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A Voronovskaya-type theorem for a certain nonlinear Bernstein operators

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Abstract. The present paper concerns with the nonlinear Bernstein operators NB_nf of the form

$$(NB_nf)(x)=\sum_{k=0}^{n}P_{n,k}\left(x,f\left(\frac{k}{n}\right)\right)\ ,\ 0\leq x\leq 1\ ,\quad n\in\mathbb{N},$$

acting on bounded functions on an interval $[0,1]$, where $P_{n,k}$ satisfy some suitable assumptions. As a continuation of the very recent paper of the authors [11], we estimate the rate of convergence by modulus of continuity and provide a Voronovskaya-type formula for these operators. We note that our results are strict extensions of the classical ones, namely, the results dealing with the linear Bernstein polynomials.

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1. Introduction

We consider the problem of approximating a given real-valued function f, defined on $[0,1]$, by means of a sequence of nonlinear Bernstein operators (NB_nf). Operators like positive linear, convolution, moment and sampling operators play an important role in several branches of Mathematics, for instance in reconstruction of signals and images, in Fourier analysis, operator theory, probability theory and approximation theory.

In this paper, we deal with nonlinear Bernstein operators generated by the classical Bernstein operators. These operators considered in the papers [1], [4] and [11], in which other kinds of convergence properties are studied.

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Let f be a function defined on the interval $[0,1]$ and let $\mathbb{N} := \{1,2,...\}$. The classical Bernstein operators $B_n f$ applied to f are defined as

$$(B_n f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x) \ , \ 0 \le x \le 1 \ , \quad n \in \mathbb{N}, \tag{1.1}$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ is the Bernstein basis. These polynomials were introduced by Bernstein [7] in 1912 to give the first constructive proof of the Weierstrass approximation theorem. Some properties of the polynomials (1.1) can be found in Lorentz [14].

We now state a brief and technical explanation of the relation between approximation by linear and nonlinear operators. Approximation with nonlinear integral operators of convolution type was introduced by J. Musielak in [15] and widely developed in [5] (and the references contained therein). In [15], the assumption of linearity of the singular integral operators was replaced by an assumption of a Lipschitz condition for the kernel function $K_\lambda(t,u)$ with respect to the second variable. Especially, nonlinear integral operators of type

$$(T_\lambda f)(x) = \int_a^b K_\lambda(t-x, f(t))\, dt, \qquad x \in (a,b),$$

and its special cases were studied by Bardaro-Karsli and Vinti [2], [3] and Karsli [10], [12] in some Lebesgue spaces.

Very recently, by using the techniques due to Musielak [15], Karsli-Tiryaki and Altin [11] considered the following type nonlinear counterpart of the well-known Bernstein operators;

$$(NB_n f)(x) = \sum_{k=0}^{n} P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right) \ , \ 0 \le x \le 1 \ , \quad n \in \mathbb{N}, \tag{1.2}$$

acting on bounded functions f on an interval $[0,1]$, where $P_{n,k}$ satisfy some suitable assumptions. They proved some existence and approximation theorems for the nonlinear Bernstein operators. In particular, they obtain some pointwise convergence for the nonlinear sequence of Bernstein operators (1.2) to some discontinuity point of the first kind of f, as $n \to \infty$.

As a continuation of the very recent paper of the authors [11], we estimate a Voronovskaya-type formula for this class nonlinear Bernstein operators on the interval $[0,1]$. Let us note that such kind of results for a general class of discrete operators were studied by Bardaro and Mantellini [4].

An outline of the paper is as follows: The next section contains basic definitions and notations.

In Section 3, the main approximation results of this study are given. They are dealing with some approximation theorems for nonlinear Bernstein operators (1.2) and rate of convergence by modulus of continuity. Also we give a Voronovskaya-type formula for this class nonlinear Bernstein operators on the interval $[0,1]$.

In Section 4, we give some certain results which are necessary to prove the main result.

The final section, that is Section 5, concerns with the proof of the main results presented in Section 3.

2. Preliminaries

In this section, we recall the following structural assumptions according to [11], which will be fundamental in proving our convergence theorems.

