Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 2, 183–192

Weighted Ostrowski-Grüss type inequalities

Ana Maria Acu and Heiner Gonska

Abstract. Several inequalities of Ostrowski-Grüss-type availabe in the literature are generalized considering the weighted case of them. Involving the least concave majorant of the modulus of continuity we provide upper bounds of our inequalities.

Mathematics Subject Classification (2010): 26D15, 26A15, 39B62.

Keywords: Ostrowki-Grüss type inequalities, modulus of smoothness, least concave majorant of modulus of continuity.

1. Introduction

Over the last decades, integral inequalities have attracted much attention because of their applications in statistical analysis and the theory of distributions. In this paper we improve the classical Ostrowski type inequality for weighted integrals and generalize some Ostrowski-Grüss type inequalities involving differentiable mappings. Also, applications to special weight functions are investigated.

For each $x \in [a,b]$ consider the linear functional

$$\mathcal{L}(f)(x) := f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right), f \in C[a,b].$$

If $f : [a,b] \to \mathbb{R}$ is differentiable with bounded derivative, then

$$\begin{aligned} |\mathcal{L}(f)(x)| &\leq \frac{1}{8}(b-a)(\Gamma-\gamma) && (1.1)\\ &\leq \frac{1}{4\sqrt{3}}(b-a)(\Gamma-\gamma) && (1.2)\\ &\leq \frac{1}{4}(b-a)(\Gamma-\gamma), && (1.3)\end{aligned}$$

where $\gamma := \inf\{f'(x)|x \in [a,b]\}$ and $\Gamma := \sup\{f'(x)|x \in [a,b]\}$.

This paper was presented at the third edition of the International Conference on Numerical Analysis and Approximation Theory (NAAT 2014), Cluj-Napoca, Romania, September 17-20, 2014.

The inequality (1.3) was proven by S.S. Dragomir and S. Wang [3] and it is known as the Ostrowski-Grüss-type inequality. This inequality was improved by M. Matić et al. [6], and we recall their result in (1.2). An improvement of this result was given by X.L. Cheng in [2], as shown in relation (1.1). He also proved that the constant 1/8 is best possible.

In [5] the authors introduced the linear functional $\mathcal{L}_c : C[a,b] \to \mathbb{R}$ defined by

$$\mathcal{L}_c(f)(x) := f(x) - \frac{1}{b-a}\int_a^b f(t)dt - c\cdot\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right), \text{ where } c \geq 0.$$

The following result gives bounds of the functional $\mathcal{L}_c$ involving differences of upper and lower bounds of first order derivatives.

Theorem 1.1. [5] *For all $x \in [a,b]$, $c \in [0,2]$ and $f \in C^1[a,b]$ we have*

$$\frac{1}{2(b-a)}\left[(x-a-u_c(x))^2\gamma - (x-b-u_c(x))^2\Gamma\right] \leq \mathcal{L}_c(f)(x) \tag{1.4}$$

$$\leq \frac{1}{2(b-a)}\left[(x-a-u_c(x))^2\Gamma - (x-b-u_c(x))^2\gamma\right],$$

where $u_c(x) := c\left(x - \frac{a+b}{2}\right)$.

Remark 1.2. a) From (1.4), with $c=1$, inequality (1.1) follows which was established by X.L. Cheng in [2].

b) As a consequence of (1.4), for $c=0$, the following inequality holds:

$$-\frac{(x-a)^2+(b-x)^2}{2(b-a)}\|f'\|_\infty \leq \frac{(x-a)^2\gamma-(b-x)^2\Gamma}{2(b-a)} \leq f(x) - \frac{1}{b-a}\int_a^b f(t)dt$$

$$\leq \frac{(x-a)^2\Gamma-(b-x)^2\gamma}{2(b-a)} \leq \frac{(x-a)^2+(b-x)^2}{2(b-a)}\|f'\|_\infty.$$

This inequality improves the classical Ostrowski inequality presented by Anastassiou in [1] in the form

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{(x-a)^2+(b-x)^2}{2(b-a)}\|f'\|_\infty.$$

Weighted versions of (1.3), (1.2) and (1.1) were established by J. Roumeliotis in [9] and [10]. These results are given below.

