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Optimal cubic Lagrange interpolation: Extremal node systems with minimal Lebesgue constant

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Abstract. In the theory of interpolation of continuous functions by algebraic polynomials of degree at most $n-1 \geq 2$, the search for explicit analytic expressions of extremal node systems which lead to the minimal Lebesgue constant is still an intriguing topic in mathematics today [33]. The first non-trivial case $n-1=2$ (quadratic interpolation) has been completely resolved, even in two alternative fashions, see [25], [27]. In the present paper we proceed to completely resolve the cubic case $(n-1=3)$ of optimal polynomial Lagrange interpolation on the unit interval $[-1,1]$. We will provide two explicit analytic expressions for the uncountable infinitely many extremal node systems $x_1^* < x_2^* < x_3^* < x_4^*$ in $[-1,1]$ which all lead to the (known) minimal Lebesgue constant of cubic Lagrange interpolation on $[-1,1]$. The descriptions of the extremal node systems (which need not be zero-symmetric) resemble the solutions for the quadratic case and incorporate two intrinsic constants expressed by radicals, of which one constant looks particularly intricate. Our results encompass earlier related work provided in [17], [23], [24], [29], [30] and are guided by symbolic computation.

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1. Introduction

Lagrange polynomial interpolation on node systems $x_1 < x_2 < \cdots < x_{n-1} < x_n$ in some interval $\mathbf{I}$ is a classical and feasible method to approximate (continuous) functions on $\mathbf{I}$ by algebraic polynomials of maximal degree $n-1$, see e.g. [9], [18], [21],

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[26] or [28] for details. The goodness of this approximation method, as compared with the best possible approximation in Chebyshev's sense, is measured by means of the Lebesgue constants which can be viewed as operator norms or condition numbers, see also [33]. They depend, for a given n, solely on the chosen configuration of interpolation points, and Lebesgue's lemma suggests that we choose them in such a manner that the corresponding Lebesgue constant becomes least. Such extremal (or: optimal) node systems are in a sense the opposite to the equidistant interpolation points which may yield disastrous approximation results, see e.g. Runge's example in [18]. Although near-optimal node systems are known, and algorithms exist to numerically compute optimal (canonical) node systems, see [1] and [18], the search for explicit analytic formulas characterizing optimal node systems remains an intriguing and challenging topic in mathematics today. Here are three quotations in this regard, see also [6], [7], [13], [14], [17], [33]:

- *The nature of the optimal set X^* remains a mystery* [15, p. xlvii]
- *The problem of analytical description of the optimal matrix of nodes is considered by pure mathematicians as a great challenge.* [4]
- *To this day there is no explicit representation for the nth row of the optimal array, and in all likelihood there never will be.* [16]

It suffices to restrict the search for optimal node systems to the unit interval $\mathbf{I} = [-1, 1]$. The first non-trivial case $n - 1 = 2$ of quadratic Lagrange interpolation on $\mathbf{I}$ has been completely resolved: All optimal node systems $x_1^* < x_2^* < x_3^*$ in $\mathbf{I}$ can be explicitly described in two alternative fashions, see [25], [27]. In the present paper we proceed to completely resolve the cubic case $n-1 = 3$. Similarly to the quadratic case we will provide two alternative, but equivalent, explicit analytic (i.e., non-numeric) descriptions of all extremal node systems $x_1^* < x_2^* < x_3^* < x_4^*$ (with $-1 \leq x_1^*$ and $x_4^* \leq 1$) which, by their definition, lead to the (known) minimal Lebesgue constant of cubic Lagrange interpolation on $\mathbf{I}$. Amplifying Theorem 2 in [17] which states that optimal node systems are not unique, we will furthermore show that actually there exist, for each $n - 1 \geq 2$, uncountable infinitely many such node systems in $\mathbf{I}$. Generally, they are zero-asymmetrically distributed in $\mathbf{I}$; however, the subset of optimal zero-symmetric node systems deserves special attention: The investigation, in the cubic case, of optimal node systems $-x_4^* < -x_3^* < x_3^* < x_4^*$ produces two intrinsic constants (b and t, as defined below, of which b is particularly intricate) which are key in describing the general distribution of all optimal node systems $x_1^* < x_2^* < x_3^* < x_4^*$. These can be geometrically visualized by a 2D-region in the plane spanned by the two outer optimal nodes.

Historically, the first investigation into optimal node systems for the cubic case seems to be [29, p. 229], [30, Problem 6.43], where an analytic, but implicit (i.e., not by radicals) formula for the optimal zero-symmetric node systems as well as for the minimal Lebesgue constant has been provided. However, the reader of [29], [30] might be left with the impression that actually all optimal node systems had been determined in this way, since optimal zero-asymmetric ones are not mentioned there. A particular case of a zero-symmetric node system in $\mathbf{I}$ (for $n - 1 = 3$) is a canonical node distribution $-1 < -x_3 < x_3 < 1$. The unique node $x_3 = x_3^* = t$ which turns it to the optimal canonical node configuration has been explicitly determined in [23], [24], where also the minimal Lebesgue constant for the cubic Lagrange interpolation on $\mathbf{I}$ has been determined in an explicit form. Both quantities can be expressed, using Cardan's formula, by means of roots of certain cubic polynomials with integer coefficients.

The manuscript is organized as follows: After providing some necessary notations and definitions, we will consider, in an ascending order of generalization, first canonical node systems, then zero-symmetric node systems, and finally arbitrary (zero-symmetric and zero-asymmetric) node systems and will establish corresponding optimal node configurations. The proofs are postponed to section 6.

We hope that this paper, along with [25], [27] and our dedicated web repository `www.math.u-szeged.hu/~vajda/Leb/` will add to the dissemination of computer-aided optimal quadratic and cubic Lagrange interpolation and may stimulate research of the higher-degree polynomial cases, see Remark 7.5.

2. Definitions and basic theoretical background

Let $C(\mathbf{I})$ denote the Banach space of continuous real functions f on $\mathbf{I}$, equipped with the uniform norm:

$$||f|| = \max_{x\in\mathbf{I}} |f(x)|. \tag{2.1}$$

We wish to approximate f by an algebraic polynomial of degree at most $n-1$, where $n \geq 3$. An old idea, named after Lagrange, see [19], is to sample f at n distinct points in $\mathbf{I}$,

$$X_n : -1 \leq x_1 < x_2 < \ldots < x_{n-1} < x_n \leq 1, \tag{2.2}$$

and to construct an interpolating polynomial of degree at most $n-1$ as follows:

$$L_{n-1}(x) = L_{n-1}(f, X_n, x) = \sum_{j=1}^{n} f(x_j)\ell_{n-1,j}(X_n, x) \tag{2.3}$$

where

$$\ell_{n-1,j}(x) = \ell_{n-1,j}(X_n, x) = \prod_{i=1, i\neq j}^{n} \frac{x - x_i}{x_j - x_i}, \tag{2.4}$$

so that

$$\ell_{n-1,j}(X_n, x_i) = \delta_{j,i} \quad \text{(Kronecker delta)} \tag{2.5}$$

and hence

$$L_{n-1}(x_i) = f(x_i), \quad 1 \leq i \leq n \text{ (interpolatory condition)} \tag{2.6}$$

If $||f|| \leq 1$ then (2.3) implies that $|L_{n-1}(x)|$ can be estimated from above by

$$\sum_{j=1}^{n} |\ell_{n-1,j}(X_n, x)| = \lambda_n(x) = \lambda_n(X_n, x). \tag{2.7}$$

Definition 2.1. We call the x_i's in (2.2) the *interpolation nodes* and the grid X_n the *node system*, the unique L_{n-1} the *Lagrange interpolation polynomial*, the $\ell_{n-1,j}$'s (of exact degree $n-1$) the *Lagrange fundamental polynomials*, and λ_n the Lebesgue function (named after Lebesgue, see [34]).