In the following we will denote by $C(I)$ the space of all uniformly continuous and bounded functions $f : I \to \mathbb{R}$, endowed with the norm $\|f\|_\infty = \sup_{t\in\mathbb{R}} |f(t)|$.

Let Ψ be the class of all functions $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that the function ψ is non-decreasing, continuous and concave with $\psi(0) = 0$, $\psi(u) > 0$ for $u > 0$, and $\lim\limits_{u\to\infty} \psi(u) = +\infty$.

We now introduce a sequence of functions. Let $\{P_{n,k}\}_{n\in\mathbb{N}}$ be a sequence of functions $P_{n,k} : [0,1] \,\mathrm{x}\, \mathbb{R} \to \mathbb{R}$ defined by

$$P_{n,k}(t,u) = p_{n,k}(t)H_n(u) \tag{2.1}$$

for every $t \in [0,1], u \in \mathbb{R}$, where $H_n : \mathbb{R} \to \mathbb{R}$ is such that $H_n(0) = 0$ and $p_{n,k}(t)$ is the Bernstein basis.

Throughout the paper we assume that $\mu : \mathbb{N} \to \mathbb{R}^+$ is an increasing sequence such that $\lim\limits_{n\to\infty} \mu(n) = \infty$.

First of all we assume that the following conditions hold:

a) $H_n : \mathbb{R} \to \mathbb{R}$ is such that

$$|H_n(u) - H_n(v)| \le \psi(|u-v|), \quad \psi \in \Psi,$$

holds for every $u, v \in \mathbb{R}$, for every $n \in \mathbb{N}$. That is, H_n satisfies a (L-ψ) Lipschitz condition.

b) We now set

$$K_n(x,u) := \begin{cases} \sum\limits_{k\le nu} p_{n,k}(x), & 0 < u \le 1 \\ 0, & u = 0 \end{cases} \tag{2.2}$$

and from (2.2) one can write

$$\lambda_n(x,t) := \int_0^t d_u K_n(x,u).$$

Similar approach and some particular examples can be found in [6], [9], [11], [13] and [16].

c) Denoting by $r_n(u) := H_n(u) - u$, $u \in \mathbb{R}$ and $n \in \mathbb{N}$. Assume that for n sufficiently large

$$\sup_u |r_n(u)| \leq \frac{1}{\mu(n)},$$

holds.

3. Convergence Results

Let X be the set of all bounded Lebesgue measurable functions $f : [0,1] \to \mathbb{R}$.

We will consider the following type nonlinear Bernstein operators,

$$(NB_n f)(x) = \sum_{k=0}^{n} P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right)$$

defined for every $f \in X$ for which $NB_n f$ is well-defined, where $P_{n,k}(x,u)$ satisfies (2.1) for every $x \in [0,1]$, $u \in \mathbb{R}$.

Definition 3.1. *Let $f \in C[a,b]$ and $\delta > 0$ be given. Then the modulus of continuity is given by;*

$$\omega_\psi(f;\delta) = \sup_{|t-x|\leq\delta\ ,\ t,x\in[a,b]} \psi(|f(t) - f(x)|). \tag{3.1}$$

Definition 3.2. *We will say that the sequence $(P_{n,k})_{n\in\mathbb{N}}$ is $\psi-$singular if the following assumptions are satisfied;*

(P.1) For every $x \in I$ and $\delta > 0$ there holds

$$\psi\left(\sum_{\left|\frac{k}{n}-x\right|\geq\delta} \left|\frac{k}{n} - x\right| p_{n,k}(x)\right) = o\left(n^{-1}\right)\ ,\ (n \to \infty).$$

(P.2) For every $u \in \mathbb{R}$ and for every $x \in I$ we have

$$\lim_{n\to\infty} n\ \left[\sum_{k=0}^{n} P_{n,k}(x,u) - u\right] = 0.$$

We are now ready to establish the main results of this study:

Theorem 3.3. *Let $f : I \to \mathbb{R}$, $f \in C(I)$ and suppose that a kernel satisfies (a), (b) and (c). Then*

$$\|NB_n f - f\|_\infty \to 0 \text{ as } n \to \infty$$

where $I = [0,1]$ and $\psi \in \Psi$.