Definition 1.3. *Let $w : (a,b) \to (0,\infty)$ be integrable, i.e., $\int_a^b w(t)dt < \infty$, then $m(\alpha,\beta) := \int_\alpha^\beta w(t)dt$ and $M(\alpha,\beta) := \int_\alpha^\beta tw(t)dt$ are the first moments, for $[\alpha,\beta] \subseteq [a,b]$. Define the mean of the interval $[\alpha,\beta]$ with respect to the weight function w as $\sigma(\alpha,\beta) := \frac{M(\alpha,\beta)}{m(\alpha,\beta)}$.*

The weighted variant of the functional $\mathcal{L}$ can be written in the following way:

$$\mathcal{L}_w(f)(x) := f(x) - \frac{1}{m(a,b)} \int_a^b f(t)w(t)dt - \frac{f(b)-f(a)}{b-a} \left(x - \sigma(a,b)\right).$$

Theorem 1.4. [9] *Let $f : [a,b] \to \mathbb{R}$ be differentiable with bounded derivative and let $w : (a,b) \to (0,\infty)$ be integrable. Then*

$$\begin{aligned} |\mathcal{L}_w(f)(x)| &\le \frac{1}{2}(\Gamma - \gamma)\frac{\sqrt{b-a}}{m(a,b)} \left\{ \int_a^b K^2(x,t)dt - \frac{m(a,b)^2(x-\sigma(a,b))^2}{b-a} \right\}^{\frac{1}{2}} \\ &\le \frac{1}{4}(\Gamma - \gamma)(b-a), \end{aligned}$$

for all $x \in [a,b]$, where $K(x,t) = \begin{cases} \int_a^t w(u)du, a \le t \le x \\ \int_b^t w(u)du, x < t \le b. \end{cases}$

Theorem 1.5. [10] *Let $f : [a,b] \to \mathbb{R}$ be differentiable with bounded derivative and let $w : (a,b) \to (0,\infty)$ be integrable. Then*

$$|\mathcal{L}_w(f)(x)| \le \frac{\Gamma - \gamma}{m(a,b)} \int_x^{t^*} (t-x)w(t)dt, \tag{1.5}$$

for all $x \in [a,b]$, where $t^* \in [a,b]$ is unique and verifies

$$\frac{m(a,b)}{b-a}|x - \sigma(a,b)| = \begin{cases} m(t^*,b), a \le x \le \sigma(a,b) \\ m(a,t^*), \sigma(a,b) < x \le b. \end{cases}$$

For each $x \in [a,b]$ consider the linear functional $\mathcal{L}_{w,c} : C[a,b] \to \mathbb{R}$ defined by

$$\mathcal{L}_{w,c}(f)(x) := f(x) - \frac{1}{m(a,b)} \int_a^b f(t)w(t)dt - c \cdot \frac{f(b)-f(a)}{b-a} \left(x - \sigma(a,b)\right),$$

where $c \ge 0$.

In this paper we propose the weighted analogue of (1.4). Also, new inequalities of $\mathcal{L}_{w,c}$ will be considered involving the least concave majorants of the first order moduli of continuity.

2. Generalized Ostrowski-Grüss type inequalities

In this section we will give the upper bounds of $\mathcal{L}_{w,c}$ involving differences of upper and lower bounds of first order derivatives. First, we need the following lemma.

Lemma 2.1. *For $c \in [0,1]$ and $x \in [0,1]$, there exists a unique $t^* = t^*(x) \in [a,b]$ satisfying*

$$u_{w,c}(x) = \begin{cases} \dfrac{1}{m(a,b)} \displaystyle\int_b^{t^*} w(u)du, \ a \le x \le \sigma(a,b) \\ \dfrac{1}{m(a,b)} \displaystyle\int_a^{t^*} w(u)du, \ \sigma(a,b) < x \le b, \end{cases}$$

where $u_{w,c}(x) = \dfrac{c}{b-a}\left(x - \sigma(a,b)\right)$.