Three properties of λ_n are summarized in the following statement, see [4], [17], [28, p. 95]:

Proposition 2.2.

i) *λ_n is a piecewise polynomial satisfying $\lambda_n(x) \geq 1$ with equality only if $x = x_i$ $(1 \leq i \leq n)$.*

ii) *λ_n has precisely one local maximum, which we will denote by $\mu_i = \mu_i(X_n)$, in each open sub-interval (x_i, x_{i+1}) of X_n $\quad (1 \leq i \leq n-1)$. The extremum point in (x_i, x_{i+1}), at which the maximum μ_i is attained, we will denote by $\xi_i = \xi_i(X_n)$ so that $\lambda_n(\xi_i) = \mu_i$ holds.*

iii) *λ_n is strictly decreasing and convex in $(-\infty, x_1)$ and strictly increasing and convex in (x_n, ∞).*

Definition 2.3. The largest value of λ_n in $\mathbf{I}$, denoted by $\Lambda_n = \Lambda_n(X_n)$, is called the *Lebesgue constant*:

$$\Lambda_n = \max_{x \in \mathbf{I}} \lambda_n(x). \tag{2.8}$$

The importance of Λ_n in interpolation theory stems from the following inequality ("Error Comparison Theorem") which can be viewed as a version of Lebesgue's lemma, but can also be proved directly [26, Theorem 4.1]:

$$||f - L_{n-1}|| \leq (1 + \Lambda_n)||f - P_{n-1}^*||, \tag{2.9}$$

where $f \in C(\mathbf{I})$, and P_{n-1}^* denotes the polynomial of best uniform approximation to f out of the linear space of all algebraic polynomials of degree at most $n-1$. Usually, P_{n-1}^* is much harder to determine than L_{n-1}, and of course there always holds $||f - P_{n-1}^*|| \leq ||f - L_{n-1}||$. The estimate (2.9), which is sharp for some f, tells us that a small Lebesgue constant implies that the approximation to f by the Lagrange interpolation polynomial is nearly as good as the best uniform approximation to f by means of P_{n-1}^*. Therefore, it is desirable to minimize Λ_n which can be achieved by a strategic placement of the interpolation nodes.

It is known [26, p. 100] that for each $n \geq 3$ there exists, in $\mathbf{I}$, an extremal (or optimal) node system $X_n = X_n^* : -1 \leq x_1^* < x_2^* < \cdots < x_{n-1}^* < x_n^* \leq 1$ such that there holds

$$\Lambda_n^* = \Lambda_n(X_n^*) \leq \Lambda_n = \Lambda_n(X_n) \text{ for all possible choices of node systems } X_n. \tag{2.10}$$

Definition 2.4. The Lebesgue constant Λ_n^* is called *minimal*.

It is furthermore known that for a given $n \geq 3$ an extremal node system is not unique, see [17, Theorem 2], and there in particular exists an extremal node system which includes the endpoints of $\mathbf{I}$ as interpolation nodes, see [26, p. 100]. Obviously, all extremal node systems, for a given n, generate the same minimal Lebesgue constant. We take the opportunity to amplify [17, Theorem 2] in the following way:

Theorem 2.5. For each $n \geq 3$ there exist uncountable infinitely many optimal node systems $X_n^* : x_1^* < x_2^* < \cdots < x_{n-1}^* < x_n^*$ in $\mathbf{I}$ which all yield (2.10).

Definition 2.6. The construction of $L_{n-1}(f, X_n^*, x)$ is called *optimal* Lagrange polynomial interpolation on $\mathbf{I}$ since it furnishes, for a given n, the minimal interpolation error in the sense of (2.9).

Definition 2.7. A node system which includes the endpoints of $\mathbf{I}$ as interpolation nodes (that is, $x_1 = -1$ and $x_n = 1$) is called a *canonical* node system (abbreviated CNS).

In answering a conjecture which goes back to [3], it was proved in [8] and in [14] that the following deep result holds true:

Proposition 2.8. If a Lebesgue function corresponding to a CNS X_n satisfies the so-called equioscillation property

$$\mu_1 = \mu_2 = \ldots = \mu_{n-2} = \mu_{n-1}, \tag{2.11}$$

then X_n is an extremal node system, i.e. $X_n = X_n^*$ with $\Lambda_n(X_n^*) = \Lambda_n^*$.

Thus, the fulfillment of (2.11) is a sufficient condition for a CNS to be extremal. Actually, it was additionally proved that a CNS which satisfies (2.11) is unique and *zero-symmetric*, i.e., $x_i^* = -x_{n-i+1}^*$ for $1 \le i \le n$.
To the best of our knowledge, currently the optimal CNS $X_n = X_n^* : -1 = -x_n^* < -x_{n-1}^* < \cdots < x_{n-1}^* < x_n^* = 1$ and the associated minimal Lebesgue constant $\Lambda_n^* = \Lambda(X_n^*)$ are explicitly known only if $n = 3$ ($X_3^* : -1 < 0 < 1$ and $\Lambda_3^* = 1.25$, see [3], [25], [27] or [33]), or if $n = 4$ (see next section).
In the next three sections we will focus on cubic interpolation on $\mathbf{I}$ by the Lagrange interpolation polynomial L_3, i.e., we set $n = 4$.

3. The optimal canonical node system and the minimal Lebesgue constant for cubic Lagrange interpolation

We consider first optimal cubic Lagrange interpolation on a CNS. It must be zero-symmetric if it has to be extremal, so that the goal is to find, analytically and explicitly, the unique node $x_3 = x_3^* = t$ in $X_4 : -1 = -x_4 < -x_3 < x_3 < x_4 = 1$ which turns X_4 into the unique extremal node system $X_4^* : -1 < -t < t < 1$ and to determine the associated minimal Lebesgue constant $\Lambda_4^* = \Lambda_4(X_4^*)$ for cubic Lagrange interpolation on $\mathbf{I}$. This problem, which is also addressed in [21, Example 2.5.3] and [22, Exercise 4.10], has been solved in [23], [24] by means of roots of certain cubic polynomials with integer coefficients: *The minimal Lebesgue constant, Λ_4^*, is the unique real root of the polynomial*

$$P_3^*(x) = -11 + 53x - 93x^2 + 43x^3. \tag{3.1}$$

This root can be explicitly expressed, with the aid of Cardan's formula, as

$$\begin{aligned} \Lambda_4^* &= \frac{1}{129}\left(93 + \sqrt[3]{125172 + 11868\sqrt{69}} + \sqrt[3]{125172 - 11868\sqrt{69}}\right) && (3.2) \\ &= 1.4229195732\ldots . && (3.3) \end{aligned}$$

The square of the (unique positive) optimal node $x_3^* = t$ is the unique real root of the polynomial

$$Q_3^*(x) = -1 + 2x + 17x^2 + 25x^3. \tag{3.4}$$

Hence t itself can be explicitly expressed, with the aid of Cardan's formula, as

$$t = \frac{1}{5\sqrt{3}}\sqrt{-17+\sqrt[3]{\frac{14699+1725\sqrt{69}}{2}}+\sqrt[3]{\frac{14699-1725\sqrt{69}}{2}}} \tag{3.5}$$

$$= 0.4177913013\ldots\ , \tag{3.6}$$

so that

$$X_4^* : -1 < -t < t < 1, \quad \textit{with}\, t \textit{ from } (3.5), \tag{3.7}$$

is the (unique and zero-symmetric) optimal CNS in **I**.