Theorem 3.4. *If $f(x)$ is continuous and $\omega_\psi(f;\delta)$ the modulus of continuity of $f(x)$ given in (3.1), then*

$$|NB_n f(x) - f(x)| \leq \psi(\epsilon) + \frac{5}{4}\omega_\psi(f;\delta) + \frac{1}{\mu(n)}$$

where $\delta = n^{-\frac{1}{2}}$.

Theorem 3.5. *Let $f \in L_1[0,1]$ be a function such that $f'(x)$ exists at a point $x \in (0,1)$. Let us assume that the sequence $(P_{n,k})_{n\in\mathbb{N}}$ is $\psi-$singular and*

$$\limsup_{n\to\infty} n\ \psi\left(M_1\left(p_{n,k},x\right)\right) = l_1(x) \in \mathbb{R}, \tag{3.2}$$

where M_1 is the first order absolute moment of Bernstein polynomials given in Lemma 4.3. Then,

$$\limsup_{n\to\infty} n\ \left|\left(NB_nf\right)(x) - f(x)\right| \leq M\ l_1(x),$$

where $M > 0$ be a sufficiently large integer.

4. Auxilary Results

In this section we give certain results, which are necessary to prove our theorems.

Lemma 4.1. *$\omega_\psi(f;\delta)$ has the following properties,*

i) $\omega_\psi(f;\delta) \geq 0$,
ii) If $\delta_1 \leq \delta_2$, then $\omega_\psi(f;\delta_1) \leq \omega_\psi(f;\delta_2)$,
iii) Let $m \in \mathbb{N}$, then $\omega_\psi(f;m\delta) \leq m\ \omega_\psi(f;\delta)$,
iv) Let $\lambda \in \mathbb{R}^+$, then $\omega_\psi(f;\lambda\delta) \leq (\lambda+1)\,\omega_\psi(f;\delta)$,
v) $\lim\limits_{\delta\to0^+} \omega_\psi(f;\delta) = 0$,
vi) $\psi(|f(t)-f(x)|) \leq \omega_\psi(f;|t-x|)$,
vii) $\psi(|f(t)-f(x)|) \leq \left(\frac{|t-x|}{\delta}+1\right)\omega_\psi(f;\delta)$, and they can be proven as similar with the classical ones.

Lemma 4.2. *It is well known that for $(B_nt^s)(x)$, $s = 0,1,2$, one has*

$$(B_n1)(x) = 1, (B_nt)(x) = x, (B_nt^2)(x) = x^2 + \frac{x(1-x)}{n}.$$

For proof of this Lemma see [14].

By direct calculation, we find the following equalities:

$$(B_n(t-x)^2)(x) = \frac{x(1-x)}{n}\quad,\quad (B_n(t-x))(x) = 0.$$

Lemma 4.3. *The first order absolute moment for Bernstein polynomial is defined as*

$$M_1(p_{n,k},x) = \sum_{k=0}^{n}\left|\left(\frac{k}{n}-x\right)\right| p_{n,k}(x)$$

and

$$M_1(p_{n,k},x) \leq \left(\frac{2x(1-x)}{\pi}\right)^{\frac{1}{2}} \frac{1}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)$$

which can be found [8].

5. Proof of the Theorems

Proof of Theorem 3.3. We evaluate $\|NB_n f - f\|_\infty$. We have

$$\begin{aligned} |NB_n f(x) - f(x)| &= \left|\sum_{k=0}^{n} P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right) - f(x)\right| \\ &= \left|\sum_{k=0}^{n} \left\{H_n\left(f\left(\frac{k}{n}\right)\right) - f(x)\right\} p_{n,k}(x)\right| \\ &\leq \sum_{k=0}^{n} \left|H_n\left(f\left(\frac{k}{n}\right)\right) - H_n(f(x))\right| p_{n,k}(x) \\ &\quad + \sum_{k=0}^{n} |H_n(f(x)) - f(x)|\, p_{n,k}(x) \\ &= I_{n,1}(x) + I_{n,2}(x). \end{aligned}$$