Proof. Let us consider $a \le x \le \sigma(a,b)$ and

$$f(t) = \frac{1}{m(a,b)} \int_b^t w(u)du - u_{w,c}(x), t \in [x,b].$$

Since f is strictly increasing on (x,b), $f(b) = -u_{w,c}(x) \ge 0$, then to show that $t^* \in [x,b]$ exists, it will suffice to establish that $f(x) \le 0$, where

$$f(x) = \frac{1}{m(a,b)} \int_b^x w(u)du - u_{w,c}(x).$$

It follows

$$\begin{aligned}
u_{w,c}(x) &= \frac{c}{b-a}(x - \sigma(a,b)) = \frac{c}{b-a}\left(x - \frac{M(a,b)}{m(a,b)}\right) \\
&= -\frac{c}{(b-a)m(a,b)} \int_a^b (t-x)w(t)dt \\
&= -\frac{c}{(b-a)m(a,b)} \left\{ \int_a^x (t-x)w(t)dt + \int_x^b (t-x)w(t)dt \right\} \\
&\ge -\frac{c}{(b-a)m(a,b)} \int_x^b (t-x)w(t)dt \ge -\frac{c}{(b-a)m(a,b)}(b-x) \int_x^b w(t)dt \\
&\ge -\frac{c}{m(a,b)} \int_x^b w(t)dt \ge \frac{1}{m(a,b)} \int_b^x w(t)dt.
\end{aligned}$$

In a similar way for $\sigma(a,b) < x \le b$ follows that there exists a unique $t^* \in [a,x]$ such that $u_{w,c}(x) = \frac{1}{m(a,b)} \int_a^{t^*} w(u)du$. □

Denote

$$\mathcal{P}(x,t) = \begin{cases} \frac{1}{m(a,b)} \int_a^t w(u)du - u_{w,c}(x), a \le t < x, \\ \frac{1}{m(a,b)} \int_b^t w(u)du - u_{w,c}(x), x \le t \le b. \end{cases}$$

It is easy to verify that, for all $f \in C^1[a,b]$, $\mathcal{L}_{w,c}(f)(x) = \int_a^b \mathcal{P}(x,t) f'(t)dt$.

Theorem 2.2. *For all $x \in [a,b], c \in [0,1]$ and $f \in C^1[a,b]$ we have*

$$\begin{aligned}
(1-c)(x-\sigma(a,b))\gamma + (\gamma - \Gamma)\nu(x,t^*) &\le \mathcal{L}_{w,c}(f)(x) \qquad (2.1)\\
&\le (1-c)(x-\sigma(a,b))\Gamma + (\Gamma - \gamma)\nu(x,t^*),
\end{aligned}$$

where $\nu(x,t^*) := \frac{1}{m(a,b)} \int_x^{t^*} (t-x)w(t)dt.$

Proof. If $a \le x \le \sigma(a,b)$, then

$$\mathcal{P}(x,t) \ge 0, \text{ for } t \in [a,x) \cup [t^*,b] \text{ and } \mathcal{P}(x,t) < 0, \text{ for } t \in [x,t^*).$$

Also, if $\sigma(a,b) < x \le b$, it follows

$$\mathcal{P}(x,t) \le 0, \text{ for } t \in [a,t^*] \cup [x,b] \text{ and } \mathcal{P}(x,t) > 0, \text{ for } t \in (t^*,x).$$

Let $a \le x \le \sigma(a,b)$. It follows

$$\mathcal{L}_{w,c}(f)(x) \le \Gamma\left(\int_a^x \mathcal{P}(x,t)dt + \int_{t^*}^b \mathcal{P}(x,t)dt\right) + \gamma\int_x^{t^*} \mathcal{P}(x,t)dt.$$

