Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 1, 123–133

On some generalizations of Nadler's contraction principle

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Abstract. The purpose of this work is to present some generalizations of the well known Nadler's contraction principle. More precisely, using an axiomatic approach of the Pompeiu-Hausdorff metric we will study the properties of the fractal operator generated by a multivalued contraction.

Mathematics Subject Classification (2010): 47H25, 54H10.

Keywords: H^+-type multivalued mapping, Lipschitz equivalent metric, multivalued operator, contraction.

1. Introduction

Let (X,d) be a metric space and $\mathcal{P}$(X) be the set of all subsets of X. Consider the following families of subsets of X:

P(X):={Y∈ $\mathcal{P}$(X)| Y≠ ∅}, $P_{b,cl}$(X):={Y∈ $\mathcal{P}$(X)|Y is bounded and closed}

The following (generalized) functionals are used in the main sections of the paper.

1. The gap functional generated by d:

$$D_d : \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{X}) \to \mathbb{R}_+ \cup \{\infty\}, D_d(A, B) = inf\{d(a, b) \mid a \in A, b \in B\}$$

2. The diameter generalized functional:

$$\delta : \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{X}) \to \mathbb{R}_+ \cup \{\infty\}, \delta(A, B) = \sup\{d(a, b)|a \in A, b \in B\}$$

3. The excess generalized functional:

$$\rho_d : \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{X}) \to \mathbb{R}_+ \cup \{\infty\}, \rho_d(A, B) = sup\{D(a, B)|a \in A\}$$

4. The Pompeiu-Hausdorff generalized functional:

$$H_d : \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{X}) \to \mathbb{R}_+ \cup \{\infty\}, H_d(A, B) = \max\{\sup_{a\in A} D_d(a, B), \sup_{b\in B} D_d(b, A)\}$$

This paper was presented at the 10th Joint Conference on Mathematics and Computer Science (MaCS 2014), May 21-25, 2014, Cluj-Napoca, Romania.

5. The H^+-generalized functional:

$$H^+ : \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{X}) \to \mathbb{R}_+ \cup \{\infty\}, H^+(A,B) := \frac{1}{2}\{\rho(A,B) + \rho(B,A)\}$$

Let (X,d) be a metric space. If $T : X \to P(X)$ is a multivalued operator, then $x \in X$ is called fixed point for T if and only if $x \in T(x)$. The following concepts are well-known in the literature.

Definition 1.1. [7] *Let (X,d) be a metric space. A mapping $T : X \to P_{b,cl}(X)$ is called a multivalued contraction if there exist a constant $k \in (0,1)$ such that:*

$$H_d(T(x), T(y)) \leq kd(x,y), \text{ for all } x, y \in X.$$

Definition 1.2. [5] *Let X be a nonempty set and d, ρ two metrics on X. Then, by definition, d, ρ are called strongly(or Lipschitz) equivalent if there exists $c_1, c_2 > 0$ such that:*

$$c_1\rho(x,y) \leq d(x,y) \leq c_2\rho(x,y), \text{ for all } x, y \in X.$$

Definition 1.3. [7] *Let (X,d) be a metric space. Then, by definition, the pair (d, H_d) has the property (p^*) if for $q > 1$, for all $A, B \in P(X)$ and any $a \in A$, there exists $b \in B$ such that:*

$$d(a,b) \leq qH_d(A,B).$$

Definition 1.4. [6] *Let (X,d) be a metric space. $T : X \to P_{b,cl}(x)$ is called H_d- upper semi-continuous in $x_0 \in X$ (H_d-u.s.c) respectively H_d- lower semi-continuous (H_d-l.s.c) if and only if for each sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that*

$$\lim_{n\to\infty} x_n = x_0$$

we have

$$\lim_{n\to\infty} \rho_d(T(x_n), T(x_0)) = 0 \text{ respectively } \lim_{n\to\infty} \rho_d(T(x_0), T(x_n)) = 0.$$

2. Main results

Concerning the functional H^+ defined below, we have the following properties.

