n e_i,\ i = 0, ..., r,$$

The convergence of the operators is assured by the Korovkin theorem taking into account the convergence of the Durrmeyer operators. □

For $m = n$ we get

$$D^{GL}_{n,n} e_i = M_n e_i,\ i = 0, ..., n+1.$$

We observe that the operator $D^{GJ}_{n,0}$ preserves the moments of the Durrmeyer operator up to order one while the operator $D^{GL}_{n,n}$ keeps the moments up to order $n+1$. Both operators use the same amount of information about the approximated function ($n+1$ evaluations of the function).

We approximate the function $f : [0,1] \to \mathbb{R}$, $f(x) = Exp(x^2)$ using the associated discrete operators $D^{GJ}_{n,0}$ and $D^{GL}_{n,n}$ for $n = 5$.

Operator	$\max_{x\in[0,1]} \lvert D_{n,m} f(x) - M_n f(x)\rvert$	$\max_{x\in[0,1]} \lvert D_{n,m} f(x) - f(x)\rvert$
$D^{GJ}_{n,0}$	$7.5 \cdot 10^{-2}$	$6.5 \cdot 10^{-1}$
$D^{GL}_{n,n}$	$4 \cdot 10^{-5}$	$5.8 \cdot 10^{-1}$

3. Discrete operators generated by quadratures obtained using positive linear operators

We consider the linear positive operators $L_n : C[0,1] \to C[0,1]$, $n \geq 1$ of the form

$$L_n f(t) = \sum_{j=0}^{n} w_{n,j}(t) f(t_{j,n}),\ f \in C[0,1],\ t \in [0,1],$$

where $w_{n,j} \in C[0,1]$, $w_{n,j} \geq 0$, and the corresponding approximation formula

$$f(t) = L_n f(t) + R_n f(t).$$

For $k = 0, ..., n$ we get the quadrature formulas

$$(n+1)\int_0^1 p_{n,k}(t) f(t)dt = \sum_{j=0}^{n} A_j^{n,k} f(t_{j,n}) + (n+1)\int_0^1 p_{n,k}(t) R_n f(t)dt,$$

where

$$A_j^{n,k} = (n+1)\int_0^1 p_{n,k}(t) w_{n,j}(t)dt,\ j,k = 0, ..., n.$$

We get the following associated discrete operator

$$D_n^{PL} f(x) = \sum_{k=0}^{n} p_{n,k}(x) \sum_{j=0}^{n} A_j^{n,k} f(t_{j,n}),\ x \in [0,1].$$

Theorem 3.1. *If the sequence $(L_n f)_{n\geq 1}$ converges uniformly to the function $f \in C[0,1]$, then the sequence $\left(D_n^{PL} f\right)_{n\geq 1}$ also converges uniformly to the function f and*

$$|D_n^{PL} f(x) - M_n f(x)| \leq \sup_{x\in[0,1]} |R_n f(x)|,\ x \in [0,1].$$

Proof. For every $x \in [0,1]$ we have

$$\left|D_n^{PL} f(x) - M_n f(x)\right| \leq (n+1) \int_0^1 p_{n,k}(t)\, |R_n f(t)|\, dt \leq \sup_{x\in[0,1]} |R_n f(x)|\, .$$

The convergence of the associated operators follows from the inequality

$$\left|D_n^{PL} f(x) - f(x)\right| \leq \left|D_n^{PL} f(x) - M_n f(x)\right| + |M_n f(x) - f(x)|\, . \qquad \square$$

We present two examples.

Example 3.2. For $0 \leq \alpha \leq \beta$ the Bernstein-Stancu operator is given by (see [4])

$$P_n^{\alpha,\beta} f(t) = \sum_{j=0}^{n} p_{n,j}(t) f\left(\frac{j+\alpha}{n+\beta}\right),\ t \in [0,1].$$

We get the associated discrete operator

$$D_n^{BS} f(x) = \frac{1}{2n+1} \sum_{k=0}^{n} \binom{n}{k} p_{n,k}(x) \sum_{j=0}^{n} \frac{\binom{n}{j}}{\binom{2n}{k+j}} f\left(\frac{j+\alpha}{n+\beta}\right),\ x \in [0,1].$$

For the case of the Bernstein operator ($\alpha = \beta = 0$) we have that the associated operator preserves the moments of the Durrmeyer operator up to the order 1, i.e.

