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The generalization of Mastroianni operators using the Durrmeyer's method

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Abstract. In the present paper, we define a sequence of Durrmeyer's type operators associated with Mastroianni operators and introduce a new operator.

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1. Introduction

In [5], [6] G. Mastroianni defined and studied a general class of linear positive approximation operators, which was generalized by O. Agratini, B. Della Vecchia [1]. In brief we recall this construction.

Taking $[0,\infty) := \mathbb{R}_+$, we consider the next spaces of functions:
$B(\mathbb{R}_+) = \{f : \mathbb{R}_+ \longrightarrow \mathbb{R} | (\exists) M_f > 0 : |f(x)| \leq M_f\}$, a normed space with the uniform norm $\|f\|_B = \sup\{|f(x)| : x \in \mathbb{R}_+\}$;
$B_\rho(\mathbb{R}_+) = \left\{f : \mathbb{R}_+ \longrightarrow \mathbb{R} |\, |f(x)| \leq N_f \rho(x),\, N_f > 0,\, \rho(x) = 1 + x^2\right\}$, a normed space with the norm $\|f\|_\rho = \sup\left\{\dfrac{|f(x)|}{\rho(x)} : x \geq 0\right\} = \sup\left\{\dfrac{|f(x)|}{1+x^2} : x \geq 0\right\}$;
$C_\rho(\mathbb{R}_+) = \{f \in B_\rho(\mathbb{R}_+) | f \text{ continuous function}\}$;
$C^*_\rho(\mathbb{R}_+) = \left\{f \in C_\rho(\mathbb{R}_+) | (\exists) \lim\limits_{x\to\infty} \dfrac{|f(x)|}{1+x^2} < \infty\right\}$.
The space $C^*_\rho(\mathbb{R}_+)$ endowed with the norm $\|f\|_\rho$ is a Banach space.
In our estimations we use the first modulus of continuity on a finite interval $[0,b]$, $b > 0$, $\omega_{[0,b]}(f;\delta) = \sup\{|f(x+h) - f(x)| : 0 < h \leq \delta,\, x \in [0,b]\}$ and the Peetre's K-functional defined as

$$K_2(f;\delta) = \inf\left\{\|f-g\|_B + \delta\|g''\|_B : g \in W^2_\infty\right\},\ \delta > 0,$$

where $W^2_\infty = \{g \in C_B(\mathbb{R}_+) : g',\, g'' \in C_B(\mathbb{R}_+)\}$.

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It is known (see [9] p.177, th. 2.4) that, there exists a positive constant C such that $K_2(f;\delta) \le C\omega_2\left(f;\sqrt{\delta}\right)$, where

$$\omega_2\left(f;\sqrt{\delta}\right) = \sup_{0<h\le\sqrt{\delta}} \sup_{x\in[0,\infty)} \left\{|f(x+2h) - 2f(x+h) + f(x)|\right\}.$$

Let $(\Phi_n)_{n\ge1}$ be a sequence of real functions defined on $[0,\infty) := \mathbb{R}_+$ which are infinitely differentiable on $\mathbb{R}_+$ and satisfy the conditions:

(i) $\Phi_n(0) = 1$, $n \in \mathbb{N}$;

(ii) for every $n \in \mathbb{N}$, $x \in \mathbb{R}_+$ and $k \in \mathbb{N}\cup\{0\} := \mathbb{N}_0$,

$$(-1)^k \Phi_n^{(k)}(x) \ge 0; \tag{1.1}$$

(iii) for each $(n,k) \in \mathbb{N}\times\mathbb{N}_0$ there exists a number $p(n,k) \in \mathbb{N}$ and a function $\alpha_{n,k} \in \mathbb{R}^{\mathbb{R}}$ such that $\Phi_n^{(i+k)}(x) = (-1)^k \Phi_{p(n,k)}^{(i)}(x)\alpha_{n,k}(x)$, $i \in \mathbb{N}_0$, $x \in \mathbb{R}_+$ and

(iv) $\lim\limits_{n\to\infty} \dfrac{n}{p(n,k)} = \lim\limits_{n\to\infty} \dfrac{\alpha_{n,k}(x)}{n^k} = 1.$