We have

$$\int_a^x \mathcal{P}(x,t)dt + \int_{t^*}^b \mathcal{P}(x,t)dt = \int_a^x \left(\frac{1}{m(a,b)}\int_a^t w(u)du - u_{w,c}(x)\right)dt$$
$$+\int_{t^*}^b \left(\frac{1}{m(a,b)}\int_b^t w(u)du - u_{w,c}(x)\right)dt$$
$$= \frac{1}{m(a,b)}\left\{x\int_a^x w(u)du - \int_a^x tw(t)dt - t^*\int_b^{t^*} w(u)du - \int_{t^*}^b tw(t)dt\right\}$$
$$-u_{w,c}(x)(x-a+b-t^*) = \frac{1}{m(a,b)}\left\{x\int_a^x w(u)du - \int_a^x tw(t)dt - \int_{t^*}^b tw(t)dt\right\}$$
$$-u_{w,c}(x)(x-a+b) + t^*\left(u_{w,c}(x) - \frac{1}{m(a,b)}\int_b^{t^*} w(u)du\right)$$
$$= \frac{1}{m(a,b)}\left\{x\int_a^x w(u)du - \int_a^x tw(t)dt - (x-a+b)\int_b^{t^*} w(u)du - \int_{t^*}^b tw(t)dt\right\}$$
$$= \frac{1}{m(a,b)}\left\{\int_{t^*}^b (x-a+b-t)w(t)dt + \int_a^x (x-t)w(t)dt\right\}$$
$$= \frac{1}{m(a,b)}\left\{(b-a)\int_{t^*}^b w(t)dt + \int_a^b (x-t)w(t)dt - \int_x^{t^*} (x-t)w(t)dt\right\}$$
$$= \frac{1}{m(a,b)}\left\{-(b-a)m(a,b)u_{w,c}(x) + xm(a,b) - M(a,b) + \int_x^{t^*} (t-x)w(t)dt\right\}$$
$$= (1-c)(x-\sigma(a,b)) + \frac{1}{m(a,b)}\int_x^{t^*} (t-x)w(t)dt.$$

$$\int_x^{t^*} \mathcal{P}(x,y)dt = \int_x^{t^*}\left\{\frac{1}{m(a,b)}\int_b^t w(u)du - u_{w,c}(x)\right\}dt$$
$$= \frac{1}{m(a,b)}\left\{t^*\int_b^{t^*} w(u)du - x\int_b^x w(u)du - \int_x^{t^*} tw(t)dt\right\} - u_{w,c}(x)(t^*-x)$$
$$= t^*\left\{\frac{1}{m(a,b)}\int_b^{t^*} w(u)du - u_{w,c}(x)\right\} - \frac{1}{m(a,b)}\left\{x\int_b^x w(u)du + \int_x^{t^*} tw(t)dt\right\}$$
$$+x\cdot\frac{1}{m(a,b)}\int_b^{t^*} w(u)du = -\frac{1}{m(a,b)}\int_x^{t^*} (t-x)w(t)dt = -\nu(x,t^*).$$

Therefore,

$$\begin{aligned}\mathcal{L}_{w,c}(f)(x) &\leq \Gamma\left[(1-c)(x-\sigma(a,b))+\nu(x,t^*)\right]-\gamma\nu(x,t^*) \\ &= (1-c)(x-\sigma(a,b))\Gamma+(\Gamma-\gamma)\nu(x,t^*).\end{aligned} \tag{2.2}$$

By similar reasoning it follows that (2.2) is also valid if $\sigma(a,b)\leq x\leq b$.

