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Coincidence point and fixed point theorems for rational contractions

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Abstract. The purpose of this work is to present some coincidence point theorems for singlevalued and multivalued rational contractions. A comparative study of different rational contraction conditions is also presented. Our results extend some recent theorems in the literature.

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1. Introduction

In this first section, for the convenience of the reader, we will recall the standard terminologies and notations in non-linear analysis. See, for example [4], [11], [6], [9].

Let (X, d) be a metric space, $x_0 \in X$ and $r > 0$.

Denote $\widetilde{B}(x_0, r) := \{x \in X | d(x_0, x) \leq r\}$ the closed ball centered at x_0 with radius r.

If $S : X \to X$ is an operator, then we denote by $F(S) := \{x \in X | x = S(x)\}$ the fixed point set of S.

An operator $f : Y \subseteq X \to Y$ is said to be an α-contraction if $\alpha \in [0, 1]$ and $d(f(x), f(y)) \leq \alpha d(x, y)$, for all $x, y \in Y$.

Definition 1.1. *Let $(X, \leq)$ be an partially ordered set and A, B be two nonempty subsets of X. Then we will wrote $A \leq_s B$ if and only for all $a \in A$ exists $b \in B$ satisfying $a \leq b$.*

We denote by $P(X)$ the family of all nonempty subsets of X. Also $P_p(X)$ will denote the family of all nonempty subsets of X having the property "p", where "p"

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could be: b = bounded, cl = closed, cp = compact etc. We consider the following functionals:

$$D : P(X) \times P(X) \to \mathbb{R}_+, \quad D(A, B) = \inf\{d(a, b) | a \in A, b \in B\}$$
$$\rho : P_b(X) \times P_b(X) \to \mathbb{R}_+, \quad \rho(A, B) = \{sup\{D(a, B) | a \in A\}$$
$$H : P_b(X) \times P_b(X) \to \mathbb{R}_+, \quad H(A, B) = \max\{\sup_{a\in A} D(a, B), \sup_{b\in B} D(b, A)\}.$$

Definition 1.2. *Let $(X, \preceq)$ be a partially ordered set and $T : X \to P(X)$ be a multivalued mapping, satisfying the following implication*

$$x \preceq y \Rightarrow Tx \preceq_s Ty.$$

Then T is said to be increasing.

Definition 1.3. ([6]) *A function $\psi : \mathbb{R}_+ \to \mathbb{R}_+ := [0, +\infty)$ is called an altering distance function if the following properties are satisfied:*
(Ψ_1) $\psi(t) = 0 \Leftrightarrow t = 0$.
(Ψ_2) ψ is monotonically non-decreasing.
(Ψ_3) ψ is continuous.
By Ψ we denote the set of all altering distance functions.

The following theorem is an result proved by B.K. Das and S Gupta, in 1975.

Theorem 1.4. *Let (X, d) be a metric space and let $S : X \to X$ be a given mapping such that,*

i) there exist $a, b \in \mathbb{R}^*_+$ with $a + b < 1$ for which $d(Sx, Sy) \leq ad(x, y) + bm(x, y)$ for all $x, y \in X$ where

$$m(x, y) = d(y, Sy)\frac{1 + d(x, Sx)}{1 + d(x, y)}.$$

ii) there exists $x_0 \in X$, such that the sequence of iterates $(S^n x_0)$ has a subsequence $(S^{n_k} x_0)$ with $\lim_{k\to\infty} (S^{n_k} x_0) = z_0$. Then z_0 is the unique fixed point of S.

Definition 1.5. *Let S be a self mapping of a metric space (M, d) with a nonempty fixed point set $F(S)$. Then S is said to satisfy the property (P) if $F(S) = F(S^n)$ for each $n \in \mathbb{N}$.*

Definition 1.6. *Let $(X, \preceq)$ be a partially ordered set endowed with a metric d on X. We say that X is regular if and only if the following hypothesis holds:*
If $\{z_n\}$ is an non-decreasing sequence in X with respect to $\preceq$ such that $\lim_{n\to\infty} z_n = z \in X$ then $z_n \preceq z$ for all $n \in \mathbb{N}$.