Remark 1.1. There is easy to see that

$$\lim_{n\to\infty} \frac{\Phi_n^{(k)}(0)}{n^k} = \lim_{n\to\infty} \frac{(-1)^k \alpha_{n,k}(0)}{n^k} = (-1)^k. \tag{1.2}$$

The Mastroianni operators $M_n : C_B(\mathbb{R}_+) \longrightarrow C(\mathbb{R}_+)$ are defined by the following formula

$$M_n(f,x) = \sum_{k=0}^{\infty} m_{n,k}(x) f\left(\frac{k}{n}\right) \tag{1.3}$$

with the basis functions,

$$m_{n,k}(x) = \frac{(-x)^k \Phi_n^{(k)}(x)}{k!}. \tag{1.4}$$

For these operators and for the test functions $e_r(x) = x^r$, $r = 0, 1, 2$ the following results were obtained [5]:

$$\begin{aligned} M_n(e_0;x) &= \Phi_n(0), \\ M_n(e_1;x) &= -\frac{\Phi_n'(0)}{n}x, \\ M_n(e_2;x) &= \frac{\Phi_n''(0)x^2 - \Phi_n'(0)x}{n^2}. \end{aligned} \tag{1.5}$$

In terms of the hypergeometric and confluent hypergeometric functions, recent results, about Durrmeyer type operators [2], [3], [4], [8], have considered in the definition of the basis functions, the family's functions:

$$\Phi_{n,c}(x) = \begin{cases} e^{-nx}, & c = 0,\ x \ge 0 \\ (1+cx)^{-\frac{n}{c}}, & c \in \mathbb{N},\ x \ge 0 \end{cases}$$

For these functions we have

$$\Phi_{n,c}^{(k+1)}(x) = -n\Phi_{n+c,c}^{(k)}(x),\ n > \max\{0,-c\}$$

respectively

$$\Phi_{n,c}^{(i+k)}(x) = (-1)^k n_{[k,-c]} \Phi_{n+kc,c}^{(i)}(x)$$

where $n_{[k,-c]} = n(n+c)(n+2c)\cdots(n+\overline{k-1}c)$ is the factorial power of order k of n with the increment $-c$ and $n_{[0,-c]} = 1$.

So, the conditions (iii)-(iv) are true, for $p(n,k) = n + kc$ and $\alpha_{n,k}(x) = n_{[k,-c]}$.

In the next section we propose a Mastroianni–Durrmeyer operator, when the sequence of functions $(\Phi_n)_{n\geq 1}$ satisfy the conditions (i)-(iv) and other supplementary conditions, is non-nominated.

2. Main results

Let $(\Phi_n)_{n\geq 1}$ be the sequence of functions which satisfy the conditions (i)-(iv) and the next supplementary conditions, for any $n \in \mathbb{N}$ and $r,\ k \in \mathbb{N}_0$:

(v) $\lim\limits_{x\to\infty} x^r \Phi_n^{(k)}(x) = 0$

(vi) $(\exists) J_{n,k,r} := \int\limits_0^\infty x^r \Phi_n^{(k)}(x)dx < \infty$, $(\exists) J_{n,0,0} := \int\limits_0^\infty \Phi_n(x)dx \neq 0$.

We define the operators of Durrmeyer type associated with Mastroianni operators (1.3)-(1.4) for each real value function $f \in \mathbb{R}^{\mathbb{R}}$ for which the series exists:

$$DM_n(f;x) = \sum_{k=0}^{\infty} m_{n,k}(x) \frac{\int\limits_0^\infty m_{n,k}(t)f(t)dt}{\int\limits_0^\infty m_{n,k}(t)dt} = \int\limits_0^\infty K_n(t,x)f(t)dt \tag{2.1}$$

with the kernel

$$K_n(t,x) = \frac{1}{I_{n,0,0}} \sum_{k=0}^{\infty} m_{n,k}(x)m_{n,k}(t), I_{n,0,0} = J_{n,0,0} = \int\limits_0^\infty \Phi_n(t)dt \neq 0. \tag{2.2}$$

Lemma 2.1. *The next identity is true for any $n \in \mathbb{N}$ and $r,\ k \in \mathbb{N}_0$*

$$I_{n,k,r} = \frac{(r+1)_k}{k!} I_{n,0,r},$$

where

$$I_{n,k,r} := \int\limits_0^\infty t^r m_{n,k}(t)dt = \frac{(-1)^k}{k!} J_{n,k,r+k}$$

and

$$(n)_k = n(n+1)(n+2)\cdots(n+k-1) = n_{[k,-1]},\ (n)_0 = 1$$

is the Pochhammer symbol or the factorial power of order k of n and the increment -1. So, $(1)_k = k!$, $(2)_k = (k+1)!$.

