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Improved error analysis of Newton's method for a certain class of operators

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Abstract. We present an improved error analysis for Newton's method in order to approximate a locally unique solution of a nonlinear operator equation using Newton's method. The advantages of our approach under the same computational cost – as in earlier studies such as [15, 16, 17, 18, 19, 20] – are: weaker sufficient convergence condition; more precise error estimates on the distances involved and an at least as precise information on the location of the solution. These advantages are obtained by introducing the notion of the center γ_0-condition. A numerical example is also provided to compare the proposed error analysis to the older convergence analysis which shows that our analysis gives more precise error bounds than the earlier analysis.

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1. Introduction

Let $\mathbf{X}$, $\mathbf{Y}$ be Banach spaces. Let $U(x,r)$ and $\overline{U}(x,r)$ stand, respectively, for the open and closed ball in $\mathbf{X}$ with center x and radius $r>0$. $\mathbb{L}(\mathbf{X},\mathbf{Y})$ denotes the space of bounded linear operators from $\mathbf{X}$ into $\mathbf{Y}$. In the present paper we are concerned with the problem of approximating a locally unique solution $x^\star$ of nonlinear operator equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet continuously differentiable operator defined on $\overline{U}(x_0,R)$ for some $R>0$ with values in $\mathbf{Y}$.

Several problems from various disciplines such as Computational Sciences can be brought in the form of equation (1.1) using Mathematical Modelling [13, 14, 21, 5, 7, 17, 18]. The solution of these equations can rarely be found in closed form. That is why the solution methods for these equations are iterative. In particular, the practice of numerical functional analysis and operator theory for finding such solutions is essentially connected to variants of Newton's method [1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22].

The study about convergence matter of Newton methods is usually centered on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give criteria ensuring the convergence of Newton methods; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. We find in the literature several studies on the weakness and/or extension of the hypothesis made on the underlying operators.

There is a plethora on local as well as semi-local convergence results, we refer the reader to [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]. The most famous among the semi-local convergence of iterative methods is the celebrated Kantorovich theorem for solving nonlinear equations. This theorem provides a simple and transparent convergence criterion for operators with bounded second derivatives F'' or the Lipschitz continuous first derivatives [2, 7, 8, 11, 13, 14, 22]. Another important theorem inaugurated by Smale at the International Conference of Mathematics [17, 18], where the concept of an approximate zero was proposed and the convergence criteria were provided to determine an approximate zero for analytic function, depending on the information at the initial point. Wang [20] generalized Smale's result by introducing the γ-condition (see (1.3)). For more details on Smale's theory, the reader can refer to the excellent Dedieu's book [10, Chapter 3.3]. Newton's method defined by

$$\begin{cases} x_0 \text{ is an initial point} \\ x_{n+1} = x_n - F'(x_n)^{-1} F(x_n) \qquad \text{for each} \quad n = 0, 1, 2, \cdots \end{cases} \tag{1.2}$$

is undoubtedly the most popular iterative process for generating a sequence $\{x_n\}$ approximating $x^\star$ [8]. Here, $F'(x) \in \mathbb{L}(\mathbf{X}, \mathbf{Y})$ denotes the Fréchet-derivative of F at $x \in \overline{U}(x_0, R)$.

In the present paper motivated by the works in [9, 15, 16, 17, 18, 19, 20, 21] and optimization considerations, we expand the applicability of Newton's method under the γ-condition by introducing the notion of the center γ_0-condition (to be precised in Definition 3.1) for some $\gamma_0 \leq \gamma$. This way we obtain more precise upper bounds on the norms of $\| F'(x)^{-1} F'(x_0) \|$ for each $x \in \overline{U}(x_0, R)$ (see (1.3), (2.2) and (2.3)) leading to weaker sufficient convergence conditions and a tighter convergence analysis than in earlier studies such as [15, 16, 17, 18, 19, 20, 21]. Our approach of introducing center-Lipschitz condition has already been fruitful for expanding the applicability of Newton's method under the Kantorovich-type theory [2, 3, 4, 5, 6, 7, 13, 14, 22]. Wang [20] used the $\gamma-$Lipschitz condition which is given by

$$\left\| F'(x_0)^{-1} F''(x) \right\| \leq \frac{2\gamma}{\Big(1 - \gamma \|x - x_0\|\Big)^3} \quad \text{for each} \quad x \in U(x_0, r),\ 0 < r \leq R \tag{1.3}$$

where $\gamma > 0$ and x_0 are such that $\gamma \parallel x - x_0 \parallel < 1$ and $F'(x_0)^{-1} \in \mathbb{L}(\mathbf{Y}, \mathbf{X})$ to show the following semi-local convergence result for Newton's method.