Furthermore, also the three extremum points $\xi_i = \xi_i^*$, with $\xi_1^* = -\xi_3^* < \xi_2^* = 0 < \xi_3^*$, of the cubic Lebesgue function $\lambda_4^*(x) = \lambda_4^*(X_4^*, x)$ are expressed in an analogous way in [24], but we only provide here the numerical value $\xi_3^* = 0.7331726239\ldots$, see Figure 1.

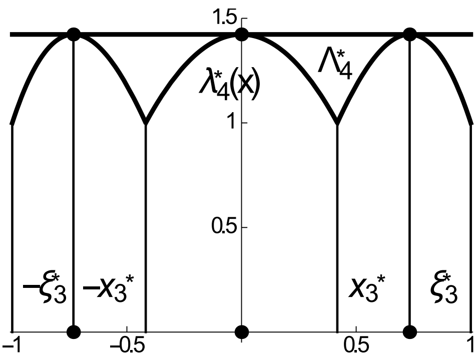

Figure 1

An analytical, but implicit, solution for t and Λ_4^* had been provided earlier in [29, p. 229], [30, Problem 6.43], see also [23]: t is the unique positive root of the polynomial

$$S_6(x) = Q_3^*(x^2) \tag{3.8}$$

and

$$\Lambda_4^* = \frac{1+t^2}{1-t^2}. \tag{3.9}$$

The polynomial (3.1) and the explicit expressions (3.2), (3.5) were not given there. Numerical values for t and/or Λ_4^* are given, for example, in [1], [2], [4], [11], [20], [21] and [33].

4. Optimal zero-symmetric node systems for cubic Lagrange interpolation

We consider next optimal cubic Lagrange interpolation on zero-symmetric node systems in **I**.

Problem 4.1. Find, analytically and explicitly, all optimal zero-symmetric node systems $X_4^* : -x_4^* < -x_3^* < x_3^* < x_4^*$ in $\mathbf{I}$!

Our solution to this problem will be expressed by means of the already deployed explicit constant t according to (3.5) and by means of a real parameter $\beta \in [1, b]$ where the right-hand endpoint $b > 1$ of that interval has still to be determined. We will first state our solution to Problem 4.1 and then turn to the task of determining the constant b.

Theorem 4.2. *All optimal zero-symmetric node systems for the cubic Lagrange interpolation on* $\mathbf{I}$ *are given by* $X_4^* : -x_4^* < -x_3^* < x_3^* < x_4^*$, *where*

$$-x_4^* = -\frac{1}{\beta},\ -x_3^* = -\frac{t}{\beta},\ x_3^* = \frac{t}{\beta},\ x_4^* = \frac{1}{\beta}. \tag{4.1}$$

Here, t *is given by* (3.5) *and* β *is an arbitrary number from the interval* $[1, b]$. *The right-hand endpoint* $b > 1$ *of that interval will be specified implicitly and numerically in Lemma* 4.3, *and explicitly in Lemma* 4.4.

Note that the choice $\beta = 1$ takes us back to the optimal CNS as given in (3.7). For the constant b in Theorem 4.2 we will now provide an implicit and an explicit analytical description. However, both of them are intricate.

Lemma 4.3. *The constant* b *in Theorem* 4.2 *can be expressed implicitly as the unique positive root of the following polynomial of degree 18 with integer coefficients:*

$$\begin{aligned} P_{18}(x) = {} & -121 + 220x - 1014x^2 + 1344x^3 + 3283x^4 - 5166x^5 + 4502x^6 \\ & + 15692x^7 - 84178x^8 + 7868x^9 + 210676x^{10} - 25694x^{11} - 310732x^{12} \\ & + 34154x^{13} + 255377x^{14} - 8450x^{15} - 124700x^{16} + 26875x^{18}. \end{aligned} \tag{4.2}$$

The numerical evaluation of this root by computer algebra systems yields

$$b = 1.0433133411\ldots\ . \tag{4.3}$$

However, the computer algebra systems we have checked do not render the real root b of P_{18} by means of radicals. But such an explicit representation of b is possible:

Lemma 4.4. *The constant* b *in Theorem* 4.2 *can be expressed explicitly in terms of radicals as follows:*

$$\begin{aligned} b = b(t) = {} & \left(\frac{t+t^3}{2+2t} - \sqrt{\left(\frac{-1}{27}\right)(1+(-1+t)t)^3 + \frac{(t+t^3)^2}{4(1+t)^2}} \right)^{1/3} \\ & + \left(\frac{t+t^3}{2+2t} + \sqrt{\left(\frac{-1}{27}\right)(1+(-1+t)t)^3 + \frac{(t+t^3)^2}{4(1+t)^2}} \right)^{1/3}, \end{aligned} \tag{4.4}$$

where t *denotes the constant in* (3.5), *so that* b *still depends on* t. *Upon inserting the value* (3.5) *for* t, *one obtains in place of* (4.4) *a non-parametric explicit expression for* *b which, however, is quite intricate:*

$$
\begin{aligned}
b = {} & \Bigg(\Bigg(58 + \bigg(\frac{1}{2}\Big(14699 - 1725\sqrt{69}\Big)\bigg)^{1/3} + \bigg(\frac{1}{2}\Big(14699 + 1725\sqrt{69}\Big)\bigg)^{1/3}\Bigg) \qquad (4.5) \\
& \Big/\Big(150+750\sqrt{\Big(6\Big/\Big(-34 + 2^{2/3}\left(14699 - 1725\sqrt{69}\right)^{1/3} + 2^{2/3}\left(14699+1725\sqrt{69}\right)^{1/3}\Big)\Big)}\Big) \\
& - \sqrt{\Bigg(}\Big(\Big(58\ 2^{1/3} + \left(14699 - 1725\sqrt{69}\right)^{1/3} + \left(14699 + 1725\sqrt{69}\right)^{1/3}\Big)^2 \Big/ \\
& (3750\ 2^{2/3}\Big(\sqrt{6} + 30\Big/\Big(\sqrt{\Big(}-34 + 2^{2/3}\left(14699 - 1725\sqrt{69}\right)^{1/3} + \\
& \quad 2^{2/3}\left(14699 + 1725\sqrt{69}\right)^{1/3}\Big)\Big)\Big)^2\Big) - \\
& \tfrac{1}{91125000}\Big(116 + 2^{2/3}\left(14699 - 1725\sqrt{69}\right)^{1/3} + 2^{2/3}\left(14699 + 1725\sqrt{69}\right)^{1/3} - \\
& 5\sqrt{\Big(}6\Big(-34 + 2^{2/3}\left(14699 - 1725\sqrt{69}\right)^{1/3} + \\
& \quad 2^{2/3}\left(14699 + 1725\sqrt{69}\right)^{1/3}\Big)\Big)\Big)^3\Big)\Big)^{1/3} + \\
& \Big(\Big(58 + \left(\tfrac{1}{2}\left(14699 - 1725\sqrt{69}\right)\right)^{1/3} + \left(\tfrac{1}{2}\left(14699 + 1725\sqrt{69}\right)\right)^{1/3}\Big)\Big/ \\
& (150+750\sqrt{\Big(6\Big/\Big(-34 + 2^{2/3}\left(14699 - 1725\sqrt{69}\right)^{1/3} + 2^{2/3}\left(14699+1725\sqrt{69}\right)^{1/3}\Big)\Big)}\Big) + \\
& \sqrt{\Bigg(}\Big(\Big(58\ 2^{1/3} + \left(14699 - 1725\sqrt{69}\right)^{1/3} + \left(14699 + 1725\sqrt{69}\right)^{1/3}\Big)^2 \Big/ \\
& (3750\ 2^{2/3}\Big(\sqrt{6} + 30\Big/\Big(\sqrt{\Big(}-34 + 2^{2/3}\left(14699 - 1725\sqrt{69}\right)^{1/3} + \\
& \quad 2^{2/3}\left(14699 + 1725\sqrt{69}\right)^{1/3}\Big)\Big)\Big)^2\Big) - \\
& \tfrac{1}{91125000}\Big(116 + 2^{2/3}\left(14699 - 1725\sqrt{69}\right)^{1/3} + 2^{2/3}\left(14699 + 1725\sqrt{69}\right)^{1/3} - \\
& 5\sqrt{\Big(}6\Big(-34 + 2^{2/3}\left(14699 - 1725\sqrt{69}\right)^{1/3} + \ 2^{2/3}\left(14699 + 1725\sqrt{69}\right)^{1/3}\Big)\Big)\Big)^3\Big)\Big)^{1/3}
\end{aligned}
$$