The proof suppose an easy computation, so we have omitted them. We remark that

$I_{n,0,r} = J_{n,0,r} = \int\limits_0^\infty t^r \Phi_n(t)dt$, $r \geq 0$, (the moments of the r-th order reported to Φ_n)

$I_{n,k,0} = \int\limits_0^\infty m_{n,k}(t)dt = \frac{(-1)^k}{k!} J_{n,k,k},$
$J_{n,k,k} = (-1)^k k! J_{n,0,0},\ k \geq 0,$
$I_{n,k,0} = I_{n,0,0} = J_{n,0,0} = \int\limits_0^\infty \Phi_n(t)dt,$
$I_{n,k,r} = \int\limits_0^\infty t^r m_{n,k}(t)dt = \frac{(-1)^k}{k!} \int\limits_0^\infty t^{r+k} \Phi_n^{(k)}(t)dt = \frac{(r+1)_k}{k!} \int\limits_0^\infty t^r \Phi_n(t)dt$
$= \frac{(r+1)_k}{k!} J_{n,0,r} = \frac{(r+1)_k}{k!} I_{n,0,r}$, (the moments of the r-th order reported to $m_{n,k}$).

Lemma 2.2. *The moments of the operators $DM_n(f;x)$ are given for $e_r(x) = x^r$, $r \in \mathbb{N}_0$ as*

$$DM_n(e_r;x) = \frac{I_{n,0,r}}{I_{n,0,0}} \sum_{k=0}^{\infty} \frac{(r+1)_k}{k!} m_{n,k}(x). \tag{2.3}$$

Further, we have

$$\begin{aligned} DM_n(e_0;x) &= 1, \\ DM_n(e_1;x) &= \frac{I_{n,0,1}}{I_{n,0,0}} \left(1 - x\Phi_n'(0)\right), \\ DM_n(e_2;x) &= \frac{I_{n,0,2}}{2I_{n,0,0}} \left(x^2\Phi_n''(0) - 4x\Phi_n'(0) + 2\right), \\ DM_n\left((e_1 - xe_0)^2;x\right) &= x^2 \left(\frac{I_{n,0,2}}{2I_{n,0,0}}\Phi_n''(0) + 2\frac{I_{n,0,1}}{I_{n,0,0}}\Phi_n'(0) + 1\right) \\ &\quad -2x\left(\frac{I_{n,0,2}}{I_{n,0,0}}\Phi_n'(0) + \frac{I_{n,0,1}}{I_{n,0,0}}\right) + \frac{I_{n,0,2}}{I_{n,0,0}}. \end{aligned} \tag{2.4}$$

Proof. Using Lemma 2.1 we obtain

$$DM_n(e_r;x) = \frac{1}{I_{n,0,0}} \sum_{k=0}^{\infty} m_{n,k}(x) I_{n,k,r} = \frac{I_{n,0,r}}{I_{n,0,0}} \sum_{k=0}^{\infty} m_{n,k}(x) \frac{(r+1)_k}{k!}.$$

$$DM_n(e_0;x) = \frac{1}{I_{n,0,0}} \sum_{k=0}^{\infty} m_{n,k}(x) I_{n,k,0} = \frac{I_{n,0,0}}{I_{n,0,0}} \sum_{k=0}^{\infty} m_{n,k}(x) \frac{(1)_k}{k!} = 1,$$