If we write (2.2) for $-f$ instead of f, we obtain

$$\mathcal{L}_{w,c}(f)(x)\geq (1-c)(x-\sigma(a,b))\gamma+(\gamma-\Gamma)\nu(x,t^*).$$ □

Corollary 2.3. *For all $x\in[a,b]$ and $f\in C^1[a,b]$, we obtain*

$$|\mathcal{L}_w(f)(x)|\leq \frac{\Gamma-\gamma}{m(a,b)}\int_x^{t^*}(t-x)w(t)dt. \tag{2.3}$$

Proof. The inequality (2.3) follows from (2.1) with $c=1$. □

Remark 2.4. The coefficient $\frac{1}{m(a,b)}\int_x^{t^*}(t-x)w(t)dt$ is sharp in the sense that it cannot be replaced by a smaller one. The inequality (2.3) holds for all $x\in[a,b]$. Let $x=\sigma(a,b)$ and

$$f(t)=\begin{cases}\Gamma(t-a),\ a\leq t<\sigma(a,b)\\ \Gamma\left(\sigma(a,b)-a\right)+\gamma\left(t-\sigma(a,b)\right),\ \sigma(a,b)\leq t\leq b.\end{cases}$$

If $x=\sigma(a,b)$, then $t^*=b$ and it follows

$$\mathcal{L}_w(f)(\sigma(a,b))=\frac{\Gamma-\gamma}{m(a,b)}\int_{\sigma(a,b)}^b\left(t-\sigma(a,b)\right)w(t)dt.$$

Therefore, in (2.3) equality holds.

Corollary 2.5. *For all $x\in[a,b]$ and $f\in C^1[a,b]$, the following inequality holds*

$$\begin{aligned}&-\frac{\|f'\|_\infty}{m(a,b)}\int_a^b|t-x|w(t)dt\leq (x-\sigma(a,b))\gamma+(\gamma-\Gamma)\nu(x,t^*) \\ &\leq f(x)-\frac{1}{m(a,b)}\int_a^b f(t)w(t)dt \\ &\leq (x-\sigma(a,b))\Gamma+(\Gamma-\gamma)\nu(x,t^*)\leq \frac{\|f'\|_\infty}{m(a,b)}\int_a^b|t-x|w(t)dt.\end{aligned} \tag{2.4}$$

Proof. This result is a consequence of (2.1), for $c=0$. □

Remark 2.6. a) Inequality (2.3) was established by J. Roumeliotis [10].

b) Inequality (2.4) improves the classical weighted Ostrowski inequality

$$\left|f(x)-\frac{1}{m(a,b)}\int_a^b f(t)w(t)dt\right|\leq \|f'\|_\infty\cdot\frac{1}{m(a,b)}\int_a^b|t-x|w(t)dt.$$

3. Ostrowski-Grüss-type inequalities in terms of the least concave majorant

The aim of this section is to extend the inequalities mentioned in the previous section, by using the least concave majorant of the modulus of continuity. This approach was inspired by a paper of Gavrea & Gavrea [4] who were the first to observe the possibility of using moduli in this context.

Proposition 3.1. *The linear functional* $\mathcal{L}_{w,c} : C[a,b] \to \mathbb{R}$ *satisfies*

i) $|\mathcal{L}_{w,c}(f)(x)| \leq 4\|f\|_\infty$, *for all* $f \in C[a,b]$.

ii) $|\mathcal{L}_{w,c}(f)(x)| \leq [(c-1)|x-\sigma(a,b)| + 2\nu(x,t^*)] \|f'\|_\infty$, *for all* $f \in C^1[a,b]$, *where* $c \in [0,1]$ *and* ν *is defined in Theorem* 2.2.