Lemma 2.1. [2] *H^+ is a metric on $P_{b,cl}(X)$.*

Lemma 2.2. [1] *We have the following relations:*

$$\frac{1}{2}H_d(A,B) \leq H^+(A,B) \leq H_d(A,B), \text{ for all } A, B \in P_{b,cl}(X) \tag{2.1}$$

(i.e., H_d and H^+ are strongly equivalent metrics).

Proposition 2.3. [2] *Let $(X, ||\cdot||)$ be a normed linear space. For any λ (real or complex), $A, B \in P_{b,cl}(X)$*

1. $H^+(\lambda A, \lambda B) = |\lambda| H^+(A,B)$.
2. $H^+(A + a, B + a) = H^+(A,B)$.

Theorem 2.4. [2] *If $a, b \in X$ and $A, B \in P_{b,cl}(X)$, then the relations hold:*

1. $d(a,b) = H^+(\{a\},\{b\})$.
2. $A \subset \overline{S}(B,r_1), B \subset \overline{S}(A,r_2) \Rightarrow H^+(A,B) \leq r$ *where* $r = \frac{r_1+r_2}{2}$.

Theorem 2.5. [2] *If the metric space* (X,d) *is complete, then* $(P_{b,cl}(X),H^+)$ *and* $(P_{b,cl}(X),H_d)$ *are complete too.*

Definition 2.6. [2] *Let (X,d) be a metric space. A multivalued mapping* $T : X \to P_{b,cl}(x)$ *is called* (H^+,k)*-contraction if*

1. *there exists a fixed real number* k, $0 < k < 1$ *such that for every* $x, y \in X$
$$H^+(T(x),T(y)) \leq kd(x,y).$$
2. *for every* x *in* X, y *in* $T(x)$ *and* $\varepsilon > 0$, *there exists* z *in* $T(y)$ *such that*
$$d(y,z) \leq H^+(T(y),T(x)) + \varepsilon.$$

Theorem 2.7. [2] *Let* (X,d) *be a complete metric space,* $T : X \to P_{b,cl}(X)$ *be a multivalued* (H^+,k) *contraction. Then* $FixT \neq \emptyset$.

Remark 2.8. [1] If T is a multivalued k-contraction in the sense of Nadler then T is a multivalued (H^+,k)-contraction but not viceversa.

Example 2.9. Let $X = \{0, \frac{1}{2}, 2\}$ and $d : X \times X \to \mathbb{R}$ be a standard metric. Let $T : X \to P_{b,cl}(X)$ be such that

$$T(x) = \begin{cases} \{0, \frac{1}{2}\}, & \text{for } x = 0 \\ \{0\}, & \text{for } x = \frac{1}{2} \\ \{0,2\}, & \text{for } x = 1 \end{cases}$$

Then T is a $(H^+$, k) contraction (with $k \in \left[\frac{2}{3}, 1\right))$ but is not an k- contraction in the sense of Nadler, since

$$H_d(T(0),T(2)) = H_d(\{0,\frac{1}{2}\},\{0,2\}) = 2 \leq kd(0,2) = 2k \Rightarrow k \geq 1,$$

which is a contradiction with our assumption that $k < 1$.

Theorem 2.10. [3] *(Nadler) Let* (X,d) *be a metric space and* $T : X \to P_{cp}(X)$ *be a multivalued contraction. Then*

$$H_d(T(A),T(B)) \leq kH_d(A,B) \text{ for all } A,B \in P_{cp}(X). \tag{2.2}$$

Lemma 2.11. [4] *Let* (X,d) *be a metric space and* $A, B \in P_{cp}(X)$.
Then for all $a \in A$ *there exists* $b \in B$ *such that*

$$d(a,b) \leq H_d(A,B).$$

Theorem 2.12. *Let* (X,d) *be a metric space and* $T : X \to P_{cp}(X)$ *for which there exists* $k > 0$ *such that:*

$$H_d(T(x),T(y)) \leq kd(x,y), \text{ for all } x,y \in X$$

Then

$$H^+(T(A),T(B)) \leq 2kH^+(A,B) \text{ for all } A,B \in P_{cp}(X).$$

Proof. Let $A, B \in P_{cp}(X)$.