Definition 1.7. *Let (X, d) a complete metric space, with $T : X \to P_{cl}(X)$ and $R : X \to X$. Then $C(R, T) = \{x \in X | Rx \in Tx\}$ is called the coincidence point set of S and T. We say that a point $x \in X$ is a coincidence point of R and T if $Rx = Tx$.*

We will denote by $F(T)$ the fixed point set for T and by $SF(T)$ the strict fixed point set of T.

If Y is a nonempty subset of X and $T : Y \to P(X)$ is a multivalued operator, then by definition, an element $x \in Y$ is said to be:

(i) a fixed point of T if and only if $x \in T(x)$;
(ii) a strict fixed point of T if and only if $x = T(x)$.
The following result appeared in [9].

Theorem 1.8. ([9]) *Let $(X, \preceq)$ be a partially ordered set equipped with a metric d on X such that (X, d) is a complete metric space. Let $T, R : X \to X$ be two mappings satisfying(for pair $(x, y) \in X \times X$ where in Rx and Ry are comparable),*

$$d(Tx, Ty) \leq \frac{\alpha d(Rx, Tx) \cdot d(Ry, Ty)}{1 + d(Rx, Ry)} + \beta d(Rx, Ry) \tag{1.1}$$

where α, β are non-negative real numbers with $\alpha + \beta < 1$. Suppose that
a) X is regular and T is weakly increasing with R.
b) the pair (R, T) is commuting and weakly reciprocally continuous.
Then R *and* T *have a coincidence point.*

On the other hand, in [2] the following local fix point theorem for multivalued contraction is given.

Theorem 1.9. *Let (X, d) be a complete metric space, $x_0 \in X$ and $r > 0$. Let $T : \widetilde{B}(x_0; r) \to P_{cl}(X)$ be a multivalued α- contraction such that $D(x_0, T(x_0)) < (1-\alpha)r$. Then $F(T) \neq \emptyset$.*

We also mention that the following fixed point theorem, for the so called multivalued rational contractions was presented in [10], as follows.

Theorem 1.10. *Let (X, d) a complete metric space and $T : X \to P_{cl}(X)$ be a multivalued operator such that exists $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$ satisfying*

$$H(Tx, Ty) \leq \frac{\alpha D(y, Ty)[1 + D(x, Tx)]}{1 + d(x, y)} + \beta d(x, y), \text{ for all } x, y \in X. \tag{1.2}$$

Then T *has a fixed point.*

The purpose of this paper is twofold. First we will extend Theorem 1.8 for the case of multivalued operators. Secondly, we will present a local fixed point theorem for multivalued rational conractions.

2. Main results

Our first main result is the following coincidence point theorem.

Theorem 2.1. *Let (X, d) be a complete metric space. Let $T : X \to P_{cl}(X)$ and $R : X \to X$ be two operators satisfying*

$$\rho(Tx, Ty) \leq \frac{\alpha D(Ry, Ty)[1 + D(Rx, Tx)]}{1 + d(Rx, Ry)} + \beta d(Rx, Ry), \forall x, y \in X \tag{2.1}$$

where α, β are some non-negative real numbers with $\alpha + \beta < 1$. Suppose that R is continuous and $T(X) \subset R(X)$. Then R and T have a coincidence point.

Proof. Let $x_0 \in X$ be arbitrary. Since $T(x_0) \subset T(X) \subset R(X)$, there exists $x_1 \in X$ such that $R(x_1) \in T(x_0)$. For $R(x_1) \in T(x_0)$ and $T(x_1)$, by well-known property of the functional ρ, for any $q > 1$, there exists $u_1 \in T(x_1)$ such that

$$d(Rx_1, u_1) \le q\rho(Tx_0, Tx_1).$$

Since $u_1 \in T(x_1) \subset T(X) \subset R(X)$ there exists $x_2 \in X$ such that $u_1 = R(x_2) \in T(x_1)$. Thus

$$d(Rx_1, Rx_2) \le q\rho(Tx_0, Tx_1) \le q\left[\frac{\alpha D(Rx_1, Tx_1)[1 + D(Rx_0, Tx_0)]}{1 + d(Rx_0, Rx_1)} + \beta d(Rx_0, Rx_1)\right]$$

$$\le q\left[\frac{\alpha d(Rx_1, Rx_2)[1 + d(Rx_0, Rx_1)]}{1 + d(Rx_0, Rx_1)} + \beta d(Rx_0, Rx_1)\right].$$