Proof. Inequality i) follows immediately from definition of $\mathcal{L}_{w,c}$. The second inequality is obtained after elementary calculations as follows:

$$|\mathcal{L}_{w,c}(f)(x)| \leq \int_a^b |\mathcal{P}(x,t)|\, f'(t)dt \leq \|f'\|_\infty \int_a^b |\mathcal{P}(x,t)|\, dt$$

If $a \leq x \leq \sigma(a,b)$, then

$$\begin{aligned}\int_a^b |\mathcal{P}(x,t)|\, dt &= \int_a^x \mathcal{P}(x,t)dt + \int_{t^*}^b \mathcal{P}(x,t)dt - \int_x^{t^*} \mathcal{P}(x,t)dt \\ &= (1-c)(x-\sigma(a,b)) + 2\nu(x,t^*).\end{aligned} \tag{3.1}$$

By similar reasoning, for $\sigma(a,b) < x \leq b$, it follows

$$\begin{aligned}\int_a^b |\mathcal{P}(x,t)|\, dt &= -\int_a^{t^*} \mathcal{P}(x,t)dt - \int_x^b \mathcal{P}(x,t)dt + \int_{t^*}^x \mathcal{P}(x,t)dt \\ &= (c-1)(x-\sigma(a,b)) + 2\nu(x,t^*).\end{aligned}$$

□

Theorem 3.2. *If* $f \in C[a,b]$, $c \in [0,1]$, *then*

$$|\mathcal{L}_{w,c}(f)(x)| \leq 2\tilde{\omega}\left(f; \frac{1}{2}[(c-1)|x-\sigma(a,b)| + 2\nu(x,t^*)]\right),$$

where ν *is defined in Theorem* 2.2 *and* $\tilde{\omega}$ *is the least concave majorant of the usual modulus of continuity.*

Proof. Taking an arbitrary $g \in C^1[a,b]$ and using Proposition 3.1 we obtain

$$\begin{aligned}|\mathcal{L}_{w,c}(f)(x)| &\leq |\mathcal{L}_{w,c}(f-g)(x)| + |\mathcal{L}_{w,c}(g)(x)| \\ &\leq 4\|f-g\|_\infty + [(c-1)|x-\sigma(a,b)| + 2\nu(x,t^*)] \|g'\|_\infty.\end{aligned}$$

Passing to the inf we arrive at

$$\begin{aligned}|\mathcal{L}_{w,c}(f)(x)| &\leq 4 \inf_{g\in C^1[a,b]} \left\{\|f-g\|_\infty + \frac{1}{4}[(c-1)|x-\sigma(a,b)| + 2\nu(x,t^*)] \|g'\|_\infty\right\} \\ &= 2\tilde{\omega}\left(f; \frac{1}{2}[(c-1)|x-\sigma(a,b)| + 2\nu(x,t^*)]\right),\end{aligned}$$

so the result follows as a consequence of the relation [11]:

$$\inf_{g\in C^1([a,b])}\left(\|f-g\|_\infty+\frac{t}{2}\|g'\|\right)=\frac{1}{2}\tilde{\omega}(f;t), t\geq 0. \qquad \square$$

Corollary 3.3. *For all $x\in[a,b]$ and $f\in C[a,b]$, we obtain*

$$|\mathcal{L}_w(f)(x)|\leq 2\tilde{\omega}\left(f;\frac{2}{m(a,b)}\int_x^{t^*}(t-x)w(t)dt\right).$$

Corollary 3.4. *For all $x\in[a,b]$ and $f\in C[a,b]$, the following inequality holds*

$$\left|f(x)-\frac{1}{m(a,b)}\int_a^b f(t)w(t)dt\right|$$

$$=2\tilde{\omega}\left(f;\frac{1}{2m(a,b)}\left(\int_x^a(t-x)w(t)dt+\int_x^b(t-x)w(t)dt\right)\right).$$

4. Numerical example

In this section the inequality (2.4) is evaluated for some specific weight functions.
1. Let the weight function w be the probability density function of the Beta distribution,

$$w_{p,q}(x)=\begin{cases}\dfrac{1}{B(p,q)}x^{p-1}(1-x)^{q-1}, \; x\in[0,1],\\ 0, \; x\in\mathbb{R}\setminus[0,1],\end{cases}$$

where $B(p,q)=\displaystyle\int_0^1 x^{p-1}(1-x)^{q-1}dx$, $p,q>0$.