Theorem 1.1. [20] *Let* $F : \overline{U}(x_0, R) \subseteq \mathbf{X} \longrightarrow \mathbf{Y}$ *be twice-Fréchet differentiable. Suppose there exists* $x_0 \in U(x_0, R)$ *such that* $F'(x_0)^{-1} \in \mathbb{L}(\mathbf{Y}, \mathbf{X})$ *and*

$$\left\| F'(x_0)^{-1} F(x_0) \right\| \leq \beta; \tag{1.4}$$

condition (1.3) *holds and for* $\alpha = \gamma\,\beta$

$$\alpha \leq 3 - 2\sqrt{2}; \tag{1.5}$$

$$t^\star \leq R, \tag{1.6}$$

where

$$t^\star = \frac{1 + \alpha - \sqrt{(1+\alpha)^2 - 8\,\alpha}}{4\,\gamma} \leq \left(1 - \frac{1}{\sqrt{2}}\right)\frac{1}{\gamma}. \tag{1.7}$$

Then, sequence $\{x_n\}$ *generated by Newton's method is well defined, remains in* $\overline{U}(x_0, t^\star)$ *for each* $n = 0, 1, \cdots$ *and converges to a unique solution* $x^\star \in \overline{U}(x_0, t^\star)$ *of equation* $F(x) = 0$.

Moreover, the following error estimates hold

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n \tag{1.8}$$

and

$$\|x_{n+1} - x^\star\| \leq t^\star - t_n, \tag{1.9}$$

where scalar sequence $\{t_n\}$ *is defined by*

$$\begin{cases} t_0 = 0, \quad t_1 = \beta, \\ t_{n+1} = t_n + \dfrac{\gamma\,(t_n - t_{n-1})^2}{\left(2 - \dfrac{1}{(1-\gamma\,t_n)^2}\right)(1-\gamma t_n)(1-\gamma t_{n-1})^2} = t_n - \dfrac{\varphi(t_n)}{\varphi'(t_n)} \end{cases} \tag{1.10}$$

for each $n = 1, 2, \cdots,$ *where*

$$\varphi(t) = \beta - t + \frac{\gamma\,t^2}{1 - \gamma\,t}. \tag{1.11}$$

Notice that $t^\star$ *is the small zero of equation* $\varphi(t) = 0$, *which exists under the hypothesis* (1.5).

The rest of the paper is organized as follows. Section 2 contains the semi-local and local convergence analysis of Newton's method. A numerical example is given in the concluding Section 3.

2. Semi-local convergence of Newton's method

We need some auxiliary results. We shall use the Banach lemma on invertible operators [2, 7, 12].

Lemma 2.1. *Let A, B be bounded linear operators, where A is invertible and $\| A^{-1} \| \| B \| < 1$.*

Then, $A+B$ is invertible and

$$\| (A+B)^{-1} \| \leq \frac{\| A^{-1} \|}{1 - \| A^{-1} \| \| B \|}. \tag{2.1}$$

We shall also use the following definition of Lipschitz and local Lipschitz conditions.

Definition 2.2. *(see [9, p. 634], [22, p. 673]) Let $F : \overline{U}(x_0, R) \longrightarrow \mathbf{Y}$ be Fréchet-differentiable on $U(x_0, R)$. We say that F' satisfies the Lipschitz condition at x_0 if there exists an increasing function $\ell : [0, R] \longrightarrow [0, +\infty)$ such that*

$$\begin{aligned} &\| F'(x_0)^{-1} (F'(x) - F'(y)) \| \leq \ell(r) \; \| x - y \| \\ &\textit{for each } x, y \in \overline{U}(x_0, r), \; 0 < r \leq R. \end{aligned} \tag{2.2}$$

In view of (2.2), there exists an increasing function $\ell_0 : [0, R] \longrightarrow [0, +\infty)$ such that the center-Lipschitz condition

$$\begin{aligned} &\| F'(x_0)^{-1} (F'(x) - F'(x_0)) \| \leq \ell_0(r) \; \| x - x_0 \| \\ &\textit{for each } x \in \overline{U}(x_0, r), \; 0 < r \leq R \end{aligned} \tag{2.3}$$

holds. Clearly,

$$\ell_0(r) \leq \ell(r) \text{ for each } r \in (0, R] \tag{2.4}$$

holds in general and $\ell(r)/\ell_0(r)$ can be arbitrarily large [2, 3, 4, 5, 6, 7].

Lemma 2.3. *(see [9, p. 638]) Let $F : \overline{U}(x_0, R) \longrightarrow \mathbf{Y}$ be Fréchet-differentiable on $U(x_0, R)$. Suppose $F'(x_0)^{-1} \in \mathbb{L}(\mathbf{Y}, \mathbf{X})$ and there exist $0 < \gamma_0 \leq \gamma$ such that $\gamma_0 R < 1$, $\gamma R < 1$. Then, F' satisfies conditions (2.2) and (2.3), respectively, with*

$$\ell(r) := \frac{2\gamma}{(1 - \gamma r)^3} \tag{2.5}$$

and

$$\ell_0(r) := \frac{\gamma_0 (2 - \gamma_0 r)}{(1 - \gamma_0 r)^2}. \tag{2.6}$$

Notice that with preceding choices of functions ℓ and ℓ_0, we have that

$$\ell_0(r) < \ell(r) \quad \text{for each} \quad r \in (0, R]. \tag{2.7}$$

We also need a result by Zabrejko and Nguen.