The first few decimal digits of b read, more precisely as in (4.3), *as follows:*

$$b = 1.04331334111159291363108683195449367479847946940399 5\ldots . \qquad (4.6)$$

To give an example, we consider the shortest sub-interval $[-x_4^*, x_4^*]$ of $\mathbf{I}$ which allows optimal cubic Lagrange interpolation. According to Theorem 4.2 we have to choose $\beta = b$, so that we get, as a kind of counterpart to the CNS,

Example 4.5. The configuration

$$-x_4^* = -\frac{1}{b},\ -x_3^* = -\frac{t}{b},\ x_3^* = \frac{t}{b},\ x_4^* = \frac{1}{b} \qquad (4.7)$$

is the unique optimal zero-symmetric node system in $\mathbf{I}$ with the shortest interval length $x_4^* - (-x_4^*) = \frac{2}{b}$. The corresponding numerical values are:

$$-0.9584848200\ldots < -0.4004466202\ldots < 0.4004466202\ldots < 0.9584848200\ldots \qquad (4.8)$$

and

$$\frac{2}{b} = 1.9169696400\ldots\,. \tag{4.9}$$

We note that (4.7) is also unique in having the property that the corresponding Lebesgue function $\lambda_4(x)$ equioscillates on $\mathbf{I}$ most, that is, five $(= n + 1)$ times, see Figure 2.

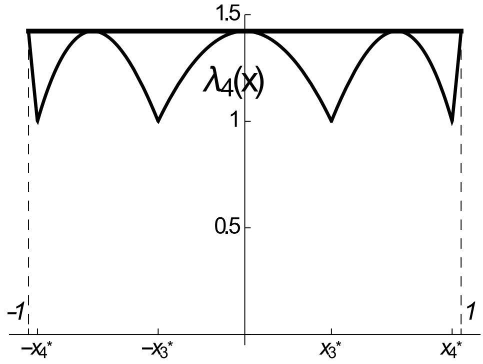

Figure 2

It is interesting to remark that the analogous configuration for the quadratic case $(n = 3)$ is $-\frac{2\sqrt{2}}{3} < 0 < \frac{2\sqrt{2}}{3}$, and the corresponding Lebesgue function $\lambda_3(x)$ equioscillates on $\mathbf{I}$ most, that is, four $(= n + 1)$ times. The extremum points $-1 < -\frac{\sqrt{2}}{3} < \frac{\sqrt{2}}{3} < 1$ in $\mathbf{I}$ of that particular $\lambda_3(x)$ were considered by Bernstein in [3] and motivated him to state his famous equioscillation conjecture. We therefore call (4.7) the Bernstein-type node system (BNS).

An analytical, but implicit, solution similar to (4.1) had been provided earlier in [29, p. 229] (misprinted) and [30, Problem 6.43]: The optimal zero-symmetric nodes in $\mathbf{I}$ are $-a < -at < at < 1$, where $a \in [a_0, 1]$ and t is, as before, the unique positive root of the polynomial $S_6(x) = Q_3^*(x^2)$ and a_0 is the unique positive root of the (parameterized) polynomial

$$S_3(x) = -(t + 1) + (t^3 + 1)x^2 + (t^3 + t)x^3. \tag{4.10}$$

However, Tureckii did not express t and a_0 by radicals, and in [29, p. 229] the term $(t^3+1)x^2$ of $S_3(x)$ was misprinted as $(t^3+1)x$. We observe that the left-hand endpoint a_0 of the interval $[a_0, 1]$ coincides with $\frac{1}{b}$, where the constant b is given in Lemma 4.3 and 4.4. After all, we point out that optimal zero-asymmetric node systems are not mentioned in [29], [30], [31].

5. Optimal arbitrary node systems for cubic Lagrange interpolation

Finally we consider optimal cubic Lagrange interpolation on arbitrary (zero-symmetric and zero-asymmetric) node systems in $\mathbf{I}$.

Problem 5.1. Find, analytically and explicitly, all optimal node systems $X_4^* : x_1^* < x_2^* < x_3^* < x_4^*$ in $\mathbf{I}$!

Our solution to this problem is twofold: Building on the already deployed constants b in (4.4) and t in (3.5), the first solution describes all four optimal nodes by means of two parameters, $\alpha \in [-b, -1]$ and $\beta \in [1, b]$, whereas the second equivalent solution gives only the selection range for the outer optimal nodes x_1^* and x_4^* and expresses the inner optimal nodes x_2^* and x_3^* as functions of them. The first solution is similar to the one given in [25], whereas the second solution is similar to the one given in [27], for the quadratic case.

Theorem 5.2. *All optimal node systems for the cubic Lagrange interpolation on* $\mathbf{I}$ *are given by* $X_4^* : x_1^* < x_2^* < x_3^* < x_4^*$, *where*

$$x_1^* = \frac{-2-\alpha-\beta}{-\alpha+\beta},\ x_2^* = \frac{-2t-\alpha-\beta}{-\alpha+\beta},\ x_3^* = \frac{2t-\alpha-\beta}{-\alpha+\beta},\ x_4^* = \frac{2-\alpha-\beta}{-\alpha+\beta}, \tag{5.1}$$

in which $\alpha \in [-b, -1]$ *and* $\beta \in [1, b]$ *are arbitrary numbers, and the constants* b *and* t *are defined in* (4.4) *respectively* (3.5).

Note that the choice $\alpha = -\beta$ takes us back to the zero-symmetric case (4.1). To illustrate Theorem 5.2, we give an analytic example:

Example 5.3. Choose $\alpha = -1.04 \in [-b, -1]$ and $\beta = 1.03 \in [1, b]$, say. According to (5.1) we get

$$X_4^* : x_1^* = -\frac{199}{207} < x_2^* = \frac{100}{207}\left(\frac{1}{100} - 2t\right) < x_3^* = \frac{100}{207}\left(\frac{1}{100} + 2t\right) < x_4^* = \frac{201}{207} \tag{5.2}$$

which is an optimal (zero-asymmetric) node system in $\mathbf{I}$. A little computation reveals that indeed

$$\max_{x\in\mathbf{I}} \lambda_4(X_4^*, x) = \Lambda_4^* = \frac{1}{129}\left(93 + \sqrt[3]{125172 + 11868\sqrt{69}} + \sqrt[3]{125172 - 11868\sqrt{69}}\right)$$

holds, e.g., $\lambda_4(X_4^*, \frac{1}{207}) = \Lambda_4^*$ and $\lambda_4'(X_4^*, x)|_{x=\frac{1}{207}} = 0$. The numerical values in (5.2) read, after inserting t from (3.6):

$$\begin{aligned} x_1^* = -0.9613526570\cdots < x_2^* = -0.3988321752\cdots <\\ < x_3^* = 0.4084940109\cdots < x_4^* = 0.9710144927\ldots. \end{aligned} \tag{5.3}$$

The corresponding Lebesgue function $\lambda_4(x) = \lambda_4(X_4^*, x)$ is depicted in Figure 3.