From (2.2) we have $\rho(T(A), T(B)) \leq kH_d(T(A), T(B))$

Combining the previous result and $Lemma(2.2)$ we obtain

$$\rho_d(T(A), T(B)) \leq kH_d(A, B) \leq 2kH^+(A, B) \tag{2.3}$$

Interchanging the roles of A and B, we get

$$\rho_d(T(B), T(A)) \leq kH_d(B, A) \leq 2kH^+(B, A) \tag{2.4}$$

Adding (2.3) and (2.4), and then dividing by 2, we get

$$H^+(T(A), T(B)) \leq 2kH^+(A, B).$$

□

Let us recall the relations between u.s.c and $H_d - u.s.c$ of a multivalued operator. If (X, d) is a metric space, then $T : X \to P_{cp}(X)$ is $u.s.c$ on X if and only if T is $H_d - u.s.c$.

Theorem 2.13. *Let (X, d) be a metric space and $T : X \to P_{cp}(X)$ be a multivalued (H^+, k)-contraction. Then*

(a) T is H_d-l.s.c and u.s.c on X.
(b) for all $A \in P_{cp}(X) \Rightarrow T(A) \in P_{cp}(X)$
(c) there exists $k > 0$ such that

$$H^+(T(A), T(B)) \leq 2kH^+(A, B) \ for \ all \ A, B \in P_{cp}(X).$$

Proof. (a) Let $x \in X$ such that $x_n \to x$. We have:

$$\rho_d(T(x), T(x_n)) \leq H_d(T(x), T(x_n)) \leq 2 \cdot H^+(T(x), T(x_n) \leq 2k \cdot d(x, x_n) \to 0$$

In conclusion, T is H_d-l.s.c on X.
Using the relation:

$$\rho_d(T(x_n), T(x)) \leq H_d(T(x_n), T(x)) \leq 2 \cdot H^+(T(x_n), T(x) \leq 2k \cdot d(x, x_n) \to 0$$

we obtain that T is H_d-u.s.c on X.

(b) Let $A \in P_{cp}(X)$. From (a) we obtain the conclusion.

(c) If $u \in T(A)$, then there exists $a \in A$ such that $u \in T(a)$.
From Lemma 2.11 we have that there exists $b \in T(B)$ such that

$$d(a, b) \leq H_d(A, B) \leq 2H^+(A, B).$$

Since

$$D(u, T(B)) \leq D(u, T(b)) \leq \rho_d(T(a), T(b)) \tag{2.5}$$

taking $\sup_{u \in T(A)}$ in (2.5), we have

$$\rho_d(T(A), T(B)) \leq \rho_d(T(a), T(b)) \tag{2.6}$$

Interchanging the roles of A and B, we get

$$\rho_d(T(B), T(A)) \leq \rho_d(T(a), T(b)) \tag{2.7}$$

Adding (2.6) and (2.7), and then dividing by 2, we get for all $A, B \in P_{cp}(X)$ the following result:

$$H^+(T(A), T(B)) \leq H^+(T(a), T(b)) \leq kd(a, b) \leq 2kH^+(A, B).$$

□

As a consequence of the previous result we obtain the following fixed set theorem for a multivalued contraction with respect to H^+.

Theorem 2.14. *Let (X, d) be a complete metric space and $T : X \to P_{cp}(X)$ be a multivalued operator for which there exists $k \in [0, \frac{1}{2})$ such that*

$$H^+(T(x), T(y)) \le kd(x, y),\ \textit{for all } x, y \in X$$

Then, there exists a unique $A^* \in P_{cp}(X)$ such that $T(A^*) = A^*$.