Lemma 2.4. *(see [22, p. 673]) Let $F : \overline{U}(x_0, R) \longrightarrow \mathbf{Y}$ be Fréchet-differentiable on $U(x_0, R)$. Suppose $F'(x_0)^{-1} \in \mathbb{L}(\mathbf{Y}, \mathbf{X})$ and*

$$\begin{aligned} &\| F'(x_0)^{-1} (F'(x) - F'(y)) \| \leq \lambda(r) \; \| x - y \| \\ &\textit{for each } x, y \in \overline{U}(x_0, r), \; 0 < r \leq R \end{aligned}$$

for some increasing function $\lambda : [0, R] \longrightarrow [0, +\infty)$. Then, the following assertion holds

$$\begin{aligned} &\| F'(x_0)^{-1} (F'(x+p) - F'(x)) \| \leq \Lambda(r + \| p \|) - \Lambda(r) \\ &\textit{for each } x \in \overline{U}(x_0, r), \; 0 < r \leq R \textit{ and } \; \| p \| \leq R - r, \end{aligned}$$

where

$$\Lambda(r) = \int_0^R \lambda(t)\, dt.$$

In particular, if

$$\begin{array}{l} \| F'(x_0)^{-1} (F'(x) - F'(x_0)) \| \leq \lambda_0(r) \ \| x - x_0 \| \\ for\ each\ x \in \overline{U}(x_0, r),\ 0 < r \leq R \end{array}$$

for some increasing function $\lambda_0 : [0, R] \longrightarrow [0, +\infty)$. *Then, the following assertion holds*

$$\begin{array}{l} \| F'(x_0)^{-1} (F'(x_0 + p) - F'(x_0)) \| \leq \Lambda_0(\| p \|) \\ for\ each\ 0 < r \leq R\ and\ \| p \| \leq R - r, \end{array}$$

where

$$\Lambda_0(r) = \int_0^R \lambda_0(t)\, dt.$$

Using the center-Lipschitz condition and Lemma 2.3, we can show the following result on invertible operators.

Lemma 2.5. *Let* $F : \overline{U}(x_0, R) \longrightarrow \mathbf{Y}$ *be Fréchet-differentiable on* $U(x_0, R)$. *Suppose* $F'(x_0)^{-1} \in \mathbb{L}(\mathbf{Y}, \mathbf{X})$ *and* $\gamma_0\, R < 1$ *for some* $\gamma_0 > 0$ *and* $x_0 \in \mathbf{X}$; *center-Lipschitz (*2.3*) holds on* $U_0 = U(x_0, r_0)$, *where* $\ell_0(r)$ *is given by (*2.6*) and* $r_0 = (1 - \frac{1}{\sqrt{2}})\, \frac{1}{\gamma_0}$. *Then* $F'(x)^{-1} \in \mathbb{L}(\mathbf{Y}, \mathbf{X})$ *on* U_0 *and satisfies*

$$\| F'(x)^{-1}\, F'(x_0) \| \leq \left(2 - \frac{1}{(1 - \gamma_0\, r)^2} \right)^{-1}. \tag{2.8}$$

Proof. We have by (2.3), (2.6) and $x \in U_0$ that

$$\| F'(x_0)^{-1} (F'(x) - F'(x_0)) \| \leq \ell_0(r)\, r \leq \frac{1}{(1 - \gamma_0\, r)^2} - 1 < 1.$$

The result now follows from Lemma 2.1. The proof of Lemma 2.5 is complete. □

Using (1.3) a similar to Lemma 2.1, Banach lemma was given in [20] (see also [9]).

Lemma 2.6. *Let* $F : \overline{U}(x_0, R) \longrightarrow \mathbf{Y}$ *be twice Fréchet-differentiable on* $U(x_0, R)$. *Suppose* $F'(x_0)^{-1} \in \mathbb{L}(\mathbf{Y}, \mathbf{X})$ *and* $\gamma\, R < 1$ *for some* $\gamma > 0$ *and* $x_0 \in \mathbf{X}$; *condition (*1.3*) holds on* $V_0 = U(x_0, \overline{r_0})$, *where* $\overline{r_0} = (1 - \frac{1}{\sqrt{2}})\, \frac{1}{\gamma}$. *Then* $F'(x)^{-1} \in \mathbb{L}(\mathbf{Y}, \mathbf{X})$ *on* $\overline{V_0}$ *and satisfies*

$$\| F'(x)^{-1}\, F'(x_0) \| \leq \left(2 - \frac{1}{(1 - \gamma\, r)^2} \right)^{-1}. \tag{2.9}$$

Remark 2.7. It follows from (2.8), (2.9) and $\gamma_0 \leq \gamma$ that (2.8) is more precise upper bound on the norm of $F'(x)^{-1}\, F'(x_0)$. This observation leads to a tighter majorizing sequence for $\{x_n\}$ (see Proposition 2.11).

We need an auxiliary result on majorizing sequences for Newton's method (1.2).

Lemma 2.8. *Let $\beta > 0$, $\gamma_0 > 0$, $\gamma > 0$ with $\gamma_0 > \gamma$ be given parameters. Define parameters s_i for $i = 0, 1, 2$ by*

$$s_0 = 0, \quad s_1 = \beta, \quad s_2 = \beta + \frac{\gamma_0\,\beta^2}{\left[2 - \dfrac{1}{(1-\gamma_0\,\beta)^2}\right](1-\gamma_0\,\beta)} \tag{2.10}$$

and function Ψ on $[\beta, 1/\gamma_0]$ by

$$\Psi(t) = (t-\beta)\gamma\,\beta\,(1-\gamma_0\,t)^2 - \Big(t - \beta - (s_2 - s_1)\Big)\Big[2(1-\gamma_0\,t)^2 - 1\Big](1-\gamma\,t)^3. \tag{2.11}$$