$$\begin{aligned} DM_n(e_1;x) &= \frac{1}{I_{n,0,0}} \sum_{k=0}^{\infty} m_{n,k}(x) I_{n,k,1} = \frac{I_{n,0,1}}{I_{n,0,0}} \sum_{k=0}^{\infty} m_{n,k}(x) \frac{(2)_k}{k!} \\ &= \frac{I_{n,0,1}}{I_{n,0,0}} \sum_{k=0}^{\infty} m_{n,k}(x) \frac{(k+1)!}{k!} = \frac{I_{n,0,1}}{I_{n,0,0}} \sum_{k=0}^{\infty} m_{n,k}(x)(k+1) \\ &= \frac{nI_{n,0,1}}{I_{n,0,0}} \left(M_n(e_1;x) + \frac{1}{n}\right) = \frac{nI_{n,0,1}}{I_{n,0,0}} \left(-\frac{\Phi_n'(0)}{n}x + \frac{1}{n}\right) \end{aligned}$$

$$
\begin{aligned}
DM_n\left(e_2;x\right) &= \frac{1}{I_{n,0,0}}\sum_{k=0}^{\infty} m_{n,k}(x) I_{n,k,2} = \frac{I_{n,0,2}}{I_{n,0,0}}\sum_{k=0}^{\infty} m_{n,k}(x)\frac{(3)_k}{k!} \\
&= \frac{I_{n,0,2}}{2I_{n,0,0}}\sum_{k=0}^{\infty} m_{n,k}(x)\frac{(k+2)!}{k!} \\
&= \frac{n^2 I_{n,0,2}}{2I_{n,0,0}}\left(M_n\left(e_2;x\right)+\frac{3}{n}M_n\left(e_1;x\right)+\frac{2}{n^2}\right) \\
&= \frac{n^2 I_{n,0,2}}{2I_{n,0,0}}\left(\frac{\Phi_n''(0)x^2-\Phi_n'(0)x}{n^2}-\frac{3\Phi_n'(0)x}{n^2}+\frac{2}{n^2}\right).
\end{aligned}
$$

Because $DM_n\left((e_1-xe_0)^2;x\right) = DM_n\left(e_2;x\right) - 2xDM_n\left(e_1;x\right) + x^2 DM_n\left(e_0;x\right)$ is easy to obtain the relation of enunciation. □

Lemma 2.3. *Let*

$$
\overline{DM_n}(f;x) = DM_n(f;x) - f\left(\frac{nI_{n,0,1}}{I_{n,0,0}}\left(\frac{1}{n}-\frac{\Phi_n'(0)}{n}x\right)\right) + f(x). \tag{2.5}
$$

The following assertions hold:

$$
\begin{aligned}
\overline{DM_n}\left(e_0;x\right) &= 1, \\
\overline{DM_n}\left(e_1;x\right) &= x, \\
\overline{DM_n}\left(e_1-xe_0;x\right) &= 0.
\end{aligned}
$$

The proof suppose an easy computation, so we have omitted them.

Further, we consider the next conventions:

$$
\left|DM_n\left(e_1-xe_0;x\right)\right| = \left|\frac{I_{n,0,1}}{I_{n,0,0}}\left(1-x\Phi_n'(0)\right)-x\right| := \lambda_n(x), \tag{2.6}
$$

$$
\begin{aligned}
DM_n\left((e_1-xe_0)^2;x\right) &= x^2\left(\frac{I_{n,0,2}}{2I_{n,0,0}}\Phi_n''(0)+2\frac{I_{n,0,1}}{I_{n,0,0}}\Phi_n'(0)+1\right) \\
&\quad -2x\left(\frac{I_{n,0,2}}{I_{n,0,0}}\Phi_n'(0)+\frac{I_{n,0,1}}{I_{n,0,0}}\right)+\frac{I_{n,0,2}}{I_{n,0,0}} \\
&= \beta_n(x).
\end{aligned} \tag{2.7}
$$

From (1.1) we have $\Phi_n(x)\geq 0$, $\Phi_n'(x)\leq 0$, $\Phi_n''(x)\geq 0$, $x\in\mathbb{R}_+$ and so

$$
\beta_n(x) \leq x^2\left(\frac{I_{n,0,2}}{I_{n,0,0}}\Phi_n''(0)+1\right) - 2x\frac{I_{n,0,2}}{I_{n,0,0}}\Phi_n'(0)+\frac{I_{n,0,2}}{I_{n,0,0}} := \eta_n(x). \tag{2.8}
$$

