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Spline and fractal spline interpolation

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Abstract. The classical methods of real data interpolation can be generalized by fractal interpolation. These fractal interpolation functions provide new methods of approximation of experimental data. This paper presents an application of these interpolation methods.

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1. Spline interpolation

Let $H^{m,2}[a,b]$, $m \in \mathbb{N}^*$ be the set of functions $f \in C^{m-1}[a,b]$ with $f^{(m-1)}$ absolutely continuous on $[a,b]$ and $f^{(m)} \in L^2[a,b]$, $\Lambda = \{\lambda_i \mid \lambda_i : H^{m,2}[a,b] \to \mathbb{R}, i = 1...,n\}$ a set of linear functionals, $y \in \mathbb{R}^n$ and

$$U = U_y = \{f \in H^{m,2}[a,b] | \lambda_i(f) = y_i, i = 1,...,n\}.$$

Definition 1.1. *The problem that consists of determining the elements $s \in U$ such that*

$$\|s^{(m)}\|_2 = \inf_{u \in U} \|u^{(m)}\|_2$$

is called a polynomial spline interpolation problem.

For the solution of a spline interpolation problem we can give the following structural characterization theorem ([3]) in the most general case, when we have Birkhoff type functionals. The set of Birkhoff type functionals is given by:

$$\Lambda = \{\lambda_{ij} | \lambda_{ij} f = f^{(j)}(x_i), i = 1,...,n, j \in I_i\},$$

for $I_i \subseteq \{0,...,r_i\}$, $r_i \in \mathbb{N}$, $r_i < m$, and $x_i \in [a,b], i = 1,...,k$.

Theorem 1.2. *Let Λ be a set of Birkhoff type functionals and let U be the corresponding interpolatory set. The functions $s \in U$ is a solution of the spline interpolation problem if and only if:*

1. $s^{(2m)}(x) = 0, \quad x \in [x_1, x_k] \setminus \{x_1, ...x_k\},$

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2. $s^{(m)}(x) = 0, \quad x \in (a, x_1) \cup (x_k, b)$,
3. $s^{(2m-\mu-1)}(x_i - 0) = s^{(2m-\mu-1)}(x_i + 0), \quad \mu \in \{0, 1, ..., m-1\} - I_i$ *for* $i = 1, ..., k$.

The characterization theorem states that the solution s of the polynomial spline interpolation problem is a polynomial of $2m - 1$ degree on each interior interval (x_i, x_{i+1}) and it is a polynomial of $m - 1$ degree on the intervals $[a, x_1)$ and $(x_k, b]$. Furthermore, the derivative of order $2m - \mu - 1$ is continuous in x_i if the value of the νth ordin derivative in x_i does not belong to Λ.

Definition 1.3. *The solution s of the polynomial spline interpolation problem is called a natural spline function of order $2m - 1$.*

When $\Lambda = \{\lambda_i | \lambda_i(f) = f(x_i), i = 1, ..., n\}$ is the set of Lagrange type functionals, with $x_i \in [a, b], i = 1, ..., n$ and $n \geq m$, then for every $f \in H^{m,2}[a, b]$ the interpolation spline function $S_L f$ exists and is unique.
The function $S_L f$ may be written in the form

$$S_L f = \sum_{k=1}^{n} s_k f(x_k),$$

where $s_k, k = 1, .., n$ are the fundamental interpolation spline functions. To determine these functions we can use the characterization theorem and we have

$$s_k(x) = \sum_{i=0}^{m-1} a_i^k x_i + \sum_{j=1}^{n} b_j^k (x - x_j)_+^{2m-1}, k = 1, ..., n,$$

with $a_i^k, i = 0, ..., m-1$ and $b_j^k, j = 1, ..., n$ obtained as the solution of the following systems:

$$s_k^p(\alpha) = 0, \ p = m, ..., 2m-1, \text{ and } \alpha > x_n$$
$$s_k(x_\nu) = \delta_{k\nu}, \ \nu = 1, ..., n$$

for $k = 1, ..., n$.
We collect form the server of our university some data regarding the internet traffic. We process these data with Lagrange type cubic spline function, and we obtain the following figure:

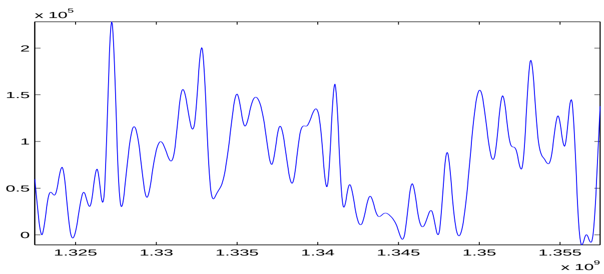

FIGURE 1. Spline interpolation: for internet traffic data

2. Fractal functions

Let (X, d) be a complete metric space, and $D(X)$ be the class of all non-empty closed bounded subsets of X. Then $(D(X), h)$ is a complete metric space with the Hausdorff metric: $h : D(X) \times D(X) \to R$

$$h(A, B) := \sup\{\sup_{a\in A} \inf_{b\in B} d(a, b), \sup_{b\in B} \inf_{a\in A} d(a, b)\}$$