Suppose that

$$\beta < b := \min\left\{\frac{1}{\gamma}, \frac{0.25331131}{\gamma_0}\right\} \tag{2.12}$$

then, the following hold

$$0 < s_2 - s_1 < \frac{1-\gamma_0\beta}{\gamma_0}, \tag{2.13}$$

$$\gamma_0\beta < 1 - \frac{1}{\sqrt{2}} \tag{2.14}$$

and function ψ has zeros in $(\beta, 1/\gamma_0)$. Denote by ρ the smallest zero of the function Ψ in $(\beta, 1/\gamma_0)$. Moreover suppose that

$$s_2 \le \rho < b \tag{2.15}$$

where s_2 and b are given in (2.10) and (2.12), respectively. Then, for

$$\delta = 1 - \frac{s_2 - s_1}{\rho - \beta} \tag{2.16}$$

the following hold

$$0 < \frac{\gamma\beta(1-\gamma_0\rho)^2}{\Big(2(1-\gamma_0\rho)^2 - 1\Big)(1-\gamma\rho)^3} = \delta < 1, \tag{2.17}$$

$$\frac{s_2 - s_1}{1-\delta} + \beta = \rho \tag{2.18}$$

and the iteration $\{s_n\}$ defined by

$$s_{n+2} = s_{n+1} + \frac{\gamma(s_{n+1} - s_n)^2}{\left[2 - \dfrac{1}{(1-\gamma_0\,s_{n+1})^2}\right](1-\gamma\,s_{n+1})(1-\gamma\,s_n)^2} \tag{2.19}$$

for each $n = 1, 2, \ldots$ is strictly increasing, bounded from above by ρ and converges to its unique least upper bound $s^\star$ which satisfies

$$s_2 \le s^\star \le \rho. \tag{2.20}$$

Furthermore, the following estimates hold

$$s_{n+2} - s_{n+1} \le \delta^n (s_2 - s_1) \quad \text{for each} \quad n = 1, 2, \ldots. \tag{2.21}$$

Proof. The left hand side inequality in (2.13) is true by the definition of s_1, s_2 and since $2(1-\gamma_0\beta)^2-1>0$ and $1-\gamma_0\beta>0$ by (2.12). The right hand side of (2.13) shall be true, if

$$\frac{\gamma_0(s_1-s_0)^2}{\Big[2(1-\gamma_0\,s_1)^2-1\Big](1-\gamma_0\beta)}<\frac{1-\gamma_0\beta}{\gamma_0}$$

or $(\gamma_0\beta)^2<(1-\gamma_0\beta)^2\Big(2(1-\gamma_0\,\beta)^2-1\Big)$ or for $z=1-\gamma_0\beta$ if $2z^4-2z^2+2z-1>0$, or if $z>0.74668869$ which is true by (2.12). Estimate (2.14) follows from (2.12) since $1-1/\sqrt{2}=0.292853219>0.25331131\ldots$. Using (2.11) we have that

$$\Psi(\beta)=(s_2-s_1)\Big[2(1-\gamma_2\beta)^2-1\Big](1-\gamma\beta)^3>0,$$

since $s_2-s_1>0$, $2(1-\gamma_0\beta)^2-1>0$ and $1-\gamma\beta>0$. We also have that

$$\Psi(\frac{1}{\gamma_0})=\Big[\frac{1}{\gamma_0}-\beta-(s_2-s_1)\Big]\Big(1-\frac{\gamma}{\gamma_0}\Big)\Big(1-\frac{\gamma}{\gamma_0}\Big)^2<0$$

by (2.13) and $\gamma_0<\gamma$. It follows from the Intermediate mean value theorem applied to function Ψ on the interval $(\beta,1/\gamma_0)$ that function Ψ has zeros on $(\beta,1/\gamma_0)$. Denote by ρ the smallest such zero. Then, it follows from the definition of δ,ρ that the equality (2.17) holds. The left hand inequality holds by (2.12) and (2.15). The right hand side inequality in (2.17) holds by (2.13) and (2.15) since $\beta<s_2\leq\rho$. Moreover, we have by (2.16), (2.17) and (2.19) that

$$0<s_3\quad\text{and}\quad 0<s_3-s_2\leq\delta(s_2-s_1). \tag{2.22}$$

Then, we also have by (2.22) that

$$\begin{aligned}s_3&\leq s_2+\delta(s_2-s_1)-s_1+s_1=s_1+(1+\delta)(s_2-s_1)\\&=\beta+\frac{1-\delta^2}{1-\delta}(s_2-s_1)\leq\beta+\frac{s_2-s_1}{1-\delta}=\rho.\end{aligned}$$

Hence, we deduce that

$$s_3\leq\rho. \tag{2.23}$$

Suppose that

$$0<s_{n+1},\quad 0<s_{n+}-s_n\leq\delta^n(s_2-s_1)\quad\text{and}\quad s_{n+1}\leq\rho. \tag{2.24}$$