First we consider $I_{n,2}(x)$. From (c) we have

$$\begin{aligned} I_{n,2}(x) &= \sum_{k=0}^{n} |H_n(f(x)) - f(x)|\, p_{n,k}(x) \\ &\leq \frac{1}{\mu(n)}. \end{aligned}$$

Next we consider $I_{n,1}(x)$

$$\begin{aligned} I_{n,1}(x) &= \sum_{k=0}^{n} \left|H_n\left(f\left(\frac{k}{n}\right)\right) - H_n(f(x))\right| p_{n,k}(x) \\ &\leq \sum_{k=0}^{n} \psi\left(\left|f\left(\frac{k}{n}\right) - f(x)\right|\right) p_{n,k}(x) \\ &= \int_0^1 \psi(|f(t) - f(x)|)\, d_t(K_n(x,t)) \\ &= \int_{|t-x|\leq\delta} \psi(|f(t) - f(x)|)\, d_t(K_n(x,t)) \\ &\quad + \int_{|t-x|>\delta} \psi(|f(t) - f(x)|)\, d_t(K_n(x,t)) \\ &\leq \psi(\epsilon) + \psi(2\|f\|_\infty)\ \epsilon \end{aligned}$$

holds true, since ψ is non-decreasing and concave function. Finally we have

$$|NB_n f(x) - f(x)| \leq \psi(\epsilon) + \psi(2\|f\|_\infty)\ \epsilon + \frac{1}{\mu(n)}$$

and so, since $\frac{1}{\mu(n)} \to 0$ when $n \to \infty$, we obtain

$$\limsup_{n\to\infty} |NB_n f(x) - f(x)| \leq \psi(\epsilon) + \psi\left(2\|f\|_\infty\right)\ \epsilon.$$

Hence the assertion follows, $\epsilon > 0$ being arbitrary.

Proof of Theorem 3.4. We can write the difference as in the previous theorem

$$\begin{aligned} |NB_n f(x) - f(x)| &= \left|\sum_{k=0}^{n} P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right) - f(x)\right| \\ &\leq I_{n,1}(x) + I_{n,2}(x) \end{aligned}$$

where

$$\begin{aligned} I_{n,2}(x) &= \sum_{k=0}^{n} |H_n(f(x)) - f(x)|\, p_{n,k}(x) \\ &\leq \frac{1}{\mu(n)}. \end{aligned}$$

First we consider $I_{n,1}(x)$. If we think $I_{n,1}(x)$ as two sum,

$$\begin{aligned} I_{n,1}(x) &\leq \sum_{\left|\frac{k}{n}-x\right|\leq\delta} \psi\left(\left|f\left(\frac{k}{n}\right) - f(x)\right|\right) p_{n,k}(x) \\ &\quad + \sum_{\left|\frac{k}{n}-x\right|>\delta} \psi\left(\left|f\left(\frac{k}{n}\right) - f(x)\right|\right) p_{n,k}(x) \\ &= \psi(\epsilon) + I_{n,1,2}(x). \end{aligned}$$

Now we will consider $I_{n,1,2}(x)$. Taking into account that $\omega_\psi(f;\delta)$ is the modulus of continuity

$$\begin{aligned} I_{n,1,2}(x) &= \sum_{\left|\frac{k}{n}-x\right|>\delta} \psi\left(\left|f\left(\frac{k}{n}\right) - f(x)\right|\right) p_{n,k}(x) \\ &\leq \omega_\psi(f;\delta) \sum_{\left|\frac{k}{n}-x\right|>\delta} \left(\frac{\left|\frac{k}{n}-x\right|}{\delta} + 1\right) p_{n,k}(x) \\ &\leq \omega_\psi(f;\delta) \left\{1 + \delta^{-1} \sum_{\left|\frac{k}{n}-x\right|>\delta} \left|\frac{k}{n} - x\right| p_{n,k}(x)\right\} \\ &\leq \omega_\psi(f;\delta) \left\{1 + \delta^{-2} \sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^2 p_{n,k}(x)\right\} \\ &\leq \omega_\psi(f;\delta) \left\{1 + \left(4n\delta^2\right)^{-1}\right\}. \end{aligned}$$

In conclusion;

$$\begin{aligned} |NB_n f(x) - f(x)| &\le \psi(\epsilon) + \omega_\psi(f;\delta)\left\{1 + \left(4n\delta^2\right)^{-1}\right\} + \frac{1}{\mu(n)} \\ &\le \psi(\epsilon) + \frac{5}{4}\omega_\psi(f;\delta) + \frac{1}{\mu(n)} \end{aligned}$$

where $\delta = n^{-\frac{1}{2}}$.