Substituting $w_{p,q}$ in relation (2.4), it follows

$$\left(x-\frac{p}{p+q}\right)\gamma+(\gamma-\Gamma)\tilde{\nu}(x) \tag{4.1}$$

$$\leq f(x)-\int_a^b w_{p,q}(t)f(t)dt\leq\left(x-\frac{p}{p+q}\right)\Gamma+(\Gamma-\gamma)\tilde{\nu}(x),$$

where

$$\tilde{\nu}(x)=\begin{cases}\dfrac{p}{p+q}-x-B(x;p+1,q)+xB(x;p,q), \; 0\leq x\leq\dfrac{p}{p+q},\\ xB(x;p,q)-B(x;p+1,q), \; \dfrac{p}{p+q}<x\leq 1,\end{cases}$$

and $B(x;p,q)=\dfrac{1}{B(p,q)}\displaystyle\int_0^x t^{p-1}(1-t)^{q-1}dt$, $0\leq x\leq 1$ is the incomplete Beta function.

In the below table , for $p=q=\dfrac{1}{2}$ and $f(t)=\dfrac{t^2}{2}, t\in[0,1]$ we calculate the left hand side and the right hand side of inequality (4.1):

Table 1. Error estimate of $E(x) = f(x) - \int_0^1 w_{p,q}(t)f(t)dt$

x	$\tilde{\nu}(x)$	l.h.s of (4.1)	r.h.s of (4.1)	$E(x)$
0	0.500000000000000	-0.500000000000000	0	-0.1875
0.1	0.406636443481054	-0.406636443481054	0.006636443481054	-0.1825
0.2	0.318514120706339	-0.318514120706339	0.018514120706339	-0.1675
0.3	0.233428745882118	-0.233428745882118	0.033428745882118	-0.1425
0.4	0.150335250602855	-0.150335250602855	0.050335250602855	-0.1075
0.5	0.068309886183791	-0.068309886183791	0.068309886183791	-0.0625
0.6	0.086241033753880	-0.086241033753880	0.186241033753880	-0.0075
0.7	0.102438865447664	-0.102438865447664	0.302438865447664	0.0575
0.8	0.113681356007205	-0.113681356007205	0.413681356007205	0.1325
0.9	0.111469208180188	-0.111469208180188	0.511469208180188	0.2175
1	0	0	0.500000000000000	0.3125

2. Let the weight w be the probability density function of the normal distribution

$$w_{m,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}, \; m, \sigma \in \mathbb{R}, \sigma > 0, x \in \mathbb{R}.$$

Then we have

$$m(a,b) = F(b) - F(a), \text{where } F \text{ is the cumulative distribution,}$$

$$\sigma(a,b) = m - \frac{\sigma}{\sqrt{2\pi}} \cdot \frac{e^{-\frac{(b-m)^2}{2\sigma^2}} - e^{-\frac{(a-m)^2}{2\sigma^2}}}{F(b) - F(a)},$$

$$\nu(x,t^*) = \begin{cases} \frac{1}{F(b)-F(a)} \left[\frac{-\sigma}{\sqrt{2\pi}} \left(e^{-\frac{(b-m)^2}{2\sigma^2}} - e^{-\frac{(x-m)^2}{2\sigma^2}} \right) + (m-x)\left(F(b)-F(x)\right) \right], \\ a \le x \le \sigma(a,b), \\ \frac{1}{F(b)-F(a)} \left[\frac{-\sigma}{\sqrt{2\pi}} \left(e^{-\frac{(a-m)^2}{2\sigma^2}} - e^{-\frac{(x-m)^2}{2\sigma^2}} \right) + (m-x)\left(F(a)-F(x)\right) \right], \\ \sigma(a,b) < x \le b. \end{cases}$$