Then, we have by (2.31), (2.27) and (2.36) that

$$s_{n+2}\geq 0,$$

$$\begin{aligned}s_{n+2}-s_{n+1}&=\frac{\gamma(s_{n+1}-s_n)(s_{n+1}-s_n)}{\Big[2-\dfrac{1}{(1-\gamma_0\,s_{n+1})^2}\Big](1-\gamma\,s_{n+1})(1-\gamma\,s_n)^2}\\&\leq\frac{\gamma\beta}{\Big[2-\dfrac{1}{(1-\gamma_0\,\rho)^2}\Big](1-\gamma\rho)^3}(s_{n+1}-s_n)=\delta(s_{n+1}-s_n)\\&\leq\delta^{n+1}(s_2-s_1)\end{aligned}$$

and

$$\begin{aligned}
s_{n+2} &\leq s_{n+1} + \delta^{n+1}(s_2 - s_1) \leq s_n + \delta^n(s_2 - s_1) + \delta^{n+1}(s_2 - s_1) \\
&\leq s_1 + (1+\delta)(s_2 - s_1) + \cdots + \delta^n(s_2 - s_1) + \delta^{n+1}(s_2 - s_1) \\
&= s_1 + (1 + \delta + \cdots + \delta^n + \delta^{n+1})(s_2 - s_1) \\
&= s_1 + \frac{1-\delta^{n+2}}{1-\delta}(s_2 - s_1) \leq \rho.
\end{aligned}$$

Hence, by mathematical induction the proof for (2.24) is finished. Hence, sequence $\{s_n\}$ is monotonically increasing, bounded from above by ρ and as such it converges to $s^\star$. □

We can show the main following semi-local convergence theorem for Newton's method.

Theorem 2.9. *Suppose that*

(a) There exist $x_0 \in \mathbf{X}$ and $\beta > 0$ such that

$$F'(x_0)^{-1} \in \mathbb{L}(\mathbf{Y}, \mathbf{X}) \quad \text{and} \quad \left\| F'(x_0)^{-1} F(x_0) \right\| \leq \beta;$$

(b) Operator F' satisfies Lipschitz and center-Lipschitz conditions (2.2) and (2.3) on $U(x_0, r_0)$ with $\ell(r)$ and $\ell(r)$ are given by (2.5) and (2.6), respectively;

(c) $U_0 \subseteq \overline{U}(x_0, R)$;

(d) Hypotheses of Lemma 2.8 hold for sequence $\{s_n\}$ defined by (2.19).

Then, the following assertions hold: sequence $\{x_n\}$ generated by Newton's method is well defined, remains in xoverlineU$(x_0, s^\star)$ for each $n = 0, 1, \cdots$ and converges to a unique solution $x^\star \in \overline{U}(x_0, s^\star)$ of equation $F(x) = 0$. Moreover, the following estimates hold

$$\|x_{n+1} - x_n\| \leq s_{n+1} - s_n \tag{2.25}$$

and

$$\|x_n - x^\star\| \leq s^\star - s_n \quad \text{for each} \quad n = 0, 1, 2, \cdots. \tag{2.26}$$

Proof. We use Mathematical Induction to prove that

$$\| x_{k+1} - x_k \| \leq s_{k+1} - s_k \tag{2.27}$$

and

$$\overline{U}(x_{k+1}, s^\star - s_{k+1}) \subseteq \overline{U}(x_k, s^\star - s_k) \quad \text{for each} \quad k = 1, 2, \cdots. \tag{2.28}$$

Let $z \in \overline{U}(x_1, s^\star - s_1)$. Then, we obtain that

$$\| z - x_0 \| \leq \| z - x_1 \| + \| x_1 - x_0 \| \leq s^\star - s_1 + s_1 - s_0 = s^\star - s_0,$$

which implies $z \in \overline{U}(x_0, s^\star - s_0)$. Note also that

$$\| x_1 - x_0 \| = \| F'(x_0)^{-1} F(x_0) \| \leq \eta = s_1 - s_0.$$

Hence, estimates (2.27) and (2.28) hold for $k = 0$. Suppose these estimates hold for natural integers $n \leq k$. Then, we have that

$$\| x_{k+1} - x_0 \| \leq \sum_{i=1}^{k+1} \| x_i - x_{i-1} \| \leq \sum_{i=1}^{k+1} (s_i - s_{i-1}) = s_{k+1} - s_0 = s_{k+1}$$

and

$$\| x_k + \theta\,(x_{k+1} - x_k) - x_0 \| \leq s_k + \theta\,(s_{k+1} - s_k) \leq s^\star \quad \text{for all} \quad \theta \in (0,1).$$

Using (2.2), Lemma 2.1 for $x = x_{k+1}$ and the induction hypotheses we get that

$$\begin{aligned} \| F'(x_0)^{-1}\,(F'(x_{k+1}) - F'(x_0)) \| &\leq \frac{1}{(1-\gamma_0 \, \| x_{k+1} - x_0 \|)^2} - 1 \\ &\leq \frac{1}{(1-\gamma_0\, s_{k+1})^2} - 1 < 1. \end{aligned} \tag{2.29}$$

It follows from (2.29) and the Banach lemma 2.1 on invertible operators that $F'(x_{k+1})^{-1}$ exists and

$$\| F'(x_{k+1})^{-1}\,F'(x_0) \| \leq \left(2 - \frac{1}{(1-\gamma_0\, s_{k+1})^2} \right)^{-1}. \tag{2.30}$$