Proof. From *Theorem* 2.13 we obtain that:

$$H^+(T(A), T(B)) \le 2kH^+(A, B),\ \textit{for all } A, B \in P_{cp}(X)$$

Since $k < \dfrac{1}{2}$ we obtain that T is a $2k$-contraction on the complete metric space $(P_{cp}(X), H^+)$. By Banach contraction principle we get the conclusion. □

In the second part of this section, we will study when the property (p^*) given in *Definition* 1.3 can be translated between equivalent metrics on a nonempty set X.

Lemma 2.15. *Let X be a nonempty set, d_1, d_2 two Lipschitz equivalent metrics such that there exists $c_1, c_2 > 0$ with $c_1 \le c_2$ i.e*

$$c_1 d_1(x, y) \le d_2(x, y) \le c_2 d_1(x, y),\ \textit{for all } x, y \in X \tag{2.8}$$

If the pair (d_1, H_{d_1}) has the property (p^*), then the pair (d_2, H_{d_2}) has the property (p^*).

Proof. Let c_1, c_2 such that

$$c_1 d_1(a, b) \le d_2(a, b) \le c_2 d_1(a, b)\ \textit{for all } a \in A, b \in B \tag{2.9}$$

and for all $q > 1$, for all $A, B \in P(X)$ and for all $a \in A$, there exists $b^* \in B$ such that

$$d_1(a, b^*) \le qH_{d_1}(A, B) \tag{2.10}$$

From (2.9) and (2.10) we obtain:

$$d_2(a, b^*) \le c_2 d_1(a, b^*) \le c_2 q H_{d_1}(A, B).$$

If, in $c_1 d_1(a, B) \le d_2(a, B)$ we take $\inf_{b \in B}$, then

$$c_1 D_{d_1}(a, B) \le D_{d_2}(a, B) \mid \sup_{a \in A} \Leftrightarrow c_1 \rho_{d_1}(A, B) \le \rho_{d_2}(A, B).$$

In a similar way,

$$c_1 \rho_{d_1}(B, A) \le \rho_{d_2}(B, A).$$

Taking maximum, we get

$$c_1 H_{d_1}(A, B) \le H_{d_2}(A, B).$$

Therefore,

$$d_2(a, b^*) \le \frac{c_2}{c_1} q H_{d_2}(A, B),$$

which means that there exists $b' = b^* \in B$ such that

$$d_2(a, b^*) \le q_1 H_{d_2}(A, B),$$

where $q_1 := \frac{c_2}{c_1} q > 1$. □

Lemma 2.16. *Let X be a nonempty set, d_1, d_2 two metrics on X such that:*

$$\text{there exists } c > 0\text{: } d_2(x,y) \leq c d_1(x,y) \quad \text{for all } x, y \in X \tag{2.11}$$

and G_1, G_2 two metrics on $P_{b,cl}(X)$ such that:

$$\text{there exists } e > 0\text{: } eG_{d_1}(A,B) \leq G_{d_2}(A,B), \text{ for all } A, B \in P_{b,cl}(X) \tag{2.12}$$

with $e \leq c$. If the pair (d_1, G_1) has the property (p^*) then, the property (p^*) is also true for the pair (d_2, G_2).

Proof. Let $A, B \in P_{b,cl}(X)$. The pair (d_1, G_{d_1}) has the property (p^*) i.e for all $q > 1$ and for all $a \in A$ there exists $b^* \in B$ such that

$$d_1(a, b^*) \leq qH_{d_1}(A,B) \tag{2.13}$$

From (2.11), (2.12) and (2.13) we obtain:

$$d_2(a,b') \leq cd_1(a,b') \leq cqG_{d_1}(A,B) \leq \frac{c}{e} qG_{d_2}(A,B).$$

