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Iterates of increasing linear operators, via Maia's fixed point theorem

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Abstract. Let X be a Banach lattice. In this paper we give conditions in which an increasing linear operator, $A : X \to X$ is weakly Picard operator (see I.A. Rus, *Picard operators and applications*, Sc. Math. Japonicae, 58(2003), No. 1, 191-219). To do this we introduce the notion of "invariant linear partition of X with respect to A" and we use contraction principle and Maia's fixed point theorem. Some applications are also given.

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1. Introduction

There are many techniques to study the iterates of a linear and of increasing linear operators:

(1) for linear operators on a Banach space see: [16], [22], [23], [25], ...
(2) for linear increasing operators on an ordered Banach space see: [4], [8], [11], [12], [21], [23], [38], ...
(3) for some classes of positive linear operators see: [1]-[6], [9], [13]-[15], [17]-[20], [27], [30], [33], [35], ...

In the paper [36] we studied the problem in terms of the following notions:

Definition 1.1. *Let X be a nonempty set and $A : X \to X$ be an operator with $F_A \neq \emptyset$, where $F_A := \{x \in X \mid A(x) = x\}$. By definition, a partition of X, $X = \bigcup_{x^* \in F_A} X_{x^*}$, is a <u>fixed point partition</u> of X with respect to A iff:*

(*i*) $A(X_{x^*}) \subset X_{x^*}$, $\forall\, x^* \in F_A$;
(*ii*) $F_A \cap X_{x^*} = \{x^*\}$, $\forall\, x^* \in F_A$.

Definition 1.2. *Let $(X, +, \mathbb{R})$ be a linear space and $A : X \to X$ be a linear operator with $F_A \setminus \{\theta\} \neq \emptyset$. By definition, a fixed point partition, $X = \bigcup_{x^* \in F_A} X_{x^*}$ is a* <u>*linear fixed point partition*</u> *of X with respect to A iff:*

$$X_{x^*} = \{x^*\} + X_\theta, \ \forall \ x^* \in F_A.$$

If there exists a norm on X_θ, $\|\cdot\| : X_\theta \to \mathbb{R}_+$, and $\|A(x)\| \leq l\|x\|$, for all $x \in X_\theta$ with some $l > 0$, then $d_{\|\cdot\|} : X_{x^*} \times X_{x^*} \to \mathbb{R}_+$, $d_{\|\cdot\|}(x, y) := \|x - y\|$ is a metric on X_{x^*} and the restriction of A to X_{x^*}, $A\big|_{X_{x^*}}$, is a Lipschitz operator with constant l. If $l < 1$, in this case we can use the following variant of contraction principle:

Weak contraction principle. *Let (X, d) be a metric space and $A : X \to X$ be an operator. We suppose that:*

(i) $F_A \neq \emptyset$;
(ii) *A is a l-contraction.*

Then:

(a) $F_A = \{x^*\}$;
(b) *$A^n(x) \to x^*$ as $n \to \infty$, $\forall \ x \in X$, i.e., A is a Picard operator;*
(c) $d(x, x^*) \leq \frac{1}{1-l} d(x, A(x))$, $\forall \ x \in X$.

In this paper we do not suppose that $F_A \setminus \{\theta\} \neq \emptyset$. So, we introduce the following notion:

Definition 1.3. *Let $(X, +, \mathbb{R}, \to)$ be a linear L-space (see [36]) and $A : X \to X$ be a linear operator. By definition, a partition of X, $X = \bigcup_{\lambda \in \Lambda} X_\lambda$, is an* <u>*invariant linear partition*</u> *(ILP) of X with respect to A iff:*

(i) *there exists $\lambda_0 \in \Lambda$ such that X_{λ_0} is a linear subspace of X and*

$$X/_{X_{\lambda_0}} = \{X_\lambda \mid \lambda \in \Lambda\};$$

(ii) $A(X_\lambda) \subset X_\lambda$, $\forall \ \lambda \in \Lambda$;
(iii) $X_\lambda = \overline{X}_\lambda$, $\forall \ \lambda \in \Lambda$.