Using (1.2), we obtain the approximation

$$\begin{aligned} F(x_{k+1}) &= F(x_{k+1}) - F(x_k) - F'(x_k)\,(x_{k+1} - x_k) \\ &= \int_0^1 (F'(x_k^\tau) - F'(x_k))\,d\tau\,(x_{k+1} - x_k), \end{aligned} \tag{2.31}$$

where $x_k^\tau = x_k + \tau\,(x_{k+1} - x_k)$ and $x_k^{\tau\,s} = x_k + \tau\,s\,(x_{k+1} - x_k)$ for each $0 \leq \tau, s \leq 1$. Using (2.9) for $k = 0$ we obtain

$$\begin{aligned} \left\| F'(x_0)^{-1} F(x_1) \right\| &\leq \int_0^1 \left\| F'(x_0) \Big[F'(x_0 + \tau(x_1 - x_0)) - F'(x_0) \Big] \right\| \mathrm{d}\tau \|x_1 - x_0\| \\ &\leq \int_0^1 \Big[\frac{1}{(1-\gamma_0 \tau \|x_1 - x_0\|)^2} - 1 \Big] \mathrm{d}\theta \|x_1 - x_0\| \\ &\leq \frac{\gamma_0 \|x_1 - x_0\|^2}{1 - \gamma_0 \|x_1 - x_0\|} \leq \frac{\gamma_0 (s_1 - s_0)}{1 - \gamma_0\, s_1}. \end{aligned}$$

Then, using (2.9) for $k = 1, 2, \ldots$, we get

$$\begin{aligned} &\| F'(x_0)^{-1}\,F(x_{k+1}) \| \\ &\leq \int_0^1 \| F'(x_0)^{-1}\,(F'(x_k^\tau) - F'(x_k)) \| \; d\tau \; \| x_{k+1} - x_k \| \\ &\leq \int_0^1 \int_0^1 \frac{2\,\gamma\,\tau\,ds\,d\tau \;\; \| x_{k+1} - x_k \|^2}{(1-\gamma \;\| x_k^{\tau\,s} - x_0 \|)^3} \\ &\leq \int_0^1 \int_0^1 \frac{2\,\gamma\,\tau\,ds\,d\tau \;\; \| x_{k+1} - x_k \|^2}{(1-\gamma \;\| x_k - x_0 \| - \gamma\,\tau\,s \;\| x_{k+1} - x_k \|)^3} \\ &= \frac{\gamma \;\| x_{k+1} - x_k \|^2}{(1-\gamma \;\| x_k - x_0 \| - \gamma \;\| x_{k+1} - x_k \|)\,(1-\gamma \;\| x_k - x_0 \|)^2} \\ &\leq \frac{\gamma\,(s_{k+1} - s_k)^2}{(1-\gamma\,s_{k+1})\,(1-\gamma\,s_k)^2} \left(\frac{\| x_{k+1} - x_k \|}{s_{k+1} - s_k} \right)^2 \leq \frac{\gamma\,(s_{k+1} - s_k)^2}{(1-\gamma\,s_{k+1})\,(1-\gamma\,s_k)^2}. \end{aligned} \tag{2.32}$$

(see also [16, p. 33, estimate (3.19)]) Then, in view of (1.2), (2.19), (2.30) and (2.32) we obtain that

$$\begin{aligned}\|x_2 - x_1\| &\leq \Big\|F'(x_1)^{-1}F'(x_0)\Big\|\Big\|F'(x_0)^{-1}F(x_1)\Big\| \\ &\leq \frac{1}{2 - \dfrac{1}{(1-\gamma_0 s_1)^2}} \frac{\gamma_0 (s_1 - s_0)^2}{1 - \gamma_0 s_1} = s_2 - s_1\end{aligned}$$

and furthermore for $k = 1, 2, \ldots$ we obtain

$$\begin{aligned}&\| x_{k+2} - x_{k+1} \| = \| (F'(x_{k+1})^{-1}\, F'(x_0))\, (F'(x_0)^{-1}\, F(x_{k+1})) \| \\ &\leq \| F'(x_{k+1})^{-1}\, F'(x_0) \| \, \| F'(x_0)^{-1}\, F(x_{k+1}) \| \\ &\leq \frac{1}{2 - \dfrac{1}{(1-\gamma_0\, s_{k+1})^2}} \frac{\gamma\, (s_{k+1} - s_k)^2}{(1 - \gamma\, s_{k+1})\, (1-\gamma\, s_k)^2} = s_{k+2} - s_{k+1}.\end{aligned} \tag{2.33}$$

Hence, we showed (2.27) holds for all $k \geq 0$. Furthermore, let $w \in \overline{U}(x_{k+2}, s^\star - s_{k+2})$. Then, we have that

$$\begin{aligned}\| w - x_{k+1} \| &\leq \| w - x_{k+2} \| + \| x_{k+2} - x_{k+1} \| \\ &\leq s^\star - s_{k+2} + s_{k+2} - s_{k+1} = s^\star - s_{k+1}.\end{aligned} \tag{2.34}$$

That is $w \in \overline{U}(x_{k+1}, s^\star - s_{k+1})$. The induction for (2.27) and (2.28) is now completed.

Lemma 2.5 implies that $\{s_n\}$ is a complete sequence. It follows from (2.27) and (2.28) that $\{x_n\}$ is also a complete sequence in a Banach space $\mathbf{X}$ and as such it converges to some $x^\star \in \overline{U}(x_0, s^\star)$ (since $\overline{U}(x_0, s^\star)$ is a closed set).