Hence

$$(1 - q\alpha)d(Rx_1, Rx_2) \le q\beta d(Rx_0, Rx_1)$$

and so

$$d(Rx_1, Rx_2) \le \frac{q\beta}{1 - q\alpha} d(Rx_0, Rx_1).$$

Now, for $R(x_2) \in T(x_1)$ and $T(x_2)$, for the same arbitrary $q > 1$, there exists $u_2 \in T(x_2)$ such that

$$d(Rx_2, u_2) \le q\rho(Tx_1, Tx_2).$$

Again, since $u_2 \in T(x_2) \subset T(X) \subset R(X)$ there exists $x_3 \in X$ such that $u_2 = R(x_3) \in T(x_2)$. In this case, by a similar procedure, we obtain

$$d(Rx_2, Rx_3) \le \frac{q\beta}{1 - q\alpha} d(Rx_1, Rx_2) \le \left(\frac{q\beta}{1 - q\alpha}\right)^2 d(Rx_0, Rx_1).$$

By this procedure, we obtain a sequence $u_n := R(x_{n+1}) \in T(x_n), n \in \mathbb{N}^*$ such that

$$d(Rx_n, Rx_{n+1}) \le q\rho(Tx_{n-1}, Rx_n)$$

and

$$d(Rx_n, Rx_{n+1}) \le \left(\frac{q\beta}{1 - q\alpha}\right)^n d(Rx_0, Rx_1). \tag{2.2}$$

By choosing $1 < q < \frac{1}{\alpha+\beta}$, we obtain thus $r := \frac{q\beta}{1-q\alpha} < 1$.

By (2.2) we get that the sequence $(Rx_n)_{n\in\mathbb{N}^*}$ is Cauchy in the complete metric space (X, d). Thus, there exists x^* such that $Rx_n \to x^*, n \to \infty$. We will show that x^* is a coincidence point for R and T (i.e. $Rx^* \in Tx^*$).

We estimate

$$D(Rx^*, Tx^*) = \inf_{y\in Tx^*} d(Rx^*, y) \le d(Rx^*, R(Rx_n)) + \inf_{y\in Tx^*} d(R(Rx_n), y)$$

$$\le d(Rx^*, R(Rx_n)) + D(Rx_{n+1}, Tx^*) \le d(Rx^*, R(Rx_n)) + \rho(Tx_n, Tx^*)$$

$$\le d(Rx^*, R(Rx_n)) + \frac{\alpha D(Rx^*, Tx^*)[1 + D(Rx_n, Tx_n)]}{1 + D(Rx_n, Rx^*)} + \beta d(Rx_n, Rx^*)$$

$$\le d(Rx^*, R(Rx_n)) + \frac{\alpha D(Rx^*, Tx^*)[1 + d(Rx_n, Rx_{n+1})]}{1 + d(Rx_n, Rx^*)} + \beta d(Rx_n, Rx^*)$$

Letting $n \to \infty$ and R continuous, we obtain

$$D(Rx^*, Tx^*) \le \alpha D(Rx^*, Tx^*)$$
$$(1-\alpha)D(Rx^*, Tx^*) \le 0.$$

Since $\alpha, \beta > 0$, then T and R has a coincidence point. □

In the next paragraph we will prove Theorem 1.6 using Theorem 1.7 condition.

