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On convergence of a kind of complex nonlinear Bernstein operators

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Abstract. The present article deals with the approximation properties and Voronovskaja type results with quantitative estimates for a certain class of complex nonlinear Bernstein operators $(NB_n f)$ of the form

$$(NB_n f)(z) = \sum_{k=0}^{n} p_{k,n}(z) G_n\left(f\left(\frac{k}{n}\right)\right), \quad |z| \le 1$$

attached to analytic functions on compact disks.

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1. Introduction

Approximation properties of complex Bernstein polynomials were initially studied by Lorentz [6]. Recently S. G. Gal has done a commendable work in this direction and he compiled the important papers in his recent book [2]. Concerning the convergence of the Bernstein polynomials in the complex plane, Bernstein proved that if $f : G \to \mathbb{C}$ is analytic in the open set $G \subseteq \mathbb{C}$ with $\overline{D}_1 \subset G$ where $\overline{D}_1 = \{z \in \mathbb{C} : |z| \le 1\}$ then the complex Bernstein polynomials

$$(B_n f)(z) = \sum_{k=0}^{n} \binom{n}{k} z^k (1-z)^{n-k} f\left(\frac{k}{n}\right)$$

converge uniformly to f in $\overline{D}_1$. In the present paper we study the rate of approximation of analytic functions and give a Voronovskaja type result for the nonlinear

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complex Bernstein operator (NB_nf). Nonlinear Bernstein operator of complex variable is defined as

$$(NB_nf)(z) = \sum_{k=0}^{n} p_{k,n}(z) G_n\left(f\left(\frac{k}{n}\right)\right) \tag{1.1}$$

where $G_n : \mathbb{C} \to \mathbb{C}$ satisfies the Hölder condition i.e,

$$|G_n(u) - G_n(v)| \leq R\,|u - v|^{\gamma}$$

for every $n \in \mathbb{N}$, $0 < \gamma \leq 1$ and suitable constant $R > 0$ and

$$\lim_{n\to\infty} [G_n(u) - u] = 0 \tag{1.2}$$

for every $u \in \overline{D}_1$ where $\overline{D}_1 = \{z \in \mathbb{C} : \ |z| \leq 1\}$.

2. Convergence Results

We will consider the following nonlinear version of complex Bernstein operator,

$$(NB_nf)(z) = \sum_{k=0}^{n} p_{k,n}(z)\, G_n\left(f\left(\frac{k}{n}\right)\right), \quad |z| \leq 1$$

defined for every $f \in \overline{D}_1$ for which (NB_nf) is well-defined, where

$$D_1 = \{z \in \mathbb{C} : \ |z| < 1\}$$

The real case of above operator (1.1) and some of its properties can be found in [5].

We are now ready to establish the main results of this study:

Theorem 2.1. *Suppose that $f : D_1 \to \mathbb{C}$is analytic in D_1, that is*

$$f(z) = \sum_{k=0}^{\infty} c_k z^k$$

for all $z \in D_1$. For all $|z| \leq 1$ and $n \in \mathbb{N}$, we have

$$|(NB_nf)(z) - f(z)| \leq R\left(\frac{3}{n}C(f)\right)^{\gamma},$$

where $0 < C(f) = \sum\limits_{k=2}^{\infty} k(k-1)\,|c_k| < \infty$.

Theorem 2.2. *Suppose that $f : D_1 \to \mathbb{C}$ is analytic in D_1. We can write*

$$f(z) = \sum_{k=0}^{\infty} c_k z^k$$

for all $z \in D_1$.The following Voronovskaja-type result in the closed unit disk holds,

$$\left|(NB_nf)(z) - f(z) - \frac{z(1-z)}{n} f''(z)\right| \leq R\left(\frac{|z(1-z)|}{2n}\frac{10}{n}M(f)\right)^{\gamma}$$

for all $n \in \mathbb{N}$, $z \in \overline{D}_1$, where $0 < M(f) = \sum\limits_{k=3}^{\infty} k(k-1)(k-2)^2 c_k < \infty$ *and* $0 < \gamma \leq 1$.

The linear counterpart of Theorem 2.2 is given by Gal [4]. Notice that our theorems contain appropriate result of Gal [4] as a special case.

3. Auxiliary Result

In this section we give a certain result, which is necessary to prove our theorems.