Therefore,

$$d_2(a,b') \leq \frac{c}{e} qG_{d_2}(A,B)$$

which means that there exists $b = b' \in B$ such that

$$d_2(a,b) \leq q_1 G_{d_2}(A,B)$$

where $q_1 := \frac{c}{e} q > 1$ i.e the pair (d_2, G_{d_2}) has the property (p^*). □

Lemma 2.17. *Let X be a nonempty set, d_1, d_2 two metrics on X such that:*

$$\text{there exists } c > 0\text{: } d_2(x,y) \leq cd_1(x,y) \text{ for all } x, y \in X \tag{2.14}$$

and G_1, G_2 two metrics on $P_{b,cl}(X)$ such that:

$$\text{there exists } e > 0\text{: } G_{d_2}(A,B) \leq eG_{d_2}(A,B), \text{ for all } A, B \in P_{b,cl}(X) \tag{2.15}$$

with $c \cdot e < 1$. If the pair (d_1, G_{d_2}) has the property (p^*) then, the property (p^*) is also true for the pair (d_2, G_{d_1}).

Proof. Let $A, B \in P_{b,cl}(X)$. The pair (d_1, G_{d_2}) has the property (p^*) i.e for all $q > 1$ and for all $a \in A$ there exits $b^* \in B$ such that

$$d_1(a,b^*) \leq qG_{d_2}(A,B) \tag{2.16}$$

From (2.14), (2.15) and (2.16) we obtain:

$$d_2(a,b') \leq cd_1(a,b') \leq cqG_{d_2}(A,B) \leq c \cdot e \cdot qG_{d_1}(A,B).$$

Therefore,

$$d_2(a,b') \leq c \cdot e \cdot qG_{d_2}(A,B)$$

which means that, there exists $b = b' \in B$ such that

$$d_2(a,b) \leq q_1 G_{d_2}(A,B)$$

where $q_1 := c \cdot e \cdot q > 1$ i.e the pair (d_2, G_{d_1}) has the property (p^*). □

In the next part of this paper we will give some general abstract results for the metric space $P_{b,cl}(X)$.

Let (X, d) be a metric space, $U \subset P(X)$ and $\Psi : U \to \mathbb{R}_+$. We define some functionals on $U \times U$ as follows:

1. Let $x^* \in X$, $U \subset P_b(X)$

$$G_{\Psi_1}(A,B) = \begin{cases} 0, & A = B \\ \Psi_1(A) + \Psi_1(B), & A \neq B \end{cases}$$

where $\Psi_1(A) := \delta(A, x^*)$.

2. Let $U := P_b(X)$ and $A^* \in P_b(X)$

$$G_{\Psi_2}(A,B) = \begin{cases} 0, & A = B \\ \Psi_2(A) + \Psi_2(B), & A \neq B \end{cases}$$

Where $\Psi_2(A) = H_d(A, A^*)$.

Lemma 2.18. *Let (X, d) be a metric space and $T : X \to P_{cp}(X)$ and $A, B \in P_{cp}(X)$. Let*

$$G_{\Psi_1}(A,B) = \begin{cases} 0, & A = B \\ \Psi_1(A) + \Psi_1(B), & A \neq B \end{cases}$$

Where $\Psi_1(A) = \delta(A, A^*)$, $A^* \in P_{cp}(X)$. Then G_{Ψ_1} is a metric on $P_{cp}(X)$.

Proof. We shall prove that the three axioms of the metric hold:

a) $G_{\Psi_1}(A, B) \geq 0$ for all $A, B \in P_{cp}(X)$

$G_{\Psi_1}(A, B) = \delta(A, A^*) + \delta(B, A^*) \geq 0$

$G_{\Psi_1}(A, B) = 0 \Leftrightarrow A = B$.