Because DM_n is a linear positive operator, using the Cauchy-Schwarz's inequality we have

$$
\begin{aligned}
\lambda_n(x) &= \left|DM_n\left(e_1-xe_0;x\right)\right| \leq DM_n\left(\left|e_1-xe_0\right|;x\right) \\
&\leq \sqrt{DM_n\left((e_1-xe_0)^2;x\right)} = \sqrt{\beta_n(x)}.
\end{aligned}
$$

Let

$$
\gamma_n(x) = \beta_n(x) + \lambda_n^2(x) \leq 2\beta_n(x). \tag{2.9}
$$

Lemma 2.4. *For every $x \in \mathbb{R}_+$ and $f'' \in C_B(\mathbb{R}_+)$ we have*

$$\left|\overline{DM_n}(f;x) - f(x)\right| \leq \frac{\|f''\|_B}{2}\gamma_n(x).$$

Proof. Using Taylor's expansion

$$f(t) = f(x) + (t-x)f'(x) + \int_x^t (t-u)f''(u)du$$

we obtain with Lemma 2.3 that

$$\overline{DM_n}(f;x) - f(x) = \overline{DM_n}\left(\int_x^t (t-u)f''(u)du; x\right).$$

Because $\left|\int_x^t (t-u)f''(u)du\right| \leq \|f''\|_B \frac{(t-x)^2}{2}$ using Lemma 2.2 we get

$$\left|\overline{DM_n}(f;x) - f(x)\right| \leq DM_n\left(\int_x^t (t-u)f''(u)du; x\right)$$

$$- \int_x^{\frac{I_{n,0,2}}{I_{n,0,0}}(1-x\Phi_n'(0))} \left(\frac{I_{n,0,1}}{I_{n,0,0}}(1-x\Phi_n'(0)) - u\right) f''(u)du$$

$$\leq \frac{\|f''\|_B}{2} DM_n\left((t-x)^2;x\right) + \frac{\|f''\|_B}{2}\left(\frac{I_{n,0,1}}{I_{n,0,0}}(1-x\Phi_n'(0)) - x\right)^2$$

$$\leq \frac{\|f''\|_B}{2}\left(\beta_n(x) + \lambda_n^2(x)\right).$$

□

Theorem 2.5. *For every $x \in \mathbb{R}_+$ and $f \in C_B(\mathbb{R}_+)$ the operators* (2.1)-(2.2) *satisfy the following relations*

(i) *If* $\lim_{n\to\infty} \frac{n^r I_{n,0,r}}{r! I_{n,0,0}} = 1$, $r = 0, 1, 2$, *then* $\lim_{n\to\infty} DM_n(f;x) = f(x)$,

(ii) $|DM_n(f;x) - f(x)| \leq 2\omega\left(f, \sqrt{\beta_n(x)}\right)$,

(iii) $|DM_n(f;x) - f(x)| \leq 2C\omega_2\left(f, \sqrt{\gamma_n(x)}\right) + \omega\left(f, \lambda_n(x)\right)$.

with $\lambda_n(x)$, $\beta_n(x)$, $\eta_n(x)$, $\gamma_n(x)$ defined as (2.6), (2.7), (2.8), (2.9).

Proof. (i) Because $\lim_{n\to\infty} \frac{\Phi_n^{(k)}(0)}{n^k} = \lim_{n\to\infty} \frac{(-1)^k \alpha_{n,k}(0)}{n^k} = (-1)^k$ and $\lim_{n\to\infty} \frac{n^r I_{n,0,r}}{r! I_{n,0,0}} = 1$, $r = 0, 1, 2$ using Lemma 2.2 we have $\lim_{n\to\infty} DM_n\left(e_r;x\right) = e_r(x)$, $r = 0, 1, 2$ and the Bohmann-Korovkin assure the conclusion (i) of the theorem.

(ii) Using a result of O. Shisha, B. Mond [7] with the modulus of continuity of f we obtain a quantitative estimation of the remainder of the approximation formula. Indeed,

$$|DM_n(f;x)-f(x)| \le \left(1+\delta_n^{-1}(x)\sqrt{DM_n\left((e_1-xe_0)^2;x\right)}\right)\omega\left(f,\delta_n(x)\right).$$

Taking

$$\delta_n(x)=\sqrt{\beta_n(x)}=\sqrt{DM_n\left((e_1-xe_0)^2;x\right)}$$

$$=\left\{\frac{I_{n,0,2}}{2I_{n,0,0}}\left(x^2\Phi_n''(0)-4x\Phi_n'(0)+2\right)-2x\frac{I_{n,0,1}}{I_{n,0,0}}\left(1-x\Phi_n'(0)\right)+x^2\right\}^{\frac{1}{2}}$$

the proof of (ii) is completed.