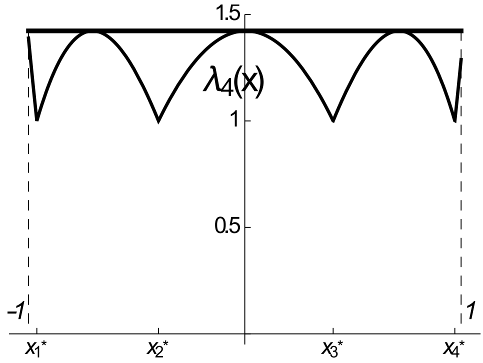

Figure 3

Our second solution to Problem 5.1 likewise builds on the deployed constants b in (4.4) and t in (3.5). The two parameters α and β and their selection ranges are now superseded by selection ranges for the two outer optimal nodes which, once fixed, uniquely determine the corresponding two inner optimal nodes.

Theorem 5.4. *(Alternative solution to Problem* 5.1*) All optimal node systems for the cubic Lagrange interpolation on* $\mathbf{I}$ *are given by* $X_4^* : x_1^* < x_2^* < x_3^* < x_4^*$*, where either*

$$-1 \leq x_1^* \leq -\frac{1}{b} = -0.9584848200\ldots \text{ and } \left(\frac{b-1}{b+1}\right)x_1^* + \frac{2}{b+1} \leq x_4^* \leq 1 \tag{5.4}$$

or

$$-\frac{1}{b} < x_1^* \leq \frac{b-3}{b+1} = -0.9576047978\ldots \text{ and } \left(\frac{b+1}{b-1}\right)x_1^* + \frac{2}{b-1} \leq x_4^* \leq 1 \tag{5.5}$$

with

$$x_2^* = \left(\frac{1+t}{2}\right)x_1^* + \left(\frac{1-t}{2}\right)x_4^*, \tag{5.6}$$

and with

$$x_3^* = \left(\frac{1-t}{2}\right)x_1^* + \left(\frac{1+t}{2}\right)x_4^*. \tag{5.7}$$

When the outer optimal nodes x_1^* and x_4^* vary in their respective ranges (5.4) and (5.5), then also the ranges of the inner optimal nodes x_2^* and x_3^* are exhausted. That ranges are substantiated in the next statement.

Theorem 5.5. *The two inner optimal nodes* x_2^* *and* x_3^* *in Theorem* 5.4 *may vary within the ranges*

$$\frac{-1}{b+1}(2t+b-1) = -0.4301327291\ldots \leq x_2^* \leq \frac{-1}{b+1}(2t-b+1) = -0.3877375269\ldots \tag{5.8}$$

and

$$\frac{1}{b+1}(2t-b+1) = 0.3877375269\cdots \leq x_3^* \leq \frac{1}{b+1}(2t+b-1) = 0.4301327291\ldots \tag{5.9}$$

To illustrate Theorems 5.4 and 5.5 we give a numerical example which can be turned into an analytical one:

Example 5.6. Choose $x_1^* = -0.99$, say. Then one gets, according to (5.4), $0.9578167738\cdots \le x_4^* \le 1$. Choose now $x_4^* = 0.96$, say. This implies, according to (5.6) and (5.7), $x_2^* = -0.4223465188\ldots$ and $x_3^* = 0.3923465188\ldots$. Hence these four numerical values constitute a zero-asymmetric optimal node system for the cubic Lagrange interpolation on **I**. The associated Lebesgue function is depicted in Figure 4. It is readily verified that this specific optimal node system corresponds to (5.1) if we insert there $\alpha = -\frac{197}{195} = -1.0102564102\ldots$ and $\beta = \frac{203}{195} = 1.0410256410\ldots$. In particular, the two inner nodes can thus be expressed explicitly as $x_2^* = \frac{1}{200}(-195t-3)$ and $x_3^* = \frac{1}{200}(195t - 3)$.

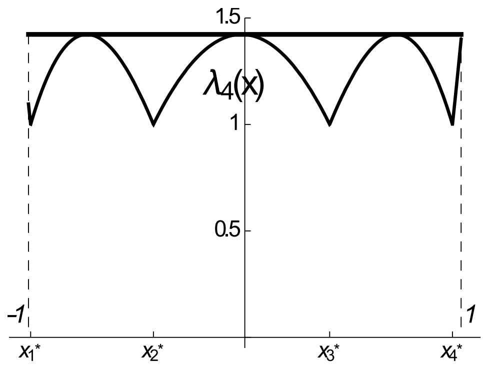

Figure 4

The somewhat curious description of the ranges of the two outer optimal nodes x_1^* and x_4^* in Theorem 5.4 can be given a geometric visualization by a 2D-quadrilateral whose one 1D-diagonal represents the zero-symmetric node systems (ZSNS) (4.1). The two endpoints of that diagonal are represented by the CNS (3.7) and by the BNS (4.7). The quadrilateral itself is not quite a square, rather two of its sides (the lower one and the right one) can be expressed as line segments of linear functions g (flat) and h (steep) of the variable x_1^*, see (5.4) and (5.5):

$$g(x_1^*) = \left(\frac{b-1}{b+1}\right) x_1^* + \frac{2}{b+1} = 0.0211976010\ldots x_1^* + 0.9788023989\ldots \tag{5.10}$$

and

$$h(x_1^*) = \left(\frac{b+1}{b-1}\right) x_1^* + \frac{2}{b-1} = 47.1751494729\ldots x_1^* + 46.1751494729\ldots. \tag{5.11}$$

The graphical representation is given in Figure 5.

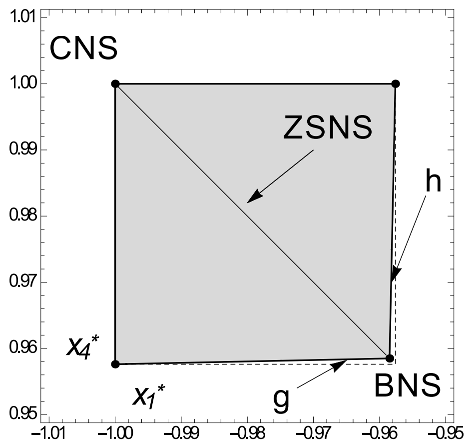

Figure 5

We are not aware of any published studies that provide explicit analytical solutions to the problem of determining all optimal node systems in the frame of optimal polynomial Lagrange interpolation on **I** with polynomials of degree ≥ 4 (i.e., $n \geq 5$), notwithstanding that this large project is vibrant, see [5], and also Remark 7.5 below. The results for the polynomial degree $= 3$ as obtained in [23], [24], and [30, Problem 6.43] do not cover zero-asymmetric node systems. Thus the present paper seems to be the first one where, for the polynomial degree $= 3$, *all* optimal node systems in **I** have been determined, explicitly and analytically. For the polynomial degree $= 2$ this has been achieved in [25] and in [27].