This is equivalent to $\Psi_1(A) = 0$ and $\Psi_1(B) = 0$ i.e

$$\delta(A, A^*) = 0 \textit{ and } \delta(B, A^*) = 0 \Leftrightarrow A = A^* \textit{ and } B = A^* \Rightarrow A = B.$$

b) $G_{\Psi_2}(A, B) = G_{\Psi_2}(B, A)$ is quite obviously.

c) For the third axiom of the metric, let consider $A, B, C \in P_{cp}(X)$. We need to show that:

$$G_{\Psi_1}(A, C) \leq G_{\Psi_1}(A, B) + G_{\Psi}(B, C) \Leftrightarrow$$
$$\Leftrightarrow \Psi_1(A) + \Psi_1(C) \leq \Psi_1(A) + \Psi_1(B) + \Psi_1(B) + \Psi_1(C) \Leftrightarrow$$
$$\Leftrightarrow 0 \leq 2\Psi_1(B) = \delta(B, A^*) \textit{ which is true.}$$ □

Lemma 2.19. *If (X, d) is a complete metric space, then $(P_{cp}(X), G_{\Psi_1})$ is complete metric space.*

Proof. We will prove that each Cauchy sequence in $(P_{cp}(X), G_{\Psi_1})$ is convergent. Let $(A_n)_{n\in\mathbb{N}}$, $(A_m)_{m\in\mathbb{N}} \in P_{cp}(X)$, we have:

$$G_{\Psi_1}(A_n, A_m) \to 0,\ m, n \to 0 \Leftrightarrow \delta(A_n, A^*) + \delta(A_m, A^*) \to 0 \Rightarrow$$
$$\Rightarrow \delta(A_n, A^*) \to 0.$$

Therefore,

$$G_{\Psi_1}(A_n, A^*) = \delta(A_n, A^*) + \delta(A^*, A^*) \to 0,\ n \to 0.$$ □

Lemma 2.20. *Let (X,d) be a metric space and $T : X \to P_{cp}(X)$ and $A, B \in P_{cp}(X)$. Let*

$$G_{\Psi_1}(A,B) = \begin{cases} 0, & A = B \\ \Psi_1(A) + \Psi_1(B), & A \neq B \end{cases}$$

where $\Psi_1 : P_{cp}(X) \to \mathbb{R}_+, \Psi_1(A) = \delta(A, A^*)$ with $A^* \in P_{cp}(X)$. Then, the pair (d, G_{Ψ_1}) has the property (p^*).

Proof. We have to show

$$d(a,b) \leq qG_{\Psi_1}(A,B) \iff d(a,b) \leq q(\Psi_1(A) + \Psi_1(B)) \Leftrightarrow$$
$$\Leftrightarrow d(a,b) \leq q(\delta(A,A^*) + \delta(A,A^*))$$

Suppose, by absurdum, that there exists $a \in A$ and there exists $q > 1$ such that for all $b \in B$ we have:

$$d(a,b) > q(\delta(A,A^*) + \delta(B,A^*)).$$

Then, $\delta(A,b) \geq d(a,b) > q(\delta(A,A^*) + \delta(B,A^*))$.
Then, taking $\sup_{b \in B}$, we obtain:

$$\delta(A,A^*) + \delta(A^*,B) \leq \delta(A,B) \geq q(\delta(A,A^*) + \delta(B,A^*))$$

which is a contradiction with $q > 1$. □

Theorem 2.21. *Let (X,d) be a metric space and $T : X \to P_{cp}(X)$ be a multivalued operator for which there exists $k \in (0,1)$ such that*

$$\delta(T(x),T(y) \leq kd(x,y).$$

For all $A, B \in P_{cp}(X)$ we consider

$$G_{\Psi_1}(A,B) = \begin{cases} 0, & A = B \\ \Psi_1(A) + \Psi_1(B), & A \neq B, \end{cases}$$

where $\Psi_1 : P_{cp}(X) \to \mathbb{R}_+$, $\Psi_1(A) = \delta(A,A^*)$ (with $A^* \in P_{cp}(X)$ is a given set satisfying $A^* = T(A^*)$). Then,

$$G_{\Psi_1}(T(A),T(B)) \leq kG_{\Psi_1}(A,B) \; for \; all \; A, B \in P_{cp}(X).$$