We also need the following fixed point result (see [28], [37], [24], ...):

Maia's fixed point theorem. *Let X be a nonempty set, d and ρ be two metrics on X and $A : X \to X$ be an operator. We suppose that:*

(i) *there exists $c > 0$ such that $d(x, y) \leq c\rho(x, y)$, $\forall \ x, y \in X$;*
(ii) *(X, d) is a complete metric space;*
(iii) *$A : (X, d) \to (X, d)$ is continuous;*
(iv) *$A : (X, \rho) \to (X, \rho)$ is an l-contraction.*

Then:

(a) $F_A = \{x^*\}$;
(b) *$A : (X, d) \to (X, d)$ is Picard operator;*
(c) *$A : (X, \rho) \to (X, \rho)$ is Picard operator;*
(d) $\rho(x, x^*) \leq \frac{1}{1-l} \rho(x, A(x))$, $\forall \ x \in X$.

The aim of this paper is to study the iterates of a linear operator and of an increasing linear operator on a Banach lattice in terms of an invariant partition of the space and using contraction principle and Maia's fixed point theorem.

2. Invariant linear partitions

In what follows we shall give some generic examples of ILP of the space.

Let $(X, +, \mathbb{R}, \|\cdot\|)$ be a normed space, $A : X \to X$ be a linear operator and $(\Lambda, +, \mathbb{R}, \tau)$ be a linear topological space and $\Phi : X \to \Lambda$ be a continuous linear and surjective operator. We suppose that Φ is an invariant operator of A (see [10], [4], [36], [26], ...), i.e., $\Phi(A(x)) = \Phi(x)$, $\forall\ x \in X$. For $\lambda \in \Lambda$, let

$$X_\lambda := \{x \in X \mid \Phi(x) = \lambda\}.$$

We remark that, $X = \bigcup\limits_{\lambda\in\Lambda} X_\lambda$, is an ILP of X with respect to A. In this case, $\lambda_0 = \theta_\Lambda$.

Here are some examples:

Example 2.1. Let $\mathbb{B}$ be a Banach space, $K \in C([0,1]^2, \mathbb{R})$ and $A : C([0,1], \mathbb{B}) \to C([0,1], \mathbb{B})$ be defined by

$$A(x)(t) := x(0) + \int_0^t K(t,s)x(s)ds,\ \forall\ t \in [0,1].$$

Let $\Lambda := \mathbb{B}$ and $\Phi : C([0,1], \mathbb{B}) \to \mathbb{B}$, be defined by, $\Phi(x) = x(0)$. It is clear that Φ is invariant for A and, $X = \bigcup\limits_{\lambda\in\Lambda} X_\lambda$, is an ILP of $(C[0,1], \mathbb{B})$ with respect to A. In this case $\lambda_0 = \theta_{\mathbb{B}}$.

Example 2.2. Let $A : C[0,1] \to C[0,1]$ be a continuous linear operator such that $A(x)(0) = x(0)$ and $A(x)(1) = x(1)$ (i.e., 0 and 1 are interpolation points of A (see [34] and the references therein)). Let $\Lambda := \mathbb{R}^2$ and $\Phi : C[0,1] \to \mathbb{R}^2$, $\Phi(x) = (x(0), x(1))$. Then Φ is invariant for A, $\lambda_0 = (0,0)$ and $C[0,1] = \bigcup\limits_{\lambda\in\mathbb{R}^2} X_\lambda$ is an ILP of $C[0,1]$ with respect to A.