By letting $k \longrightarrow \infty$ in (2.32) we get $F(x^\star) = 0$. Estimate (2.26) is obtained from (2.25) by using standard majorization techniques [2, 7, 12, 13].

Finally, to show the uniqueness part, let $y^\star \in U(x_0, s^\star)$ be a solution of equation (1.1). Using (2.3) for x replaced by $z^\star = x^\star + \tau\, (y^\star - x^\star)$ and $\mathcal{G} = \displaystyle\int_0^1 F'(z^\star)\, d\tau$ we get as in (2.9) that $\| F'(x_0)^{-1}\, (\mathcal{G} - F'(x_0)) \| < 1$. That is $\mathcal{G}^{-1} \in \mathbb{L}(\mathbf{Y}, \mathbf{X})$.

Using the identity $0 = F(x^\star) - F(y^\star) = \mathcal{G}\, (x^\star - y^\star)$, we deduce $x^\star = y^\star$. $\square$

Remark 2.10. (a) The convergence criteria in Theorem 2.9 are weaker than in Theorem 1.1. In particular, Theorem 1.1 requires that operator F is twice Fréchet-differentiable but our Theorem 2.9 requires only that F is Fréchet-differentiable.

Notice also that if F is twice Fréchet-differentiable, then (2.2) implies (1.3). Therefore, Theorem 2.9 can apply in cases when Theorem 1.1 cannot.

Notice also in practice the computation of constant γ requires the computation of constant γ_0 as a special case.

(b) Concerning the choice of constants γ and γ_0 let us suppose that the following Lipschitz conditions hold. Operator F satisfies $L-$Lipschitz condition at x_0

$$\Big\|F'(x_0)^{-1}\Big[F'(x) - F'(y)\Big]\Big\| \leq L\|x - y\| \quad \text{for each} \quad x, y \in U(x_0, R_0). \tag{2.35}$$

Operator F satisfies the center L_0-Lipschitz condition at x_0

$$\left\|F'(x_0)^{-1}\Big[F'(x)-F'(x_0)\Big]\right\| \leq L_0\|x-x_0\| \quad \text{for each} \quad x \in U(x_0,R_0). \tag{2.36}$$

Then, (2.35) implies (2.3) for $\gamma_0 = L_0/2$ and $l_0(r) = \gamma_0(2-\gamma_0\, r)/(1-\gamma_0\, r)^2$. Moreover, if F is continuously twice-Fréchet differentiable, the (2.34) implies (2.2)for $l(r) = 2\gamma/(1-\gamma\, r)^3$ and we can set $\gamma = L/2$. Examples where $\gamma_0 < \gamma$ or $L_0 < L$ can be found in [2, 3, 4, 5, 6, 7].

Proposition 2.11. *Let $F\,:\,\overline{U}(x_0,R) \longrightarrow \mathbf{Y}$ be twice Fréchet-differentiable on $U(x_0,R)$. Suppose that hypotheses of Theorem* 1.1 *and the center-Lipschitz condition (2.3) hold on $\overline{U}(x_0,r_0)$. Then, the following assertions hold*

(a) Scalar sequences $\{t_n\}$ and $\{s_n\}$ are increasingly convergent to $t^\star$, $s^\star$, respectively.

(b) Sequence $\{x_n\}$ generated by Newton's method is well defined, remains in $\overline{U}(x_0,r_0)$ for each $n=0,1,\cdots$ and converges to a unique solution $x^\star \in \overline{U}(x_0,r_0)$ of equation $F(x)=0$. Moreover, the following estimates hold for each $n=0,1,\cdots$

$$s_n \leq t_n, \tag{2.37}$$

$$s_{n+1}-s_n \leq t_{n+1}-t_n, \tag{2.38}$$

$$s^\star \leq t^\star, \tag{2.39}$$

$$\| \; x_{n+1}-x_n \; \| \leq s_{n+1}-s_n$$

and

$$\| \; x_n - x^\star \; \| \leq s^\star - s_n.$$

Proof. According to Theorems 1.1 and 2.9 we only need to show (2.37)–(2.39). Using the definition of sequences $\{t_n\}$, $\{s_n\}$ and $\gamma_0 \leq \gamma$, a simple inductive argument shows (2.37) and (2.38). Finally, (2.39) is obtained by letting $n \longrightarrow \infty$. □

Remark 2.12. (a) In view of (2.37)–(2.39), our error analysis is tighter and new information on the location of the solution $x^\star$ at least as precise as the old one. Notice also that estimates (2.37) and (2.38) hold as strict inequalities for $n>1$ if $\gamma_0 < \gamma$ (see also the numerical example) and these advantages hold under the same or less computational cost as before (see Remark 2.10).

(b) If F is an analytic operator, then a possible choice for γ_0(or γ) is given by

$$\gamma_0 = \sup_{n>1}\left\|\frac{F'(x_0)^{-1}F(x_0)^n}{n!}\right\|^{\dfrac{1}{n-1}}$$

This choice is due to Smale [17] (see also [15, 16, 17, 18, 19, 20]).

We complete this section with an useful and obvious extension.