$$D_n^B e_i = M_n e_i,\ i = 0,1.$$

Example 3.3. Let the sequence of divisions of the interval $[0,1]$ with the norm tending to 0

$$\Delta_n :\ 0 = t_{0,n} < t_{1,n} < \ldots < t_{n,n} = 1.$$

The spline linear operator is defined by

$$S_{n,1} f(t) = \sum_{j=0}^{n} s_j(t) f(t_{j,n}),\ t \in [0,1],$$

with the cardinal functions

$$s_0(t) = \begin{cases} \frac{t_{1,n}-t}{t_{1,n}-t_{0,n}},\ t \in [t_{0,n}, t_{1,n}] \\ 0,\ t \notin [t_{0,n}, t_{1,n}] \end{cases}$$

$$s_j(t) = \begin{cases} \frac{t-t_{j-1,n}}{t_{j,n}-t_{j-1,n}},\ t \in [t_{j-1,n}, t_{j,n}) \\ \frac{t_{j+1,n}-t}{t_{j+1,n}-t_{j,n}},\ t \in [t_{j,n}, t_{j+1,n}] \\ 0,\ t \notin [t_{j-1,n}, t_{j+1,n}] \end{cases},\ j = 1, ..., n-1$$

$$s_n(t) = \begin{cases} \frac{t-t_{n-1,n}}{t_{n,n}-t_{n-1,n}},\ t \in [t_{n-1,n}, t_{n,n}] \\ 0,\ t \notin [t_{n-1,n}, t_{n,n}] \end{cases}$$

The associated discrete operator is

$$D_n^{SL} f(x) = \sum_{k=0}^{n} p_{n,k}(x) \sum_{j=0}^{n} A_j^{n,k} f(t_{j,n}),\ x \in [0,1], \tag{3.1}$$

where

$$A_j^{n,k} = \int_0^1 p_{n,k}(t) s_j(t) dt,\ j,k = 0, ..., n.$$

We have

$$D_n^{SL} e_i = M_n e_i,\ i = 0, 1.$$

Next we will show an optimal property of the operator (3.1). It is known the connection between the spline functions and the optimal quadrature in sense of Sard.

We consider that the quadratures

$$\int_0^1 p_{n,k}(t) f(t) dt = \sum_{j=0}^{n} A_j^{n,k} f(t_{j,n}) + R_{n,k}(f),\ k = 0, ..., n \tag{3.2}$$

have the degree of exactness at least 0.

Let the space of functions

$$H^{1,2}[0,1] = \{g | g \in C[0,1],\ g\ \textit{absolutely continuous},\ g' \in L^2[0,1]\}.$$

If $f \in H^{1,2}[0,1]$ then the remainders of the quadrature formulas (3.2) can be written in the form

$$R_{n,k}(f) = \int_0^1 K_{n,k}(t) f'(t) dt,\ k = 0, ..., n$$

with

$$K_{n,k}(t) = R_{n,k}[(x-t)_+^0] = (x-t)_+^0 - \sum_{j=0}^{n} A_j^{n,k} (t_{j,n} - t)_+^0,$$

where

$$z_+ = \begin{cases} z,\ z \geq 0 \\ 0,\ z < 0 \end{cases}$$

We have the estimation

$$|R_{n,k}(f)| \leq \left(\int_0^1 K_{n,k}^2(t) dt \right)^{1/2} \left(\int_0^1 f'^2(t) dt \right)^{1/2}.$$

The quadratures (3.2) with the coefficients $A_j^{n,k}$ chosen such that we get

$$\inf_{A_j^{n,k}} \int_0^1 K_{n,k}^2(t) dt$$

are named optimal in sense of Sard (see [5, p. 261]). We obtain these quadratures by integration of the spline linear interpolation formula, i.e.

$$(n+1)\int_0^1 p_{n,k}(t)f(t)dt$$
$$= (n+1)\int_0^1 p_{n,k}(t)S_{n,1}f(t)dt + (n+1)\int_0^1 p_{n,k}(t)R_nf(t)dt.$$

We approximate the function $f:[0,1]\to\mathbb{R}$, $f(x)=Exp(x^2)$ using the associated discrete operators D_n^B and D_n^{SL} with equidistant nodes for $n=5$. Both operators use $n+1$ evaluations of the approximated function.

Operator	$\max\limits_{x\in[0,1]} \lvert D_nf(x)-M_nf(x)\rvert$	$\max\limits_{x\in[0,1]} \lvert D_nf(x)-f(x)\rvert$
D_n^B	$1.1\cdot 10^{-1}$	$5\cdot 10^{-1}$
D_n^{SL}	$3.5\cdot 10^{-2}$	$5.5\cdot 10^{-1}$

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