Theorem 2.2. *Let (X, d) be a complete metric space, $x_0 \in X$ and $r > 0$. Let $T : \widetilde{B}(x_0; r) \to P_{cl}(X)$ be a multivalued operator for which there exist $\alpha, \beta \in \mathbb{R}_+^*$ with $\alpha + \beta < 1$ such that*

$$H(Tx, Ty) \le \frac{\alpha D(y, Ty)[1 + D(x, Tx)]}{1 + d(x, y)} + \beta d(x, y), \text{ for all } x, y \in X. \tag{2.3}$$

We also suppose that $D(x_0, Tx_0) < \left(\frac{1-\alpha-\beta}{1-\alpha}\right) r$. Then $F(T) \neq \emptyset$.

Proof. We will inductively construct a sequence $x_n \subset \widetilde{B}(x_0; r)$ such that

i) $x_n \in Tx_{n+1}$, $\forall n \in \mathbb{N}^*$

ii) $d(x_n, x_{n-1}) < k^{n-1} r$. We denote by $k = \frac{\beta}{1-\alpha} \in [0, 1)$.

From the condition $D(x_0, Tx_0) < (\frac{1-\alpha-\beta}{1-\alpha}) r$ we have that exists $x_1 \in T(x_0)$ such that $d(x_0, x_1) < (1-k)r$. Suppose that we construct $x_1, x_2, ..., x_n \in \widetilde{B}(x_0, r)$ with properties i) and ii), now we have to prove the existence of x_{n+1}. We have

$$\begin{aligned} H(Tx_{n-1}, Tx_n) &\le \frac{\alpha D(x_n, Tx_n)[1 + D(x_{n-1}, Tx_{n-1})]}{1 + d(x_{n-1}, x_n)} + \beta d(x_{n-1}, x_n) \\ &\le \frac{\alpha D(x_n, Tx_n)[1 + d(x_{n-1}, x_n)]}{1 + d(x_{n-1}, x_n)} + \beta d(x_{n-1}, x_n) \\ &= \alpha D(x_n, Tx_n) + \beta d(x_{n-1}, x_n) < \alpha H(Tx_{n-1}, Tx_n) + \beta d(x_{n-1}, x_n) \end{aligned}$$

$$\begin{aligned} H(Tx_{n-1}, Tx_n) &\le \frac{\beta}{1-\alpha} d(x_{n-1}, x_n) \le \left(\frac{\beta}{1-\alpha}\right)^n d(x_0, x_1) \\ &< \left(\frac{\beta}{1-\alpha}\right)^n \left(1 - \frac{\beta}{1-\alpha}\right) r. \end{aligned}$$

This proves that $x_{n+1} \in Tx_n$ such that

$$d(x_{n+1}, x_n) < \left(\frac{\beta}{1-\alpha}\right)^n \left(1 - \frac{\beta}{1-\alpha}\right) r,$$

so using k we will have $d(x_{n+1}, x_n) < k^n (1-k) r$.

Moreover, we have

$$\begin{aligned} d(x_{n+p}, x_n) &\le (1 + k + ... + k^{p-1}) k^n (1-k) r \\ &\le \frac{k^p}{1-k} k^n (1-k) r \to 0 \quad as \quad n, p \to \infty. \end{aligned} \tag{2.4}$$

Therefore $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, with $\lim_{n\to\infty} x_n = x_0^* \in \widetilde{B}(x_0, r)$. Because T is closed we obtain

$$D(x_0^*, Tx_0^*) \le d(x_0^*, x_{n+1}) + H(Tx_n, Tx_0^*)$$

$$\leq d(x_0^*, x_{n+1}) + \frac{\alpha D(x_0^*, Tx_0^*)[1 + D(x_n, Tx_n)]}{1 + d(x_n, x_0^*)} + \beta d(x_n, x_0^*)$$

$$\leq d(x_0^*, x_{n+1}) + \frac{\alpha D(x_0^*, Tx_0^*)[1 + d(x_n, x_{n+1})]}{1 + d(x_n, x_0^*)} + \beta d(x_n, x_0^*).$$

Letting $n \to \infty$, we have

$$D(x_0^*, Tx_0^*) \leq \alpha D(x_0^*, Tx_0^*).$$

This proves that x_0^* is a fixed point of Tx_0^*. □

The next part of this section, is devoted to generalize Theorem 1.4 to the case of multivalued operators.