Lemma 3.1. (Lorentz [7, p. 40, Theorem 4]) *For polynomials* $P_n(z) = \sum_{k=0}^{n} a_k z^k$ *with complex coefficients on the disk* $|z| \leq 1$ *we put*

$$\|P_n\|_1 = \max_{|z|\leq 1} |P_n(z)| .$$

Then

$$\|P_n'\| \leq n \|P_n\| .$$

4. Proof of the Theorems

Proof of Theorem 2.1. We consider

$$\begin{aligned} |(NB_n f)(z) - f(z)| &= \left| \sum_{k=0}^{n} p_{k,n}(z) G_n\left(f\left(\frac{k}{n}\right)\right) - f(z) \sum_{k=0}^{n} p_{k,n}(z) \right| \\ &= \left| \sum_{k=0}^{n} p_{k,n}(z) \left\{ G_n\left(f\left(\frac{k}{n}\right)\right) - f(z) \right\} \right| \end{aligned}$$

$$\begin{aligned} &= \left| \sum_{k=0}^{n} p_{k,n}(z) \left\{ G_n\left(f\left(\frac{k}{n}\right)\right) - G_n(f(z)) + G_n(f(z)) - f(z) \right\} \right| \\ &\leq \left| \sum_{k=0}^{n} p_{k,n}(z) \left\{ G_n\left(f\left(\frac{k}{n}\right)\right) - G_n(f(z)) \right\} \right| + \left| \sum_{k=0}^{n} p_{k,n}(z) \left\{ G_n(f(z)) - f(z) \right\} \right| \end{aligned}$$

the last term in the last inequality goes to zero because of (1.2). Then we will estimate the first sum $I_1 = \left| \sum_{k=0}^{n} p_{k,n}(z) \left\{ G_n\left(f\left(\frac{k}{n}\right)\right) - G_n(f(z)) \right\} \right|$

$$\begin{aligned} I_1 &= \left| \sum_{k=0}^{n} p_{k,n}(z) \left\{ G_n\left(f\left(\frac{k}{n}\right)\right) - G_n(f(z)) \right\} \right| \\ &\leq \sum_{k=0}^{n} |p_{k,n}(z)| \left| G_n\left(f\left(\frac{k}{n}\right)\right) - G_n(f(z)) \right| . \end{aligned}$$

By using Hölder condition $0 < \gamma \leq 1$,

$$\leq R \sum_{k=0}^{n} |p_{k,n}(z)| \left| f\left(\frac{k}{n}\right) - f(z) \right|^{\gamma}$$

if we use Hölder inequality then we have

$$\leq R \left(\sum_{k=0}^{n} |p_{k,n}(z)| \left| f\left(\frac{k}{n}\right) - f(z) \right| \right)^{\gamma} .$$

Denoting $e_k(z) = z^k$, $k = 0, 1, ...$ and $\pi_{k,n}(z) = B_n(e_k)(z)$, we evidently have

$$(B_n f)(z) = \sum_{k=0}^{\infty} c_k \pi_{k,n}(z)$$

and by using this representation we get

$$= R\left(\sum_{k=0}^{\infty} |c_k| \, |\pi_{k,n}(z) - e_k(z)|\right)^{\gamma}.$$

So that we need an estimate for

$$|\pi_{k,n}(z) - e_k(z)|.$$

For this purpose we use the recurrence proved for the real variable case in Andrica [1]. It is valid for complex variable as well in [2] and [3]:

$$\pi_{k+1,n}(z) = \frac{z(1-z)}{n}\pi'_{k,n}(z) + z\pi_{k,n}(z)$$

for all $n \in \mathbb{N}$, $z \in \mathbb{C}$ and $k = 0, 1, ...$

From this recurrence we easily obtain that degree $(\pi_{k,n}(z)) = k$. Also, by replacing k with $k-1$, we get

$$\pi_{k,n}(z) - z^k = \frac{z(1-z)}{n}[\pi_{k-1,n}(z) - z^{k-1}]' + \frac{(k-1)z^{k-1}(1-z)}{n} + z[\pi_{k-1,n}(z) - z^{k-1}]$$

which by Bernstein's inequality for complex polynomials where $|z| \le r \le 1$ gives

$$\begin{aligned} |\pi_{k,n}(z) - e_k(z)| &\le (k-1)\frac{1+r}{n}\left\|\pi_{k-1,n}(z) - e_{k-1}(z)\right\|_1 \\ &\quad + \frac{r^{k-1}(1+r)(k-1)}{n} + r\left|\pi_{k-1,n}(z) - e_{k-1}(z)\right|. \end{aligned}$$