6. Proofs

6.1. Proof of Theorem 2.5

Proof. Let $\lambda_n^*(x) = \lambda_n^*(X_n^*, x)$ denote the Lebesgue function corresponding to the optimal CNS $X_n^* : -1 = -x_n^* = x_1^* < -x_{n-1}^* = x_2^* < \cdots < x_{n-1}^* < x_n^* = 1$ in **I**. We know that $\lambda_n^*(\pm 1) = 1$ and that $\lambda_n^*(x)$ is strictly decreasing resp. increasing in $(-\infty, -1)$ and $(1, \infty)$, see Proposition 2.2. As $\Lambda_n^* > 1$, there exist unique values $x = c_n < -1$ resp. $x = b_n > 1$ with the property $\lambda_n^*(c_n) = \lambda_n^*(b_n) = \Lambda_n^*$. Hence, $\max_{x \in [c_n, b_n]} \lambda_n^*(x) = \Lambda_n^*$ and in particular $\max_{x \in [\alpha, \beta]} \lambda_n^*(x) = \Lambda_n^*$ for any subinterval $[\alpha, \beta]$ of $[c_n, b_n]$ which covers **I**. Choose now an arbitrary $\alpha \in [c_n, -1]$ and an arbitrary $\beta \in [1, b_n]$ and consider the linear transformation

$$S(x) = \frac{1}{-\alpha + \beta}(2x - \alpha - \beta) \tag{6.1}$$

which maps $[\alpha, \beta]$ onto **I**, because $S(\alpha) = -1$ and $S(\beta) = 1$. In particular, the nodes of the optimal CNS X_n^* (which is contained in $[\alpha, \beta]$) will be mapped as follows:

$$\begin{aligned} S(-1) = y_1^* &= \frac{1}{-\alpha+\beta}(-2-\alpha-\beta) \\ S(-x_{n-1}^*) = y_2^* &= \frac{1}{-\alpha+\beta}(-2x_{n-1}^*-\alpha-\beta) \\ &\cdots \\ S(x_{n-1}^*) = y_{n-1}^* &= \frac{1}{-\alpha+\beta}(2x_{n-1}^*-\alpha-\beta) \\ S(1) = y_n^* &= \frac{1}{-\alpha+\beta}(2-\alpha-\beta). \end{aligned} \tag{6.2}$$

As $S(x)$ has the positive slope $\frac{2}{-\alpha+\beta}$, these images are ordered ascendingly in $\mathbf{I}$:

$$Y_n^* = Y_{n,\alpha,\beta}^* : y_1^* < y_2^* \cdots < y_{n-1}^* < y_n^*. \tag{6.3}$$

We proceed to show that Y_n^* is an extremal node system (with minimal Lebesgue constant) in $\mathbf{I}$. To this end, we observe that, for $y \in \mathbf{I}$ and $x \in [\alpha,\beta]$, we have

$$\begin{aligned} \lambda_n(Y_n^*, y) &= \sum_{j=1}^{n}\prod_{\substack{i=1\\i\neq j}}^{n}\frac{|y-y_i^*|}{|y_j^*-y_i^*|} = \sum_{j=1}^{n}\prod_{\substack{i=1\\i\neq j}}^{n}\frac{|S(x)-S(x_i^*)|}{|S(x_j^*)-S(x_i^*)|} = \\ &= \sum_{j=1}^{n}\prod_{\substack{i=1\\i\neq j}}^{n}\frac{|\frac{1}{-\alpha+\beta}(2x-\alpha-\beta)-\frac{1}{-\alpha+\beta}(2x_i^*-\alpha-\beta)|}{|\frac{1}{-\alpha+\beta}(2x_j^*-\alpha-\beta)-\frac{1}{-\alpha+\beta}(2x_i^*-\alpha-\beta)|} = \sum_{j=1}^{n}\prod_{\substack{i=1\\i\neq j}}^{n}\frac{|x-x_i^*|}{|x_j^*-x_i^*|} = \\ &= \lambda_n^*(x), \end{aligned} \tag{6.4}$$

and hence

$$\max_{y\in\mathbf{I}}\lambda_n(Y_n^*, y) = \max_{x\in[\alpha,\beta]}\lambda_n^*(x) = \Lambda_n^*,$$

see also [21, Theorem 2.5.3] and [6, Problem 1, p. 22]. Thus, $Y_n^* = Y_{n,\alpha,\beta}^*$ is indeed an optimal node system in $\mathbf{I}$, and since α and β were chosen arbitrarily, there exist uncountable infinitely many such Y_n^*'s. □

6.2. Proof of Theorem 4.2

Proof. The statement is a corollary to Theorem 5.2: insert there $\alpha = -\beta$. □

6.3. Proof of Theorem 5.2

Proof. The Lebesgue function $\lambda_4^*(x) = \lambda_4^*(X_4^*, x)$ which corresponds to the optimal CNS X_4^* in (3.7) is readily found to be given by

$$\lambda_4^*(x) = |\ell_{31}(x)| + |\ell_{32}(x)| + |\ell_{33}(x)| + |\ell_{34}(x)| = \tag{6.5}$$

$$\begin{aligned} = \quad & |(-t-x)(-1+x)(-t+x)/(2(-1-t)(-1+t))| + \\ & |(-1+x)(1+x)(-t+x)/(2(-1-t)(1-t)t)| + \\ & |(-1+x)(1+x)(t+x)/(2(-1+t)t(1+t))| + \\ & |(1+x)(-t+x)(t+x)/(2(1-t)(1+t))|, \end{aligned}$$

where t is defined in (3.5).

For $x \in (1, \infty)$, $\lambda_4^*(x)$ is strictly increasing and is representable there as

$$\lambda_4^*(x) = -\ell_{31}(x) + \ell_{32}(x) - \ell_{33}(x) + \ell_{34}(x) = \frac{x(1 - t + t^2 - x^2)}{(-1 + t)t}. \tag{6.6}$$

Let $x = b > 1$ denote the unique point on the x-axis where $\lambda_4^*(x)$ intercepts with the constant function $f(x) = \Lambda_4^*$, see (3.2). A numerical solution of the equation $\lambda_4^*(x) - \Lambda_4^* = 0$ is the value b given in (4.3). The expression of b as explicit analytical solution we will provide below (see the proof of Lemma 4.4). Similarly, for $x \in (-\infty, -1)$, $\lambda_4^*(x)$ is strictly decreasing and is representable there as

$$\lambda_4^*(x) = \ell_{31}(x) - \ell_{32}(x) + \ell_{33}(x) - \ell_{34}(x) = \frac{x(-1 + t - t^2 + x^2)}{(-1 + t)t}, \tag{6.7}$$