Another generic example is the following:

Let $(X, +, \mathbb{R}, \to)$ be a linear L-space and $A : X \to X$ be a linear operator. Let us consider the quotient space $X/_{\overline{(1-A)(X)}} = \{X_\lambda \mid \lambda \in \Lambda\}$, with $X_{\lambda_0} := \overline{(1-A)(X)}$. From a remark by Jachymski (see Lemma 1 in [22]), $A(X_\lambda) \subset X_\lambda$. From the definition of quotient space it follows that, $X = \bigcup\limits_{\lambda\in\Lambda} X_\lambda$ is an ILP of X with respect to A.

Remark 2.3. Let $(X, +, \mathbb{R}, \to)$ be a linear L-space and $A : X \to X$ be a linear operator. Let $X = \bigcup\limits_{\lambda\in\Lambda} X_\lambda$ be an ILP of X. Then $Y = \bigcup\limits_{\lambda\in\Lambda} \overline{A(X_\lambda)}$ is an ILP of Y with respect to the operator $A\big|_Y : Y \to Y$. We remark that, $F_A = F_{A|_Y}$.

3. Main results

Our abstract results are the following:

Theorem 3.1. *Let* $(X, +, \mathbb{R}, \|\cdot\|)$ *be a Banach space and* $A : X \to X$ *be a linear operator. Let* $X = \bigcup_{\lambda \in \Lambda} X_\lambda$ *be an ILP of* X *with respect to* A*, with* X_{λ_0} *a linear subspace of* X*. We suppose that there exists* $l \in]0, 1[$ *such that*

$$\|A(x)\| \le l\|x\|, \ \forall \ x \in X_{\lambda_0}.$$

Then:

(a) $F_A \cap X_\lambda = \{x^*_\lambda\}, \ \forall \ \lambda \in \Lambda$;
(b) $A^n(x) \to x^*_\lambda$ *as* $n \to \infty, \ \forall \ x \in X_\lambda, \ \lambda \in \Lambda$, *i.e.,* A *is weakly Picard operator (WPO) on* X *and* $A^\infty(x) = x^*_\lambda, \ \forall \ x \in X_\lambda$;
(c) $\|x - A^\infty(x)\| \le \frac{1}{1-l}\|x - A(x)\|, \ \forall \ x \in X.$

Proof. Let $x, y \in X_\lambda$. Then, $x - y \in X_{\lambda_0}$ and

$$\|A(x) - A(y)\| = \|A(x - y)\| \le l\|x - y\|.$$

From the contraction principle we have that $F_A \cap X_\lambda = \{x^*_\lambda\}$ and $A : X_\lambda \to X_\lambda$ is Picard operator. We also have that:

$$\|x - x^*_\lambda\| \le \frac{1}{1-l}\|x - A(x)\|, \ \forall \ x \in X_\lambda.$$

From the definition of A^∞ it follows (c). □

Theorem 3.2. *Let* $(X, +, \mathbb{R}, \|\cdot\|, \le)$ *be a Banach lattice and* $A : X \to X$ *be an increasing linear operator. We suppose that:*

(i) $X = \bigcup_{\lambda \in \Lambda} X_\lambda$ *is an ILP of* X *with respect to* A*, with* X_{λ_0} *a linear subspace of* X;
(ii) *there exists an order unit element* $u \in X$ *for* X_{λ_0}*, such that*

$$A(u) \le lu, \ \text{with some } 0 < l < 1.$$

Then:

(a) A *is WPO with respect to* $\overset{\|\cdot\|}{\to}$;
(b) $X_\lambda \cap F_A = \{x^*_\lambda\}, \ \forall \ \lambda \in \Lambda$;
(c) $A^\infty(x) = x^*_\lambda, \ \forall \ x \in X_\lambda, \ \lambda \in \Lambda$;
(d) A *is WPO with respect to* $\overset{d_{\|\cdot\|_u}}{\to}$*, where* $\|\cdot\|_u$ *is the Minkowski norm on* X_{λ_0} *with respect to* u*, i.e.,* $\|A^n(x) - A^\infty(x)\|_u \to 0$ *as* $n \to +\infty$;
(e) $\|x - A^\infty(x)\|_u \le \frac{1}{1-l}\|x - A(x)\|_u, \ \forall \ x \in X.$