As a conclusion, for all $|z| \le 1$ and $n \in \mathbb{N}$ we obtain

$$\begin{aligned} |(NB_n f)(z) - f(z)| &\le R\left(\sum_{k=0}^{\infty} |c_k| \, |\pi_{k,n}(z) - e_k(z)|\right)^{\gamma} \\ &\le R\left(\frac{r(1+r)(1+2r)}{2n}\sum_{k=2}^{\infty} k(k-1)\,|c_k|\right)^{\gamma}. \end{aligned}$$

Since $f(z) = \sum\limits_{k=0}^{\infty} c_k z^k$ is absolutely and uniformly convergent in $|z| \le 1$, then one has $f''(z) = \sum\limits_{k=2}^{\infty} k(k-1)c_k z^{k-2}$. Note that $\sum\limits_{k=2}^{\infty} k(k-1)c_k z^{k-2}$ is also absolutely convergent for $|z| \le 1$, which implies $\sum\limits_{k=2}^{\infty} k(k-1)\,|c_k| < \infty$.

Proof of Theorem 2.2. Denoting $h(z) := f(z) + \frac{z(1-z)}{n} f''(z)$

$$\begin{aligned}
&\left|(NB_n f)(z) - f(z) - \frac{z(1-z)}{n} f''(z)\right| \\
= &\left|\sum_{k=0}^{n} p_{k,n}(z)\, G_n\left(f\left(\frac{k}{n}\right)\right) - f(z) - \frac{z(1-z)}{n} f''(z)\right| \\
= &\left|\sum_{k=0}^{n} p_{k,n}(z)\, G_n\left(f\left(\frac{k}{n}\right)\right) - h(z)\right| \\
\leq &\left|\sum_{k=0}^{n} p_{k,n}(z) \left\{G_n\left(f\left(\frac{k}{n}\right)\right) - G_n(h(z))\right\}\right| \\
&+\left|\sum_{k=0}^{n} p_{k,n}(z) \left\{G_n(h(z)) - h(z)\right\}\right|
\end{aligned}$$

it is clearly seen that the last term in the last inequality goes to zero because of (1.2). Then we will estimate $I_1 = \left|\sum_{k=0}^{n} p_{k,n}(z) \left\{G_n\left(f\left(\frac{k}{n}\right)\right) - G_n(h(z))\right\}\right|$

$$\begin{aligned}
I_1 &= \left|\sum_{k=0}^{n} p_{k,n}(z) \left\{G_n\left(f\left(\frac{k}{n}\right)\right) - G_n(h(z))\right\}\right| \\
&\leq \sum_{k=0}^{n} |p_{k,n}(z)| \left|G_n\left(f\left(\frac{k}{n}\right)\right) - G_n(h(z))\right|
\end{aligned}$$

by using Hölder condition $0 < \gamma \leq 1$

$$\leq \sum_{k=0}^{n} |p_{k,n}(z)|\, R \left|f\left(\frac{k}{n}\right) - h(z)\right|^{\gamma}.$$

Substituting $h(z)$ and using the Hölder inequality,

$$\leq R\left(\sum_{k=0}^{n} |p_{k,n}(z)| \left|f\left(\frac{k}{n}\right) - f(z) - \frac{z(1-z)}{n} f''(z)\right|\right)^{\gamma}$$

Denoting $e_k(z) = z^k$, $k = 0, 1, \ldots$ and $\pi_{k,n}(z) = B_n(e_k)(z)$, we can write

$$(B_n f)(z) = \sum_{k=0}^{\infty} c_k \pi_{k,n}(z)$$

which immediately implies

$$\begin{aligned}
&\left|(NB_n f)(z) - f(z) - \frac{z(1-z)}{n} f''(z)\right| \\
\leq\ & R\left(\sum_{k=0}^{\infty} |c_k| \left|\pi_{k,n}(z) - e_k(z) - \frac{z^{k-1}(1-z)k(k-1)}{2n}\right|\right)^{\gamma}
\end{aligned}$$

for all $z \in \overline{D}_1$, $n \in \mathbb{N}$.