Let $E \subset X$, $p \geq 0$, $\epsilon > 0$, $|E|$ denote the diameter of the subset E, and define the Hausdorff p-dimensional measure of E:

$$\mathcal{H}^p(E) := \lim_{\epsilon\to 0} \mathcal{H}^p_\epsilon(E) = \sup_{\epsilon>0} \mathcal{H}^p_\epsilon(E),$$

where

$$\mathcal{H}^p_\epsilon(E) := \inf \left\{ \sum_{i=1}^{\infty} |E_i|^p, E \subset \cup_{i=1}^{\infty} E_i, |E_i| < \epsilon \right\}.$$

For each E there is a unique real number q, named the Hausdorff dimension of E, such that

$$\mathcal{H}^p(E) = \begin{cases} +\infty & if \quad 0 \leq p < q \\ 0 & if \quad q < p < \infty \end{cases}$$

B. Mandelbrot define fractal as the set of which Hausdorff dimension is noninteger.

The functions $f : I \to \mathbf{R}$, where I is a real closed interval, is named by M. F. Barnsley *fractal function* if the Hausdorff dimensions of their graphs are noninteger.

Let be N a natural number, $N > 1$, and let $w_i : \ X \to X : \ i \in \{1, ..., N\}$ be continuous functions. Then we call $\{X, w_i : i = 1, ..., N\}$ an *iterated function system (IFS).*

If, for some $0 \leq k < 1$ and all $i \in \{1, ..., N\}$,

$$d(w_i(x), w_i(x')) \leq kd(x, x'), \forall\, x, x' \in X,$$

then the IFS is named *hyperbolic.*

Define $W : \mathcal{D}(X) \to \mathcal{D}(X)$ by

$$W(A) := \cup_{i=1}^{N} w_i(A),$$

where $w_i(A) = \{w_i(x) : x \in A\}$.

W is a contraction mapping if the IFS is hyperbolic:

$$h(W(A), W(B)) \leq kh(A, B)\, \forall\, A, B \in \mathcal{D}(X).$$

Any set $G \in D(X)$ such that $W(G) = G$ is called an *attractor* for the IFS.

Theorem 2.1. (Hutchinson [4]) *Let $\{X, w_i\, i = 1, ..., N\}$ an hyperbolic IFS. There is a unique compact set $G \subset X$, such that $W(G) = G$, and*

$$G := \lim_{n\to\infty} W^n(E),\ E \in \mathcal{D}(X),\ W^0.$$

Let $\{(x_i, y_i) \in R^2, i = 0, 1, \cdots, N\}$ be given, and $I = [x_0, x_N]$. The functions $f : I \to R$, which interpolate the data according to f$(x_i) = y_i$, $i = 0, 1, ..., N$, and whose graphs are attractors of IFS are *fractal interpolation functions* .

Let be $X = I \times [a,b]$ with Euclidean metric d, $I_n = [x_{n-1}, x_n]$ $u_n : I \to I_n$, $n \in \{1, 2, ..., N\}$, contractive homeomorphism such that

$$u_n(x_0) := x_{n-1},\ u_n(x_N) := x_n,\ \forall n \in \{1, \cdots, N\}.$$

$$|u_n(c_1) - u_n(c_2)| \le l|c_1 - c_2|,\ c_1, c_2 \in I,\ 0 \le l < 1$$

$v_n : X \to [a,b]$ continuous, with

$$v_n(x_0, y_0) := y_{n-1},\ v_n(x_N, y_N) := y_n,\ \forall n \in \{1, \cdots, N\}.$$

$$|v_n(c, d_1) - v_n(c, d_2)| \le q|d_1 - d_2|,\ c \in I,\ d_1, d_2 \in [a,b], 0 \le q < 1.$$

Let $w_n : X \to X$, $n \in \{1, 2, ..., N\}$

$$w_n(x, y) = (u_n(x), v_n(x, y)).$$

Than $\{X, w_n : n = 1, 2, ..., N\}$ is an IFS but may not be hyperbolic.

Theorem 2.2. (Barnsley [1]) *For the IFS $\{X, w_n : n = 1, 2, ..., N\}$ defined above, there is a metric d equivalent to the Euclidean metric, such that the IFS is hyperbolic with respect to d. The unique attractor G of the IFS is the graph of a continuous function $f : I \to R$ which interpolates the data set $\{(x_i, y_i) \in R^2, i = 0, 1, \cdots, N\}$*

The following example given by Barnsley is used in many articles to give the iterated function system for the most widely studied fractal interpolation function.