so that $x = -b < -1$ denotes the unique point on the x-axis where $\lambda_4^*(x)$ intercepts with the constant function $f(x) = \Lambda_4^*$. Thus we have, on the interval $[-b, b]$ as well as on any subinterval $[\alpha, \beta]$ thereof which covers $\mathbf{I}$, $\max_{x\in[-b,b]} \lambda_4^*(x) = \Lambda_4^* = \max_{x\in\mathbf{I}} \lambda_4^*(x)$. We now apply the linear transformation (6.1) to X_4^* in (3.7) with arbitrary $\alpha \in [-b, -1]$ and arbitrary $\beta \in [1, b]$, and thus get (5.1), after renaming y_i^* to x_i^*, $i = 1, 2, 3, 4$ (see the proof of Theorem 2.5). In this way we generate uncountable infinitely many extremal node systems $x_1^* < x_2^* < x_3^* < x_4^*$ in $\mathbf{I}$, and in fact we so obtain *all* extremal node systems in $\mathbf{I}$.
For let $X_4^0 : x_1^0 < x_2^0 < x_3^0 < x_4^0$ be an arbitrary extremal node system in $\mathbf{I}$. The linear transformation $T(x) = \frac{2x - x_1^0 - x_4^0}{-x_1^0 + x_4^0}$ maps X_4^0 onto $\mathbf{I}$ since $T(x_1^0) = -1$ and $T(x_4^0) = 1$. Since its slope $\frac{2}{-x_1^0 + x_4^0}$ is positive, the mapped values are ordered ascendingly, that is $X_{4,T}^0 : -1 = T(x_1^0) < T(x_2^0) < T(x_3^0) < T(x_4^0) = 1$, and hence $X_{4,T}^0$ is a CNS in $\mathbf{I}$. The Lebesgue constant corresponding to X_4^0 (which equals Λ_4^* as given in (3.2)) is identical with the one corresponding to $X_{4,T}^0$, see the proof of Theorem 2.5 or [21, Theorem 2.5.3] or [6, Problem 1, p. 22]. But this implies that $X_{4,T}^0$ is necessarily the unique CNS X_4^* on $\mathbf{I}$, as given in (3.7). In particular we thus have $T(x_2^0) = -t$ and $T(x_3^0) = t$, and this implies, by the definition of $T(x)$, $x_2^0 = \left(\frac{1+t}{2}\right) x_1^0 + \left(\frac{1-t}{2}\right) x_4^0$ and $x_3^0 = \left(\frac{1-t}{2}\right) x_1^0 + \left(\frac{1+t}{2}\right) x_4^0$, where t is from (3.5).
With this information at hand we proceed as follows: The linear transformation S from (6.1) with $\alpha = \alpha^0 = \frac{-2 - x_4^0 - x_1^0}{x_4^0 - x_1^0}$ and $\beta = \beta^0 = \frac{2 - x_4^0 - x_1^0}{x_4^0 - x_1^0}$ maps the optimal CNS X_4^* onto $x_1^0 < x_2^0 < x_3^0 < x_4^0$. Indeed, $S(-1) = x_1^0$ and $S(1) = x_4^0$ as is immediately verified. Furthermore we get $S(-t) = \left(\frac{1+t}{2}\right) x_1^0 + \left(\frac{1-t}{2}\right) x_4^0$ and $S(t) = \left(\frac{1-t}{2}\right) x_1^0 + \left(\frac{1+t}{2}\right) x_4^0$, and these values are identical with x_2^0 and x_3^0 as shown above.
It remains to show that $\alpha^0 \in [-b, -1]$ and $\beta^0 \in [1, b]$. We will prove it by symbolic computation employing the built-in language symbols **InterpolatingPolynomial** and **Resolve** in Mathematica® as well as Mathematica®-specific notation. Furthermore, we will use the fact that $\lambda_n(\pm 1) \leq \Lambda_n^*$, see Proposition 2.2. Assume that the variable LF contains the Lebesgue function corresponding to the node system in (5.1). This can be defined e.g. in Mathematica® as follows:

LF=
Abs[InterpolatingPolynomial[
$\{\{x1[\alpha,\beta],1\},\{x2[\alpha,\beta],0\},\{x3[\alpha,\beta],0\},\{x4[\alpha,\beta],0\}\},x]]$ +
Abs[InterpolatingPolynomial[
$\{\{x1[\alpha,\beta],0\},\{x2[\alpha,\beta],1\},\{x3[\alpha,\beta],0\},\{x4[\alpha,\beta],0\}\},x]]$ +
Abs[InterpolatingPolynomial[
$\{\{x1[\alpha,\beta],0\},\{x2[\alpha,\beta],0\},\{x3[\alpha,\beta],1\},\{x4[\alpha,\beta],0\}\},x]]$ +
Abs[InterpolatingPolynomial[
$\{\{x1[\alpha,\beta],0\},\{x2[\alpha,\beta],0\},\{x3[\alpha,\beta],0\},\{x4[\alpha,\beta],1\}\},x]]$.

Denote the the specific parameter values α^0 and β^0 by

$$\alpha 0 = \frac{-2 - x4 - x1}{x4 - x1} \quad \beta 0 = \frac{2 - x4 - x1}{x4 - x1}. \tag{6.8}$$

Then one gets, with $\Lambda = \Lambda_4^*$ as given in (3.2) and (3.9) and b as given in (4.2) and (4.5),
[in:]

$$\begin{aligned}&\text{Resolve[ForAll}[\{x1, x4\}, (-1 \le x1 < x4 \le 1 \wedge\\&(\text{LF}/.x \to -1) \le \Lambda \wedge (\text{LF}/.x \to 1) \le \Lambda/.\{\alpha \to \alpha 0, \beta \to \beta 0\})\\&\Rightarrow\\&(-b \le \alpha 0 \le -1 \ \wedge\ 1 \le \beta 0 \le b)], \text{Reals}]\end{aligned} \tag{6.9}$$

[out:] True. □

6.4. Proof of Lemma 4.3

Proof. We want to determine the point $x = b > 1$ on the x-axis where $\lambda_4^*(x)$ intercepts with the constant function $f(x) = \Lambda_4^*$ (see the proof of Theorem 5.2). To this end, we employ a computer algebra system. The built-in language symbol **RootReduce** of Mathematica® immediately gives, in view of (3.1), (3.4) and (6.6), the claimed polynomial of degree 18:

[in:]

$$\begin{aligned}&\text{RootReduce}[x/.\text{Solve}[x(1 - t + t^2 - x^2)/((-1 + t)t) ==\\&\text{Root}[-11 + 53\#1 - 93\#1^2 + 43\#1^3\&, 1]\\&/.t \to \text{Root}[-1 + 2\#1^2 + 17\#1^4 + 25\#1^6\&, 2], x, \text{Reals}]]\end{aligned} \tag{6.10}$$

[out:]

$$\begin{aligned}&\{\text{Root}[-121 + 220\#1 - 1014\#1^2 + 1344\#1^3 + 3283\#1^4 - 5166\#1^5 +\\&4502\#1^6 + 15692\#1^7 - 84178\#1^8 + 7868\#1^9 + 210676\#1^{10} -\\&25694\#1^{11} - 310732\#1^{12} + 34154\#1^{13} + 255377\#1^{14} -\\&8450\#1^{15} - 124700\#1^{16} + 26875\#1^{18}\&, 2]\}.\end{aligned} \tag{6.11}$$

This polynomial P_{18} can also be deduced in a reverse way: If one employs in Mathematica® the built-in language symbol **FullSimplify** to the explicit expression (4.5), then one gets (6.11). □

6.5. Proof of Lemma 4.4

Proof. The equation

$$x(1-t+t^2-x^2)/((-1+t)t) = \frac{1+t^2}{1-t^2} \tag{6.12}$$

see (3.9) and (6.6), amounts to the following cubic algebraic equation in x:

$$(t+t^3)+(1+t^3)x+(-1-t)x^3 = 0. \tag{6.13}$$

Solving (6.13) with the aid of Cardan's formula yields the (parametric) solution $x = b = b(t)$ as given in (4.4). Inserting then into (4.4) the value for the constant t according to (3.5) and simplifying eventually gives (4.5). The said algebraic manipulations can be executed by pencil and paper, but it is more convenient to guide them by a computer algebra system. □

6.6. Proof of Theorem 5.4

Proof. To prove (5.4) and (5.5), we use quantifier elimination to eliminate the existentially quantified variables α and β from the parametric representation of $x_1^* = x_1^*(\alpha,\beta)$ and $x_4^* = x_4^*(\alpha,\beta)$, taking into account the range of α and β, see Theorem 5.2. This elimination can be executed by means of the built-in language symbol **Resolve** in Mathematica®:

[in :]

$$\text{Resolve[Exists[}\{\alpha,\beta\}, \big(b>1 \wedge -b \le \alpha \le -1 \wedge 1 \le \beta \le b \wedge$$
$$x_1^*(\beta-\alpha) == -2-\alpha-\beta \wedge x_4^*(\beta-\alpha) == 2-\alpha-\beta\big)], \{x_1^*, x_4^*\}, \text{Reals}]$$

[out :]

$$b>1 \wedge \Big((-1 \le x_1^* \le -\frac{1}{b} \wedge \frac{2-x_1^*+bx_1^*}{1+b} \le x_4^* \le 1) \vee$$
$$(-\frac{1}{b} < x_1^* \le \frac{-3+b}{1+b} \wedge \frac{2+x_1^*+bx_1^*}{-1+b} \le x_4^* \le 1)\Big), \tag{6.14}$$

which coincides with (5.4) and (5.5). To prove (5.6) and (5.7), consider the parametric representation of the optimal nodes $x_i^* = x_i^*(\alpha,\beta)$, $i=1,2,3,4$ in (5.1). Computing now $\left(\frac{1+t}{2}\right)x_1^*(\alpha,\beta) + \left(\frac{1-t}{2}\right)x_4^*(\alpha,\beta)$ and $\left(\frac{1-t}{2}\right)x_1^*(\alpha,\beta) + \left(\frac{1+t}{2}\right)x_4^*(\alpha,\beta)$ gives immediately (5.6) and (5.7). □

6.7. Proof of Theorem 5.5

Proof. According to (5.7), the node $x_3^* = x_3^*(x_1^*, x_4^*) = \left(\frac{1-t}{2}\right)x_1^* + \left(\frac{1+t}{2}\right)x_4^*$ is a bivariate polynomial which is linear in both variables x_1^* and x_4^*. Hence $x_3^*(x_1^*, x_4^*)$ will attain its extreme values on the boundary of the ranges of x_1^* and x_4^*, so that it suffices to investigate the values of $x_3^*(x_1^*, x_4^*)$ at five points, see (5.4), (5.5) and also Figure 5:

$$x_3^*\left(-1, \frac{3-b}{b+1}\right) = \frac{2t-b+1}{b+1} = 0.3877375269\ldots \tag{6.15}$$

$$x_3^*(-1,1) = t = 0.4177913013\ldots \tag{6.16}$$

$$x_3^*\left(\frac{-1}{b}, \frac{1}{b}\right) = \frac{t}{b} = 0.4004466202\ldots \tag{6.17}$$

$$x_3^*\left(\frac{-1}{b},1\right)=\frac{-1+b+bt+t}{2b}=0.4298765508\ldots \tag{6.18}$$

$$x_3^*\left(\frac{b-3}{b-1},1\right)=\frac{2t+b-1}{b+1}=0.4301327291\ldots . \tag{6.19}$$

Obviously, (6.15) is the minimum and (6.19) is the maximum of $x_3^*=x_3^*(x_1^*,x_4^*)$ as claimed in (5.9). The verification of (5.8) follows similar lines and will be left to the reader. □

7. Concluding remarks

Remark 7.1. Let $n=4$. According to (4.7), the largest possible value for the first optimal interpolation node in $\mathbf{I}$ is $x_1^*=-\frac{1}{b}=-0.9584848200\ldots$, if we consider zero-symmetric node systems. But if we allow arbitrary node configurations, then we can get beyond this number: the largest possible value for the first optimal interpolation node in $\mathbf{I}$ is in fact $x_1^*=\frac{b-3}{b+1}=-0.9576047978\ldots$, see (5.5).

Remark 7.2. According to section 6.6, Schurer's description in [27, Theorem 1] of the optimal arbitrary node systems $X_3^*: x_1^*<x_2^*<x_3^*$ for quadratic Lagrange interpolation on $\mathbf{I}$ can be restated almost verbatim in the form as given in our Theorem 5.4, if we replace in (5.4) and (5.5) x_4^* by x_3^* and if we replace the constant $b=1.0433133411\ldots$ by the corresponding constant $b^\circ=\frac{3}{2\sqrt{2}}=1.0606601717\ldots$. For example, (5.4) would then read

$$-1\le x_1^*\le -\frac{1}{b^\circ}(\approx -0.9428)\wedge\left(\frac{b^\circ-1}{b^\circ+1}\right)x_1^*+\frac{2}{b^\circ+1}\le x_3^*\le 1 \tag{7.1}$$

which is identical with

$$-1\le x_1^*\le -\frac{2\sqrt{2}}{3}(\approx -0.9428)\wedge(17-12\sqrt{2})x_1^*+12\sqrt{2}-16\le x_3^*\le 1 \tag{7.2}$$

as given in [27]. Note that in the quadratic case we have $x_2^*=\frac{x_1^*+x_3^*}{2}$.

Remark 7.3. The proof of Theorem 5.4 rests on Theorem 5.2. We have also found an alternative, computer-aided proof of Theorem 5.4 which avoids Theorem 5.2. It uses quantifier elimination and can be compared with section 3.5 in [25]. However, to reduce computational complexity, this automated proof requires a reduction of the variables by means of

$$x_2^*=x_1^*+x_4^*-x_3^*. \tag{7.3}$$

This equation, which is of some interest in itself, follows readily from Theorem 5.2, but, to stay independent of that theorem, we had to establish an alternative new proof for (7.3).

Remark 7.4. Suppose one chooses $x_1^*=c$ (constant) and $x_4^*=d$ (constant) from the indicated ranges (5.4) respectively (5.5) in Theorem 5.4. Then one gets, in generalizing Example 5.6, $\alpha=\frac{-2-c-d}{d-c}$ and $\beta=\frac{2-c-d}{d-c}$, and hence $x_2^*=\frac{c+d+t(c-d)}{2}$ and $x_3^*=\frac{c+d+t(d-c)}{2}$, according to Theorem 5.2, or directly from (5.6) and (5.7).

Remark 7.5. The natural question arises: How to determine the minimal Lebesgue constant Λ_n^* and all optimal node systems X_n^* in $\mathbf{I}$ for the next cases $n = 5, 6, 7, \ldots$? We have achieved some progress for $n = 5$ and $n = 6$. For example, we have implicitly determined the Lebesgue constants Λ_5^* and Λ_6^* by symbolic computation as roots of certain high-degree polynomials with integer coefficients. To be more specific, for $n = 5$ (quartic case) we have obtained

$$\Lambda_5^* = 1.5594902098\ldots \tag{7.4}$$

as a root of a polynomial of degree 73 with integer coefficients. This polynomial has the contour

$$P_{73}^*(x) = \tag{7.5}$$

$$491920844066918518676932058679834515105631225977247880376611328125 + \ldots$$
$$+14156651510438131445849849962417864414147142283963792181670336004096\, x^{73}.$$

We intend to expose our findings in a separate manuscript, see also [32].

Remark 7.6. That the topic of optimal cubic Lagrange interpolation awakens interest in the reader may be deduced from the fact that the online-version of [23] on the publishers website has received more than 60 article views subject to charge, see `http://www.tandfonline.com/doi/abs/10.1080/0020739840150312`

Remark 7.7. The desire to precisely determine the values of interesting constants (here: b, t, Λ_4^*) is reflected on in [12, p. 79].

Remark 7.8. In [10, p. 70] it is erroneously claimed that an optimal node system in $\mathbf{I}$ must necessarily be a CNS.

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