In what follows, we will use the recurrence obtained in the proof of Theorem 1.1.2 [2].

$$\pi_{k+1,n}(z)=\frac{z(1-z)}{n}\pi'_{k,n}(z)+z\pi_{k,n}(z)$$

for all $n\in\mathbb{N}$, $z\in\mathbb{C}$ and $k=0,1,....$

If we denote

$$E_{k,n}(z)=\pi_{k,n}(z)-e_k(z)-\frac{z^{k-1}(1-z)k(k-1)}{2n}$$

then it is clear that $E_{k,n}(z)$ is a polynomial of degree$\leq k$ and by a simple calculation and the use of the above recurrence we obtain the following relationship

$$\begin{aligned} E_{k,n}(z) &= \frac{z(1-z)}{n}E'_{k,n}(z)+zE_{k-1,n}(z)\\ &\quad+\frac{z^{k-2}(1-z)(k-1)(k-2)}{2n^2}\left[(k-2)-z(k-1)\right] \end{aligned}$$

for all $k\geq 2$, $n\in\mathbb{N}$ and $z\in\overline{D}_1$.

According to Bernstein's inequality $\left\|E'_{k-1,n}(z)\right\|\leq(k-1)\left\|E_{k-1,n}(z)\right\|$ the above relationship implies for all $|z|\leq 1$, $k\geq 2, n\in\mathbb{N}$ that

$$\begin{aligned} |E_{k,n}(z)| &\leq \frac{|z|\,|1-z|}{2n}\left[2\left\|E'_{k-1,n}(z)\right\|\right]+|E_{k-1,n}(z)|\\ &\quad+\frac{|z|\,|1-z|}{2n}\frac{|z|^{k-3}(k-1)(k-2)}{n}(2k-3)\\ &\leq |E_{k-1,n}(z)|+\frac{|z|\,|1-z|}{2n}\left[2(k-1)\left\|\pi_{k-1,n}(z)-e_{k-1}(z)\right\|\right.\\ &\quad\left.+2(k-1)\left\|\frac{(k-1)(k-2)\left[e_{k-2}(z)-e_{k-1}(z)\right]}{2n}\right\|+\frac{2k(k-1)(k-2)}{n}\right] \end{aligned}$$

where $\|.\|$ denotes the uniform norm in $C(\overline{D}_1)$.

It follows

$$\left\|\pi_{k,n}(z)-e_k(z)\right\|\leq\frac{3}{n}k(k-1)$$

and as a consequence, we get

$$\begin{aligned} |E_{k,n}(z)| &\leq |E_{k-1,n}(z)|+\frac{|z|\,|1-z|}{2n}\left[2(k-1)\frac{3(k-1)(k-2)}{n}\right.\\ &\quad\left.+2(k-1)\left\|\frac{(k-1)(k-2)\left[e_{k-2}(z)-e_{k-1}(z)\right]}{2n}\right\|+\frac{2k(k-1)(k-2)}{n}\right] \end{aligned}$$

which by simple calculation implies

$$|E_{k,n}(z)|\leq|E_{k-1,n}(z)|+\frac{|z|\,|1-z|}{2n}\frac{10}{n}k(k-1)(k-2).$$

Since $E_{0,n}(z) = E_{1,n}(z) = E_{2,n}(z) = 0$, for any $z \in \mathbb{C}$ it follows that from the last inequality for $k = 3, 4, ...$ we easily obtain, step by step the following

$$|E_{k,n}(z)| \leq \frac{|z|\,|1-z|}{2n} \frac{10}{n} \sum_{j=3}^{k} j(j-1)(j-2) \leq \frac{|z|\,|1-z|}{2n} \frac{10}{n} k(k-1)(k-2)^2$$

In conclusion

$$\left| (NB_n f)(z) - f(z) - \frac{z(1-z)}{n} f''(z) \right| \leq R \left(\sum_{k=3}^{\infty} |c_k|\, |E_{k,n}(z)| \right)^{\gamma}$$

$$\leq R \left(\frac{|z|\,|1-z|}{2n} \frac{10}{n} \sum_{k=3}^{\infty} |c_k|\, k(k-1)(k-2)^2 \right)^{\gamma}.$$

Note that since $f^{(4)}(z) = \sum_{k=4}^{\infty} c_k k(k-1)(k-2)(k-3) z^{k-4}$ and the series is absolutely convergent in $\overline{D}_1$, it easily follows that $\sum_{k=3}^{\infty} |c_k|\, k(k-1)(k-2)^2 < \infty$.

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