If we consider the probability density function of the standard normal distribution, namely

$$w_{0,1}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},$$

inequality (2.4) on the interval $[0,1]$ becomes

$$\left(x - \frac{1-e^{-\frac{1}{2}}}{\phi(1)\sqrt{2\pi}} \right)\gamma + (\gamma - \Gamma)\tilde{\nu}(x)$$

$$\le f(x) - \frac{1}{\phi(1)} \int_a^b f(t) w_{0,1}(t)dt \le \left(x - \frac{1-e^{-\frac{1}{2}}}{\phi(1)\sqrt{2\pi}} \right)\Gamma + (\Gamma - \gamma)\tilde{\nu}(x),$$

where

$$\tilde{\nu}(x) = \begin{cases} \dfrac{1}{\phi(1)}\left[\dfrac{-1}{\sqrt{2\pi}}\left(e^{-\frac{1}{2}} - e^{-\frac{x^2}{2}}\right) - x\left(\phi(1) - \phi(x)\right)\right], 0 \le x \le \dfrac{1 - e^{-\frac{1}{2}}}{\phi(1)\sqrt{2\pi}}, \\ \dfrac{1}{\phi(1)}\left[\dfrac{-1}{\sqrt{2\pi}}\left(1 - e^{-\frac{x^2}{2}}\right) + x\phi(x)\right], \dfrac{1 - e^{-\frac{1}{2}}}{\phi(1)\sqrt{2\pi}} < x \le 1. \end{cases}$$

Here ϕ is Laplace's function and $\phi(1) = 0.3413$.

References

[1] Anastassiou, G.A., *Ostrowski type inequalities*, Proc. Amer. Math. Soc., **123**(1995), 3775–3781.

[2] Cheng, X.L., *Improvement of some Ostrowski-Grüss type inequalities*, Comput. Math. Appl., **42**(2001), 109-114.

[3] Dragomir, S.S., Wang, S., *An inequality of Ostrowski-Grüss type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules*, Computers Math. Appl., **33**(1997), 16-20.

[4] Gavrea, B., Gavrea, I., *Ostrowski type inequalities from a linear functional point of view*, J. Inequal. Pure Appl. Math., **1**(2000), article 11.

[5] Gonska, H., Raşa, I., Rusu, M.D., *Generalized Ostrowski-Grüss-type inequalities*, Results. Math. **62**(2012), 311-318.

[6] Matić, M., Pečarić, J., Ujević, N., *Improvement and further generalization of some inequalities of Ostrowski Grüss type*, Computers Math. Applic., **39**(2000) no. 3/4, 161-175.

[7] Milovanović, G.V, *On some integral inequalities*, Univ. Beograd. Publ. Elektotehn. Fak, Ser. Mat. Fiz. no. 498-541 (1975), 119-124.

[8] Milovanović, G.V, Pecarić, J.E, *On generalization of the inequality of A. Ostrowski and some related applications*, Univ. Beograd. Publ. Elektotehn. Fak, Ser. Mat. Fiz. no. 544-576 (1976), 155-158.

[9] Roumeliotis, J., *Product inequalities and applications in numerical integration*, Ostrowski type inequalities and applications in numerical integration (S.S. Dragomir and Th. M. Rassias, Eds.), Kluwer Academic, 2001, 373-416.

[10] Roumeliotis, J., *Improved weighted Ostrowski-Grüss type inequalities*, Inequality Theory and Applications, Nova Sci Publ, Hauppauge, NY, **3**(2003), 153-160.

[11] Semenov, E.M., Mitjagin, B.S., *Lack of interpolation of linear operators in spaces of smooth functions*, Math. USSR-Izv., **11**(1977), 1229-1266.

Ana Maria Acu
Lucian Blaga University of Sibiu, Department of Mathematics and Informatics
Str. Dr. I. Ratiu, No.5-7, RO-550012 Sibiu, Romania
e-mail: acuana77@yahoo.com

Heiner Gonska
University of Duisburg-Essen, Faculty of Mathematics
Forsthausweg 2, 47057 Duisburg, Germany
e-mail: heiner.gonska@uni-due.de