Example 2.3. [1] Let $\{(x_i, y_i) \in R^2, i = 0, 1, \cdots, N\}$, $N > 1$

$$w_n(x, y) = \begin{pmatrix} a_n & 0 \\ c_n & d_n \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_n \\ f_n \end{pmatrix},$$

where $|d_n| < 1$ is given, a_n, c_n, e_n, f_n are real number such that

$$w_n(x_0, y_0) := (x_{n-1}, y_{n-1}),\ w_n(x_N, y_N) := (x_n, y_n)$$

From the above equations follows that

$$a_n = \frac{x_n - x_{n-1}}{x_N - x_0},$$

$$c_n = \frac{y_n - y_{n-1}}{x_N - x_0} - \frac{d_n(y_N - y_0)}{x_N - x_0},$$

$$e_n = \frac{x_N x_{n-1} - x_0 x_n}{x_N - x_0}$$

$$f_n = \frac{x_N y_{n-1} - x_0 y_n}{x_N - x_0} - \frac{d_n(x_N y_0 - x_0 y_N)}{x_N - x_0}.$$

w_n is a shear transformation: it maps lines parallel to the y-axis into the lines parallel to the y-axis, d_n is the vertical scaling factor.

Using the same data regarding the internet traffic, we construct the iterated function system, and we implement this in MatLab. The graph of the fractal interpolation function for these data is given in Figure 2.

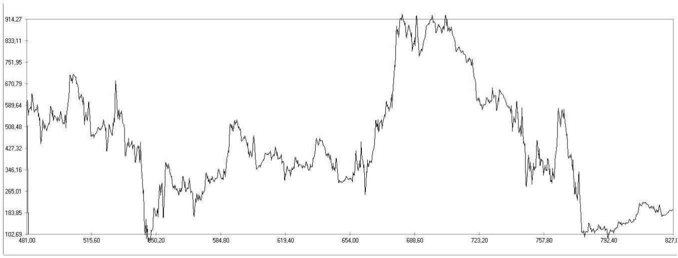

FIGURE 2. Fractal interpolation: for internet traffic data

3. Spline fractal interpolation functions

Let be

$$u_n(x) = a_n x + b_n$$

and

$$v_n(x, y) = \alpha_n y + q_n(x)$$

where a_n and b_n can be obtained from the relations for a_n, e_n, given in example and $-1 < \alpha_n < 1$.

The function f is a cubic spline fractal interpolation function, which interpolates the set of ordinates $y_0, y_1, ..., y_N$ with respect to the mesh $x_0 < x_1 < ... < x_N$ if f is of class $C^2[x_0, x_N]$, who satisfies the interpolation conditions $f(x_i) = y_i$, $i = 0, 1, ..., N$, and the graph of f is fixed point of the iterated function system $\{\mathbb{R}^2; \omega_n(x, y), n = 1, 2, ..., N\}$, where $\omega_n(x, y) = (u_n x, v_n(x, y))$, and the function $q_n(x)$ is a suitable cubic polynomial.

In order to construct the cubic spline fractal interpolation function for the internet traffic data, we use the algorithm used by Chand and Kapoor in ([2]), where the cubic spline functions are constructed by the moments $M_n = f''(x_n)$ for $n = 1, 2, ..., N$.

Let $G = \{f : I \to \mathbb{R},\ f \text{ is continuous},\ f(x_0) = y_0,\ f(x_N) = y_N\}$, and ρ be the sup-norm on G, then (G, ρ) is a complete metric space, and the fractal interpolation function is the unique fixed point of the Read-Bajraktarević operator T on (G, ρ) so that

$$Tf(x) \equiv v_n(u_n^{-1}(x), f(u_n^{-1}(x))) = f(x), n = 1, 2, ..., N. \tag{3.1}$$

In the algorithm mentioned above, Chand and Kapoor obtained from the properties of a fractal spline interpolation function, that the cubic spline fractal interpolation function in terms of the moments can be written as

$$f(u_n(x)) = a_n^2 \left\{ \alpha_n f(x) + \frac{(M_n - \alpha_n M_N)(x - x_0)^3}{6(x_N - x_0)} + \frac{(M_{n-1} - \alpha_n M_0)(x_N - x)^3}{6(x_N - x_0)} \right.$$
$$- \frac{(M_{n-1}\alpha_n M_0)(x_N - x_0)(x_N - x)}{6} - \frac{(M_N - \alpha_n M_N)(x_N - x_0)(x - x_0)}{6}$$
$$\left. + \left(\frac{y_{n-1}}{a_n^2} - \alpha_n y_0\right) \frac{x_N - x}{x_N - x_0} + \left(\frac{y_n}{a_n^2} - \alpha_n y_N\right) \frac{x - x_0}{x_N - x_0} \right\}, n = 1, 2, ..., N,$$