Proof. We shall prove that for each $A, B \in P_{cp}(X)$ we have

$$\delta(T(A),A^*) + \delta(T(B),A^*) \leq k(\delta(A,A^*)) + \delta(B,A^*)) \tag{2.17}$$

Since $A^* = T(A^*)$, we have:

$$\delta(A^*,T(A)) + \delta(A^*,T(B)) = \delta(T(A^*),T(A)) + \delta(T(B^*),T(B))$$

Since

$$\delta(T(a),T(b)) \leq kd(a,b) \; for \; all \; a \in A \; and \; b \in B$$

We have (taking $\sup_{a \in A, b \in B}$) that

$$\delta(T(A),T(B)) \leq k\delta(A,B)$$

We obtain:

$$\delta(A^*,T(A)) + \delta(A^*,T(B)) = \delta(T(A^*),T(A)) + \delta(T(A^*),T(B))$$
$$\leq k\delta(A^*,A) + k\delta(A^*,B) = kG_{\psi_1}(A,B)$$

which means:

$$G_{\Psi_1}(T(A),T(B)) \leq kG_{\Psi_1}(A,B) \; for\ all\ A,B \in P_{cp}(X). \qquad \square$$

Lemma 2.22. *Let (X,d) be a metric space and $T : X \to P_{cp}(X)$ and $A,B \in P_{cp}(X)$. Let*

$$G_{\Psi_2}(A,B) = \begin{cases} 0, & A = B \\ \Psi_2(A) + \Psi_2(B), & A \neq B \end{cases}$$

where $\Psi_2 : P_{cp}(X) \to \mathbb{R}_+$, $\Psi_2(A) = H_d(A,A^*)$ with $A^* \in P_{cp}(X)$. Then G_{Ψ_2} is a metric on $P_{cp}(X)$.

Proof. We shall prove that the three axioms of the metric hold:
a) $G_{\Psi_2}(A,B) \geq 0$ for all $A,B \in P_{cp}(X)$
$G_{\Psi_2}(A,B) = H_d(A,A^*) + H_d(B,A^*) \geq 0$
$G_{\Psi_2}(A,B) = 0 \Leftrightarrow A = B$.
This is equivalent to $\Psi_2(A) = 0$ and $\Psi_2(B) = 0$ i.e

$$H_d(A,A^*) = 0 \; and \; H_d(B,A^*) = 0 \Leftrightarrow A = A^* \; and \; B = A^* \Rightarrow A = B.$$

b) $G_{\Psi_2}(A,B) = G_{\Psi_2}(B,A)$ is quite obviously. *c*) For the third axiom of the metric, let consider $A,B,C \in P_{cp}(X)$. We need to show that:

$$G_{\Psi_2}(A,C) \leq G_{\Psi_2}(A,B) + G_{\Psi_2}(B,C) \Leftrightarrow$$

$$\Leftrightarrow \Psi_2(A) + \Psi_2(C) \leq \Psi_2(A) + \Psi_2(B) + \Psi_2(B) + \Psi_2(C) \Leftrightarrow$$

$$\Leftrightarrow 0 \leq 2\Psi_2(B) = 2H_d(B,A^*) \; which\ is\ true. \qquad \square$$

Lemma 2.23. *If (X,d) is a complete metric space, then $(P_{cp}(X), G_{\Psi_2})$ is complete metric space.*

Proof. We will prove that each Cauchy sequence in $(P_{cp}(X), G_{\Psi_2})$ is convergent. Let $(A_n)_{n\in\mathbb{N}}$, $(A_m)_{m\in\mathbb{N}} \in P_{cp}(X)$, we have:

$$G_{\Psi_2}(A_n,A_m) \to 0,\ m,n \to 0 \Leftrightarrow H_d(A_n,A^*) + H_d(A_m,A^*) \to 0 \Leftrightarrow$$

$$\Leftrightarrow H_d(A_n,A^*) \to 0$$

Therefore,

$$G_{\Psi_2}(A_n,A^*) = H_d(A_n,A^*) + H_d(A^*,A^*) \to 0,\ n \to 0. \qquad \square$$