Proof. Let $x \in X_{\lambda_0}$. Since u is order unit for X_{λ_0}, there exists $M(x) > 0$ such that

$$|x| \le M(x)u.$$

From the definition of Minkowski's norm, $\|\cdot\|_u : X_{\lambda_0} \to \mathbb{R}_+$, we have that

$$|x| \le \|x\|_u u, \ \forall \ x \in X_{\lambda_0}. \tag{3.1}$$

Since X is a Banach lattice we also have that

$$\|x\| \leq \|u\| \; \|x\|_u, \; \forall \; x \in X_{\lambda_0}. \tag{3.2}$$

But A is increasing linear operator. From (3.1) we have

$$|A(x)| \leq A(|x|) \leq \|x\|_u A(u) \leq l\|x\|_u u.$$

From this relations it follows

$$\|A(x)\|_u \leq l\|x\|_u, \; \forall \; x \in X_{\lambda_0}. \tag{3.3}$$

Now let $x, y \in X_\lambda$. Then, $x - y \in X_{\lambda_0}$ and from (3.3) we have

$$\|A(x) - A(y)\|_u \leq l\|x - y\|_u.$$

On X_λ we have two metrics, $d_{\|\cdot\|}(x,y) := \|x-y\|$ and $d_{\|\cdot\|_u}(x,y) := \|x-y\|_u$. So, by the above considerations, $(X_\lambda, d_{\|\cdot\|}, d_{\|\cdot\|_u})$ and $A\big|_{X_\lambda} : X_\lambda \to X_\lambda$ satisfy the conditions of Maia's fixed point theorem. From this theorem we have, (a)-(e). □

4. Applications

In what follows we present some applications of the above abstract results.

Example 4.1. Let $h > 0$, $b > 0$ and $p, q \in C[0,b]$. We consider the following functional differential equation (see [32])

$$x'(t) = p(t)x(t) + q(t)x(t-h), \; \forall \; t \in [0,b]. \tag{4.1}$$

By a solution of (4.1) we understand a function $x \in C[-h,b] \cap C^1[0,b]$ which satisfies (4.1). The equation (4.1) is equivalent with the following fixed point equation

$$x(t) = \begin{cases} x(t), \text{ if } t \in [-h,0] \\ x(0) + \int_0^t p(s)x(s)ds + \int_0^t q(s)x(s-h)ds, \; t \in [0,b] \end{cases} \tag{4.2}$$

with $x \in C[-h,b]$.

Let $A : C[-h,b] \to C[-h,b]$ be defined by, $A(x)(t) =$ the second part of (4.2). Let $\Lambda := C[-h,0]$ and $\Phi : C[-h,b] \to C[-h,0]$ be defined by, $\Phi(x) = x\big|_{[-h,0]}$.

We observe that, $\Phi(A(x)) = \Phi(x)$, $\forall \; x \in C[-h,b]$. So, $C[-h,b] = \bigcup\limits_{\lambda \in C[-h,b]} X_\lambda$ is an ILP of $C[-h,b]$ and λ_0 is the constant function $0 \in C[-h,0]$, i.e.,

$$X_0 = \{x \in C[-h,b] \mid x\big|_{[-h,0]} = 0\}.$$

It is clear that there exists $\tau > 0$ such that $A\big|_{X_0} : X_0 \to X_0$ is a contraction with respect to Bielecki norm $\|\cdot\|_\tau$, where

$$\|x\|_\tau := \max_{t \in [-h,b]} |x(t)| e^{-\tau t}.$$

Let us denote by, $\|\cdot\|$, the max norm on $C[-h,b]$. From Theorem 3.1 we have

Theorem 4.2. *In the above considerations we have that:*

(a) *the operator A is WPO with respect to $\xrightarrow{\|\cdot\|}$, i.e., the solution set of (4.1) is $A^\infty(C[-h,b])$;*

(*b*) $F_A \cap X_\lambda = \{x^*_\lambda\}$, $\lambda \in C[-h,0]$, *i.e.,* x^*_λ *is a unique solution of* (4.1) *which satisfies the condition* $x\big|_{[-h,0]} = \lambda$;
(*c*) *the operator* $A\big|_{X_\lambda} : X_\lambda \to X_\lambda$ *is PO,* $\forall\ \lambda \in C[-h,0]$.