Theorem 2.3. *Let (X, d) be a complete metric space, let $\psi \in \Psi$ and $T : X \to P_{cl}(X)$ be a multivalued operator for which there exist $\alpha, \beta \in \mathbb{R}_+^*$ with $\alpha + \beta < 1$ such that*

$$\psi[H(Tx, Ty)] \leq \alpha\psi[m(x, y)] + \beta\psi[d(x, y)], \text{ for all } x, y \in X \tag{2.5}$$

where

$$m(x, y) = D(y, Ty)\frac{1 + D(x, Tx)}{1 + d(x, y)}. \tag{2.6}$$

Then T has a fixed point $x^* \in X$, and there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ with $x_0 \in X$ and $x_{n+1} \in T(x_n), n \in \mathbb{N}$ such that $\lim_{n\to\infty} x_n = x^*$.

Proof. Let $x_0 \in X$ be arbitrary chosen and let (x_n) be a sequence defined as follows: $x_{n+1} \in Tx_n \subset T^{n+1}x_0$,for each $n \geq 1$. Now,

$$\psi[d(x_n, x_{n+1})] \leq \psi[qH(Tx_{n-1}, Tx_n)] \leq q\alpha\psi[m(x_{n-1}, x_n)] + q\beta\psi[d(x_{n-1}, x_n] \tag{2.7}$$

using (2.6),

$$m(x_{n-1}, x_n) = D(x_n, Tx_n)\frac{1 + D(x_{n-1}, Tx_{n-1})}{1 + d(x_{n-1}, x_n)}$$

$$\leq d(x_n, x_{n+1})\frac{1 + d(x_{n-1}, x_n)}{1 + d(x_{n-1}, x_n)} = d(x_n, x_{n+1}).$$

Substituting it into (2.7), it follows that,

$$\psi[d(x_n, x_{n+1}) \leq q\alpha\psi[d(x_n, x_{n+1})] + q\beta\psi[d(x_{n-1}, x_n)]$$

so we have,

$$\psi[d(x_n, x_{n+1})] \leq \frac{q\beta}{1 - q\alpha}\psi[d(x_{n-1}, x_n)]$$

$$\leq \left(\frac{q\beta}{1 - q\alpha}\right)^2 \psi[d(x_{n-2}, x_{n-1})] \leq ... \tag{2.8}$$

$$\leq \left(\frac{q\beta}{1 - q\alpha}\right)^n \psi[d(x_0, x_1)] \tag{2.9}$$

Since $r = \frac{q\beta}{1-q\alpha} \in (0, 1)$, from (2.8) we obtain

$$\lim_{n\to\infty} \psi[d(x_n, x_{n+1})] = 0.$$

From the fact that $\psi \in \Psi$, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Now we will show that (x_n) is a Cauchy sequence. Using (2.9), moreover, for $n < m$, we have

$$\psi[d(x_n, x_m)] \leq \psi[d(x_{n-1}, x_n)] + ... + \psi[d(x_{m-1}, x_m)] \leq (r^n + ... + r^{m-1})\psi[d(x_0, x_1)]$$
$$\leq \frac{r^n}{1-r}\psi[d(x_0, x_1)] \to 0 \quad as \quad n, m \to \infty. \tag{2.10}$$

Therefore (x_n) is a Cauchy sequence. Since (X, d) is a complete metric space, we get that $x \in X$ $\lim_{n\to\infty} x_n = x^*$.