also they give the system of equations from where the moments can be obtained:

$$A_n^* f'(x_0) + A_n M_0 + \mu M_{n-1} + 2M_n + \lambda M_{n+1} + B_n M_N + B_n^* f'(x_N)$$
$$= \frac{6[(y_{n+1} - y_n)/h_{n+1} - (y_n - y_{n-1})/h_n]}{h_n + h_{n+1}} - \frac{6(a_{n+1}\alpha_{n+1} - a_n\alpha_n)}{h_n + h_{n+1}} \frac{y_N - y_0}{x_N - x_0},$$

where

$$A_n^* = \frac{-6a_{n+1}\alpha_{n+1}}{h_n + h_{n+1}}, \ A_n = \frac{-(\alpha_n h_n + 2\alpha_{n+1} h_{n+1})}{h_n + h_{n+1}},$$

$$\alpha_n = \frac{6}{h_n + h_{n+1}}, \ \mu_n = 1 - \lambda_n,$$

$$B_n = \frac{-(2\alpha_n h_n + \alpha_{n+1} h_{n+1})}{h_n + h_{n+1}}, \ B_n^* = \frac{6a_n\alpha_n}{h_n + h_{n+1}}$$

for $n = 1, 2, ..., N - 1$ and $x_n - x_{n-1} = h_n$ for $n = 1, 2, ..., N$.

Solving the systems of equations and using the boundary conditions where the values of the first derivative are prescribed at the endpoints of the interval $[x_0, x_N]$, we have the moments $M_n, n = 0, 1, ..., N$ which are used in the construction of the iterated function system given by the relations from ([2])

$$\{\mathbb{R}^2; \omega_n(x, y) = (u_n(x), v_n(x, y)), n = 1, 2, ..., N\}, \tag{3.2}$$

where $u_n(x) = a_n x + b_n$ and

$$v_n(x, y) = a_n^2 \left\{ \alpha_n f(x) + \frac{(M_n - \alpha_n M_N)(x - x_0)^3}{6(x_N - x_0)} + \frac{(M_{n-1} - \alpha_n M_0)(x_N - x)^3}{6(x_N - x_0)} \right.$$
$$- \frac{(M_{n-1}\alpha_n M_0)(x_N - x_0)(x_N - x)}{6} - \frac{(M_N - \alpha_n M_N)(x_N - x_0)(x - x_0)}{6}$$
$$\left. + \left(\frac{y_{n-1}}{a_n^2} - \alpha_n y_0\right) \frac{x_N - x}{x_N - x_0} + \left(\frac{y_n}{a_n^2} - \alpha_n y_N\right) \frac{x - x_0}{x_N - x_0} \right\}, n = 1, 2, ..., N.$$

The graph of the cubic spline is the fixed point of the iterated function system given by (3.2).

We made a MatLab implementation of this algorithm. It can be used for arbitrary set of data, in the case of the data studied before, the cubic spline fractal interpolation function will have the form given in Figure 3.

The cubic spline interpolation function is an important tool in computer graphics, CAGD, differential equations and several engineering applications and it is of class C^2. Generally, affine fractal interpolation functions are nondifferentiable functions. Therefore the cubic spline fractal interpolation it seems to be a good method in data processing, because it has better properties and it can be used with all possible boundary conditions like in the case of classical splines.

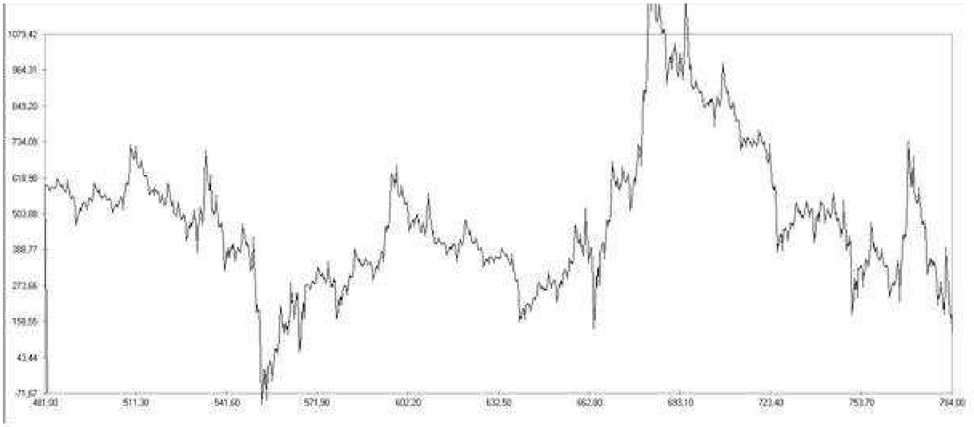

FIGURE 3. Spline fractal interpolation: for internet traffic data

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