Remark 4.3. If in addition we suppose that $p \geq 0$, $q \geq 0$, then the operator A is increasing. From the abstract Gronwall lemma (see [31]) we have that if $x \in C[-h,b] \cap C^1[0,b]$ satisfies the inequality

$$x'(t) \leq p(t)x(t) + q(t)x(t-h),\ \forall\ t \in [0,b],$$

then, $x(t) \leq A^\infty(x)(t)$, $\forall\ t \in [-h,b]$.

Example 4.4. Let $(X, +, \mathbb{R}, \|\cdot\|)$ be a Banach space and $A : X \to X$ be a linear and continuous operator. We suppose that A is l-graphic contraction, i.e.,

$$\|A(x) - A^2(x)\| \leq l\|x - A(x)\|,\ \forall\ x \in X.$$

This implies that

$$\|Au\| \leq l\|u\|,\ \forall\ u \in (1_X - A)(X).$$

Let us denote, $X_{\lambda_0} := \overline{(1_X - A)(X)}$. We consider the quotient space,

$$X/_{X_{\lambda_0}} = \{X_\lambda \mid \lambda \in \Lambda\}.$$

We remark that, $X = \bigcup\limits_{\lambda \in \Lambda} X_\lambda$ is ILP of X with respect to A. From Theorem 3.1 we have

Theorem 4.5. *In the above considerations we have:*
(*a*) $A\big|_{X_\lambda} : X_\lambda \to X_\lambda$ *is a l-contraction,* $\forall\ \lambda \in \Lambda$;
(*b*) $F_A \cap X_\lambda = \{x^*_\lambda\}$, $\lambda \in \Lambda$;
(*c*) *the attraction domain of* x^*_λ, $(AD)_A(x^*_\lambda) = X_\lambda$, $\forall\ \lambda \in \Lambda$.

Example 4.6. Let $\varphi_0, \varphi, \psi_k \in C([0,1], \mathbb{R}_+)$, $k = \overline{1,m}$ and $0 = a_0 < a_1 < \ldots < a_m = 1$. We suppose that the set $\{\varphi_0, \varphi \cdot \psi_1, \ldots, \varphi \cdot \psi_m\}$ is linearly independent. In addition we suppose that $\varphi_0(a_0) = 1$ and $\varphi(a_0) = 0$. Then the following operator

$$A : C[0,1] \to C[0,1],\ A(f) = f(a_0)\varphi_0 + \varphi \sum_{k=1}^{m} f(a_k)\psi_k$$

is increasing and linear, with $A(f)(a_0) = f(a_0)$, for all $f \in C[0,1]$. Let

$$X_\lambda := \{f \in C[0,1] \mid f(a_0) = \lambda\},\ \lambda \in \mathbb{R}.$$

It is clear that, $C[0,1] = \bigcup\limits_{\lambda \in \mathbb{R}} X_\lambda$ is an ILP of $C[0,1]$ with respect to A. From the Theorem 3.2 we have

Theorem 4.7. *In addition to the above conditions we suppose that,* $A(\varphi) \leq l\varphi$, *with* $0 < l < 1$. *Then:*
(*a*) *the operator* A *is* WPO;
(*b*) $X_\lambda \cap F_A = \{f^*_\lambda\}$, $\forall\ \lambda \in \mathbb{R}$;
(*c*) $A^\infty(f) = f^*_\lambda$, $\forall\ \lambda \in \mathbb{R}$.

Proof. We remark that φ is an order unit for $A(X_0)$. □

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