$$\psi[D(x^*, Tx^*)] = \psi[\inf_{y\in Tx^*} d(x^*, y)] \leq \psi[d(x^*, x_{n+1})] + \psi[\inf_{y\in Tx^*} d(x_{n+1}, y)]$$
$$\leq \psi[d(x^*, x_{n+1})] + \psi[H(Tx_n, Tx^*)]$$
$$\leq \psi[d(x^*, x_{n+1})] + \alpha\psi[m(x_n, x^*)] + \beta\psi[d(x_n, x^*)]$$
$$\leq \psi[d(x^*, x_{n+1})] + \alpha\psi[D(x^*, Tx^*)\frac{1 + D(x_n, Tx_n)}{1 + d(x_n, x^*}))] + \beta\psi[d(x_n, x^*)$$
$$\leq \psi[d(x^*, x_{n+1})] + \alpha\psi[D(x^*, Tx^*)\frac{1 + d(x_n, x_{n+1})}{1 + d(x_n, x^*)}] + \beta\psi[d(x_n, x^*).$$

Letting $n \to \infty$ we obtain

$$\psi[D(x^*, Tx^*)](1-\alpha) \leq 0.$$

Since $\psi \in \Psi$, we have $D(x^*, Tx^*) = 0$. This proves that $x^* \in F_T$. □

As a consequence, we obtain the following fixed point theorem.

Corollary 2.4. *Let (X, d) be a complete metric space and let $T : X \to P_{cl}(X)$ be a multivalued operator. We assume that for each $x, y \in X$,*

$$\int_0^{H(Tx,Ty)} \varphi(t)dt \leq \alpha \int_0^{D(y,Ty)\frac{1+D(x,Tx)}{1+d(x,y)}} \varphi(t)dt + \beta \int_0^{d(x,y)} \varphi(t)dt \tag{2.11}$$

where $0 < \alpha + \beta < 1$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, is a Lebesque integrable operator which is summable on each compact subset of $[0, +\infty)$, non negative and such that $\int_0^\varepsilon \varphi(t)dt > 0$ for all $\epsilon > 0$. Then T admits a fixed point $x^* \in X$ such that for each $x \in X$

$$\lim_{n\to\infty} x^n = x^*, x_n \in T^n x.$$

Proof. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, be as in the corollary, we define

$$\psi_0(t) = \int_0^t \varphi(t)dt, \ t \in \mathbb{R}_+.$$

ψ_0 is monotonically non decreasing and by hypothesis ψ_0 is continuous. Therefore, $\psi_0 \in \Psi$. So the condition (2.11) becomes

$$\psi_0[H(Tx,Ty)] \leq \alpha\psi_0 \left[D(y,Ty)\frac{1+D(x,Tx)}{1+d(x,y)}\right] + \beta\psi_0[d(x,y)]\forall x,y \in X.$$

So, from Theorem 2.3 we have that exists $x^* \in X$ such that for each $x^* \in F(T)$ and there exist a sequence$(x_n)_{n\in\mathbb{N}} \subset X$ with $x_0 \in X$ and $x_{n+1} \in T(x_n), n \in \mathbb{N}$ such that $\lim\limits_{n\to\infty} x_n = x^*$. □

Example 2.5. Let $X = \{(0,0,0),(0,0,1),(1,0,0)\}$ be endowed with the metric d. Consider the multivalued operator $T : X \to P_{cl}(X)$ and a singlevalued operator $R : X \to X$ defined by

$$T(x) = \begin{cases} \{(1,0,0)\}, & \text{if} \quad x = (0,0,1) \\ \{(0,0,0)\}, & \text{if} \quad x = (0,0,0) \\ \{(0,0,0),(1,0,0)\}, & \text{if} \quad x = (1,0,0) \end{cases}$$

$$R(x) = \begin{cases} \{(1,0,0)\}, & \text{if} \quad x = (0,0,1) \\ \{(0,0,0)\}, & \text{if} \quad x = (0,0,0) \\ \{(0,0,1)\}, & \text{if} \quad x = (1,0,0) \end{cases}$$

Then $F_T = \{(0,0,0),(1,0,0)\}$, $F_R = \{(0,0,0)\}$, $C(R,T) = \{(0,0,1),(0,0,0)\}$ and Theorem 2.1 is verified for $\alpha = \frac{1}{9}, \beta = \frac{7}{8}$, $\alpha + \beta < 1$.

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