Proof of Theorem 3.5. Since f is differentiable at the point x, then there exists a bounded function h such that $\lim_{y\to 0} h(y) = 0$. By Taylor's formula we have

$$f\left(\frac{k}{n}\right) = f(x) + \left(\frac{k}{n} - x\right) f'(x) + \left(\frac{k}{n} - x\right) h\left(\frac{k}{n} - x\right).$$

Now we can write

$$\begin{aligned} n\ |(NB_n f)(x) - f(x)| &= n\left|\sum_{k=0}^{n}\left\{H_n\left(f\left(\frac{k}{n}\right)\right) - f(x)\right\} p_{n,k}(x)\right| \\ &\le n\sum_{k=0}^{n}\psi\left(\left|f\left(\frac{k}{n}\right) - f(x)\right|\right) p_{n,k}(x) \\ &\quad + n\left|\sum_{k=0}^{n}\left\{H_n(f(x)) - f(x)\right\} p_{n,k}(x)\right| \\ &= I_1(x) + I_2(x). \end{aligned}$$

By assumption (P.2), $I_2(x)$ tends to zero. We can estimate the first term in the following way:

Let $M > 0$ be an integer such that $|f'(x)| + \left|h\left(\frac{k}{n} - x\right)\right| \le M$. Using sub-additivity of the function $\psi(x)$, $x \ge 0$, we have

$$\begin{aligned} I_1(x) &= n\sum_{k=0}^{n}\psi\left(\left|\left(\frac{k}{n} - x\right) f'(x) + \left(\frac{k}{n} - x\right) h\left(\frac{k}{n} - x\right)\right|\right) p_{n,k}(x) \\ &\le n\sum_{k=0}^{n}\psi\left(\left|\left(\frac{k}{n} - x\right)\right|\left[|f'(x)| + \left|h\left(\frac{k}{n} - x\right)\right|\right]\right) p_{n,k}(x) \\ &\le n\left\{\sum_{k=0}^{n}\psi\left(M\left|\left(\frac{k}{n} - x\right)\right|\right) p_{n,k}(x)\right\} \\ &\le n\ M\left\{\sum_{k=0}^{n}\psi\left(\left|\left(\frac{k}{n} - x\right)\right|\right) p_{n,k}(x)\right\} \end{aligned}$$

In virtue of Jensen's Inequality, we can write

$$\begin{aligned} I_1(x) &\le n\ M\ \psi\left(\sum_{k=0}^{n}\left|\left(\frac{k}{n} - x\right)\right| p_{n,k}(x)\right) \\ &= n\ M\ \psi\left(M_1(p_{n,k}, x)\right). \end{aligned}$$

In view of (3.2), one has

$$\limsup_{n\to\infty} n \ \left|(NB_n f)(x) - f(x)\right| \le M \ l_1(x).$$

This completes the proof of the theorem.

As a corollary of the Theorem 3.5 we have:

Corollary 5.1. *Let $f \in L_1[0,1]$ be a function such that $f'(x)$ exists at a point $x \in (0,1)$. Let us assume that the sequence $(P_{n,k})_{n\in\mathbb{N}}$ is $\psi-$singular satisfies (3.2) and let $\psi(x) = x^{\gamma}$ where $0 < \gamma \le 1$. Then*

$$\limsup_{n\to\infty} n \ \left|(NB_n f)(x) - f(x)\right| \le l_1(x) \left|f'(x)\right|^{\gamma}.$$

We note that to prove the above Corollary we can also use the following inequality;

$$\psi(|a|\,|b|) \le \psi(|a|)\,\psi(|b|),$$

(*see* [4]).

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