Theorem 2.24. *Let (X,d) be a metric space and $T : X \to P_{cp}(x)$ be a multivalued contraction with respect to H_d and $A,B \in P_{cp}(X)$. Let*

$$G_{\Psi_2}(A,B) = \begin{cases} 0, & A = B \\ \Psi_2(A) + \Psi_2(B), & A \neq B \end{cases}$$

Where $\Psi_2 : P_{cp}(X) \to \mathbb{R}_+$, $\Psi_2(A) = H_d(A,A^*)$ (with $A^* \in P_{cp}(X)$ is a given set satisfying $A^* = T(A^*)$). Then, there exists $k \in (0,1)$ such that

$$G_{\Psi_2}(T(A),T(B)) \leq kG_{\Psi_2}(A,B) \; for\ all\ A,B \in P_{cp}(X).$$

Proof. We shall prove that for each $A, B \in P_{cp}(X)$ we have

$$H_d(T(A), A^*) + H_d(T(B), A^*) \leq k(H_d(A, A^*)) + H_d(B, A^*)).$$

From (2.2) we have $\rho_d(T(A), T(B)) \leq H_d(T(A), T(B))$.
Then

$$\rho_d(T(A), A^*) = \rho_d(T(A), T(A^*)) \leq H_d(T(A), T(A^*)) \leq kH_d(A, A^*).$$

Interchanging the roles of A and B, we get

$$\rho_d(A^*, T(A)) = \rho_d(T(A^*), T(A)) \leq H_d(T(A^*), T(A)) \leq kH_d(A^*, A).$$

Making maximum, we get

$$H_d(T(A), A^*) \leq kH_d(A, A^*). \tag{2.18}$$

Similarly for $B \in P_{cp}(X)$, we have

$$H_d(T(B), A^*) \leq kH_d(B, A^*). \tag{2.19}$$

Adding (2.18) and (2.19) we get:

$$H_d(T(A), A^*) + H_d(T(B), A^*) \leq k(H_d(A, A^*)) + H_d(B, A^*))$$

which means:

$$G_{\Psi_2}(T(A), T(B)) \leq kG_{\Psi_2}(A, B) \; for \; all \; A, B \in P_{cp}(X). \qquad \square$$

Lemma 2.25. *Let (X, d) be a metric space and $T : X \to P_{cp}(X)$ and $A, B \in P_{cp}(X)$. Let*

$$G_{\Psi_2}(A, B) = \begin{cases} 0, & A = B \\ \Psi_2(A) + \Psi_2(B), & A \neq B, \end{cases}$$

where $\Psi_2 : P_{cp}(X) \to \mathbb{R}_+$, $\Psi_2(A) = H_d(A, A^*)$ with $A^* \in P_{cp}(X)$. Then, the pair (d, G_{ψ_2}) has the property (p^*).

Proof. We have to show

$$d(a, b) \leq qG_{\Psi_2}(A, B) \iff d(a, b) \leq q(\Psi_2(A) + \Psi_2(B)) \Leftrightarrow$$

$$\Leftrightarrow d(a, b) \leq q(H_d(A, A^*) + H_d(A, A^*))$$

Supposing again contrary: there exists $q > 1$ and there exists $a \in A$ such that for all $b \in B$ we have:

$$d(a, b) > q(H_d(A, A^*) + H_d(B, A^*)).$$

Then, taking $\inf\limits_{b \in B}$

$$H_d(A, B) \geq \rho_d(A, B) \geq D(a, B) \geq q(H_d(A, A^*) + H_d(B, A^*)).$$

But

$$H_d(A, A^*) + H_d(A^*, B) \geq H_d(A, B) \geq q(H_d(A, A^*) + H_d(B, A^*)).$$

Hence $q \leq 1$, a contradiction. $\square$

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