Theorem 2.13. *Suppose there exists an integer $N \geq 1$ such that*

$$s_0 < s_1 < \cdots < s_N < R_0 = \min\left\{\frac{1}{\gamma}, \left(1-\frac{1}{\sqrt{2}}\right)\frac{1}{\gamma_0}\right\}.$$

Let $\delta_N = \gamma\,\beta_N$ and $\beta_N = t_N - t_{N-1}$. Conditions of Lemma 2.8 are satisfied for δ_N replacing δ. Then, the conclusions of Theorem 2.9 hold. Notice that if $N = 1$ Theorem 2.13 reduces to Theorem 2.9.

3. Numerical example

We illustrate the theoretical results with a numerical example.

Example 3.1. Let $\mathbf{X} = \mathbf{Y} = \mathbb{R}^2$, $x_0 = (1, 0)$, $D = \overline{U}(x_0, 1-\kappa)$ for $\kappa \in (0,1)$. Let us define function F on D as follows

$$F(x) = (\zeta_1^3 - \zeta_2 - \kappa, \zeta_1 + 3\,\zeta_2 - \sqrt[3]{\kappa}) \quad \text{with} \quad x = (\zeta_1, \zeta_2). \tag{3.1}$$

Using (3.1) we see that the γ-Lipschitz condition is satisfied for $\gamma = 2 - \kappa$ and γ_0-Lipschitz condition is satisfied for $\gamma_0 = (3-\kappa)/2$. We also have that $\beta = (1-\kappa)/3$.

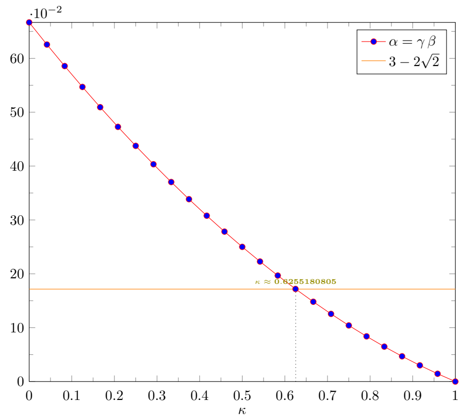

Figure 1. The condition (1.5) of the Theorem 1.1 [20].

The Figure 3.1 plots the condition (1.5) of the Theorem 1.1 [20]. In the Figure 3.1, we notice that for $\kappa \le 3/2 - \left(\sqrt{37 - 24\sqrt{2}}\right)/2$, the condition (1.5) fails. As a consequence the Theorem 1.1 [20] is not applicable. Thus according to the Theorem 1.1 [20] there is no guarantee that the Newton's method starting from x_0 will converge to the solution $x^\star = (\sqrt[3]{\kappa}, 0)$.

To compare the error bounds for the Theorem 1.1 and the Lemma 2.8, we consider $\kappa = 0.7$. From the Figure 3.1, it is clear that the condition (1.5) holds as a result the Theorem 1.1 is applicable. For the hypotheses (2.12) and (2.15) of Lemma 2.8 we obtain

$$0.1000000000 < 0.2202707044,$$

$$0.1179672 < 0.1280403078 < 0.2202707044$$

respectively. Thus our Lemma 2.8 is applicable and the Newton's method starting at $x_0 = (1, 0)$ will converge to the solution $x^\star = (\sqrt[3]{\kappa}, 0)$ for $\kappa = 0.7$. Now we compare the error bounds generated by the sequence $\{t_n\}$ given in (1.10) and the sequence $\{s_n\}$ defined in (2.19).

Table 1. Comparison between the sequences $\{s_n\}$ (2.19) and $\{t_n\}$ (1.10) [20].

n	s_n	t_n	$s_{n+1} - s_n$	$t_{n+1} - t_n$
0	$0.000000 \times 10^{+00}$	$0.000000 \times 10^{+00}$	1.000000×10^{-01}	1.000000×10^{-01}
1	1.000000×10^{-01}	1.000000×10^{-01}	1.796716×10^{-02}	2.201246×10^{-02}
2	1.179672×10^{-01}	1.220125×10^{-01}	9.900646×10^{-04}	1.683820×10^{-03}
3	1.189572×10^{-01}	1.236963×10^{-01}	3.196367×10^{-06}	1.069600×10^{-05}
4	1.189604×10^{-01}	1.237070×10^{-01}	3.341751×10^{-11}	4.338887×10^{-10}
5	1.189604×10^{-01}	1.237070×10^{-01}	3.652693×10^{-21}	7.140132×10^{-19}
6	1.189604×10^{-01}	1.237070×10^{-01}	4.364065×10^{-41}	1.933579×10^{-36}
7	1.189604×10^{-01}	1.237070×10^{-01}	6.229418×10^{-81}	1.417992×10^{-71}
8	1.189604×10^{-01}	1.237070×10^{-01}	$1.269287 \times 10^{-160}$	$7.626002 \times 10^{-142}$
9	1.189604×10^{-01}	1.237070×10^{-01}	$5.269686 \times 10^{-320}$	$2.205685 \times 10^{-282}$

In the Table 1, we notice that the error bounds given by the proposed sequence $\{s_n\}$ are tighter than those given by the older sequence $\{t_n\}$ [20].

Conclusions. Using the notion of the center γ_0-Lipschitz condition, we presented a new convergence analysis for Newton's method for approximating a locally unique solution of nonlinear equation in a Banach space setting. Under the same computational cost – as in earlier studies such as [15, 16, 17, 18, 19, 20] – new analysis provide larger convergence domain, weaker sufficient convergence conditions and better error bounds. A numerical example validating the theoretical results is also reported in this study.

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