(iii) From (2.5) we obtain for $g\in W_\infty^2$

$$\begin{aligned}
&|DM_n(f;x)-f(x)|\\
&\le \left|\overline{DM_n}(f-g;x)-(f-g)(x)+\overline{DM_n}(g;x)-g(x)\right|\\
&+\left|f\left(\frac{nI_{n,0,1}}{I_{n,0,0}}\left(\frac{1}{n}-\frac{\Phi_n'(0)}{n}x\right)\right)-f(x)\right|\\
&\le 2\left\|f-g\right\|_B+\frac{\left\|g''\right\|_B\gamma_n(x)}{2}\\
&+\left|f\left(\frac{nI_{n,0,1}}{I_{n,0,0}}\left(\frac{1}{n}-\frac{\Phi_n'(0)}{n}x\right)\right)-f(x)\right|\\
&\le 2\left\|f-g\right\|_B+\frac{\left\|g''\right\|_B\gamma_n(x)}{2}+\omega\left(f,\lambda_n(x)\right).
\end{aligned}$$

Taking infimum over $g\in W_\infty^2$ on the right hand side, we get

$$\begin{aligned}
|DM_n(f;x)-f(x)| &\le 2K_2\left(f,\gamma_n(x)\right)+\omega\left(f,\lambda_n(x)\right)\\
&\le 2C\omega_2\left(f,\sqrt{\gamma_n(x)}\right)+\omega\left(f,\lambda_n(x)\right).
\end{aligned}$$

□

Theorem 2.6. *Let $f\in C_\rho(\mathbb{R}_+)$ and $\omega_{[0,b+1]}(f;\delta)$ be its modulus of continuity on the finite interval $[0,b+1]$, $b>0$. Then*

$$\left\|DM_n(f)-f\right\|_{C[0,b]}\le 3N_f\eta_n(b)(1+b)^2+2\omega_{[0,b+1]}\left(f,\sqrt{\eta_n(b)}\right),$$

with $\eta_n(x)$ defined as (2.8).

Proof. Let $x\in\mathbb{R}_+$ and $t>b+1$ Because $f\in C_\rho(\mathbb{R}_+)$ using the growth condition of f since $t-x>1$ we have

$$\begin{aligned}
|f(t)-f(x)| &\le N_f(2+t^2+x^2)\le N_f(2+(t-x+x)^2+x^2)\\
&\le 3N_f(t-x)^2(1+b)^2.
\end{aligned}$$

For $x\in\mathbb{R}_+$, $\delta>0$ and $t<b+1$ we have

$$|f(t)-f(x)| \le \left(1+\frac{|t-x|}{\delta}\right)\omega_{[0,b+1]}\left(f,\delta\right).$$

So,

$$|f(t)-f(x)| \quad \leq \quad 3N_f(t-x)^2(1+b)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega_{[0,b+1]}\left(f,\delta\right).$$

and with (2.8) we obtain

$$\begin{aligned}|DM_n(f;x)-f(x)| &\leq 3N_f DM_n\left((e_1-xe_0)^2;x\right)(1+b)^2\\ &+\left(1+\frac{DM_n\left(|e_1-xe_0|\,;x\right)}{\delta}\right)\omega_{[0,b+1]}\left(f,\delta\right)\\ &\leq 3N_f DM_n\left((e_1-xe_0)^2;x\right)(1+b)^2\\ &+\left(1+\frac{\sqrt{DM_n\left((e_1-xe_0)^2;x\right)}}{\delta}\right)\omega_{[0,b+1]}\left(f,\delta\right)\\ &\leq 3N_f\eta_n(b)(1+b)^2+\left(1+\frac{\sqrt{\eta_n(b)}}{\delta}\right)\omega_{[0,b+1]}\left(f,\delta\right)\\ &\leq 3N_f\eta_n(b)(1+b)^2+2\omega_{[0,b+1]}\left(f,\sqrt{\eta_n(b)}\right).\end{aligned}$$

□

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