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Rodrigues formula for the Cayley transform of groups $\mathbf{SO}(n)$ and $\mathbf{SE}(n)$

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Abstract. In Theorem 3.1 we present, in the case when the eigenvalues of the matrix are pairwise distinct, a direct way to determine the Rodrigues coefficients of the Cayley transform for the special orthogonal $\mathbf{SO}(n)$ by reducing the Rodrigues problem in this case to the system (3.2). The similar method is discussed for the Euclidean group $\mathbf{SE}(n)$.

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1. Introduction

The Cayley transform of the group of rotations $\mathbf{SO}(n)$ of the Euclidean space $\mathbb{R}^n$ is defined by $\mathrm{Cay} : \mathfrak{so}(n) \to \mathbf{SO}(n), \mathrm{Cay}(A) = (I_n + A)(I_n - A)^{-1}$, where $\mathfrak{so}(n)$ is the Lie algebra of $\mathbf{SO}(n)$. Because the inverse of the matrix $I_n - A$ can be written as $(I_n - A)^{-1} = I_n + A + A^2 + \dots$ on a sufficiently small neighborhood of O_n, from the well-known Hamilton-Cayley Theorem, it follows that $\mathrm{Cay}(A)$ has the polynomial form

$$\mathrm{Cay}(A) = b_0(A)I_n + b_1(A)A + \cdots + b_{n-1}(A)A^{n-1},$$

where the coefficients $b_0, b_1, \dots, b_{n-1}$ depend on the matrix A and are uniquely defined. By analogy with the case of the exponential map (see [1] and [2]), they are called *Rodrigues coefficients* of A with respect to the Cayley transform.

Using the main idea in the articles [3] (see also [4]), in this paper we present a method to derive the Rodrigues coefficients for the Cayley transform of the group $\mathbf{SO}(n)$. The case of the Euclidean group $\mathbf{SE}(n)$ is also discussed.

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2. Cayley transform of the group $\mathbf{SO}(n)$

The matrices of the $\mathbf{SO}(n)$ group describe the rotations as movements in the space $\mathbb{R}^n$. If the matrix A belongs to the Lie Algebra $\mathfrak{so}(n)$ of the Lie group $\mathbf{SO}(n)$, then the matrix $I_n - A$ is invertible.

Indeed, the eigenvalues $\lambda_1, ..., \lambda_n$ of the matrix A are 0 or purely imaginary, so eigenvalues of the matrix $I_n - A$ are $1-\lambda_1, ..., 1-\lambda_n$. They are clearly different from 0, therefore we have $\det(I_n - A) = (1-\lambda_1)...(1-\lambda_n) \neq 0$, so $I_n - A$ is invertible.

The map $\mathrm{Cay} : \mathfrak{so}(n) \to \mathbf{SO}(n)$, defined by

$$\mathrm{Cay(A) = (I_n + A)(I_n - A)^{-1}}$$

is called the *Cayley transform* of the group $\mathbf{SO}(n)$. Let show that this map is well defined. Let be $\mathrm{Cay}(A) = R$. We have

$$\begin{aligned} R^t R &= (I_n + A)(I_n - A)^{-1t}[(I_n + A)(I_n - A)^{-1}] \\ &= (I_n + A)(I_n - A)^{-1t}[(I_n - A)^{-1}]^t(I_n + A) \\ &= (I_n + A)(I_n - A)^{-1}(I_n -^t A)^{-1}(I_n +^t A) \\ &= (I_n + A)(I_n - A)^{-1}(I_n + A)^{-1}(I_n - A) = I_n, \end{aligned}$$

because matrices and their inverses commute. Therefore $R \in \mathbf{SO}(n)$. The map Cay is obviously continuous and we have $\mathrm{Cay}(O_n) = I_n \in \mathbf{SO}(n)$, hence necessarily we have $R \in \mathbf{SO}(n)$.

Denote by $\sum$ the set of the group $\mathbf{SO}(n)$ containing the matrices with eigenvalue -1. Clearly, we have $R \in \sum$ if and only if the matrix $I_n + R$ is singular.

Theorem 2.1. *The map* $\mathrm{Cay} : \mathfrak{so}(n) \to \mathbf{SO}(n) \setminus \sum$ *is bijective and its inverse is* $\mathrm{Cay}^{-1} : \mathbf{SO}(n) \setminus \sum \to \mathfrak{so}(n)$, *where* $\mathrm{Cay}^{-1}(R) = (R + I_n)^{-1}(R - I_n)$.

Proof. If $R \in \mathbf{SO}(n) \setminus \sum$ then, the relation $\mathrm{Cay}(A) = R$ is equivalent to

$$R = (I_n + A)(I_n - A)^{-1} = (2I_n - (I_n - A))(I_n - A)^{-1} = 2(I_n - A)^{-1} - I_n.$$

Because $R \in \mathbf{SO}(n) \setminus \sum$, it follows that the matrix $R + I_n$ is invertible and from above relation we obtain that its inverse is $(R + A)^{-1} = \frac{1}{2}(I_n - A)$. Using this relation we have

$$(R + I_n)^{-1}(R - I_n) = \frac{1}{2}(I_n - A)(2(I_n - A)^{-1} - 2I_n) = I_n - I_n + A = A,$$

so $\mathrm{Cay}^{-1}(R) = (R + I_n)^{-1}(R - I_n)$.

In addition, a simple computation shows that if the matrix R is orthogonal, then the matrix $A = (R + I_n)^{-1}(R - I_n)$ is antisymmetric. Indeed, we have

$$\begin{aligned} {}^tA &= ({}^tR - I_n)({}^tR + I_n)^{-1} = (R^{-1} - I_n)(R^{-1} + I_n)^{-1} \\ &= (I_n - R)R^{-1}R(I_n + R)^{-1} = -(R + I_n)^{-1}(R - I_n) = -A, \end{aligned}$$

because the matrices $R - I_n$ and $(R + I_n)^{-1}$ commute. $\square$

3. Rodrigues type formulas for Cayley transform

Because the inverse of the matrix $I_n - A$ can be written in the form

$$(I_n - A)^{-1} = I_n + A + A^2 + ...$$

for a sufficiently small neighborhood of O_n, from Hamilton-Cayley theorem, it follows that the Cayley transform of A can be written in the polynomial form

$$\mathrm{Cay}(A) = b_0(A)I_n + b_1(A)A + ... + b_{n-1}(A)A^{n-1} \tag{3.1}$$

where the coefficients $b_0, ..., b_{n-1}$ are uniquely determined and depend on the matrix A. We will call these numbers, by analogy with the situation of the exponential map, Rodrigues coefficients of A with respect to the application Cay.

As in the case of the exponential map, an important property of the Rodrigues coefficients is the invariance with respect to equivalent matrices, i.e. for any invertible matrix U, the following relations hold

$$b_k(UAU^{-1}) = b_k(A), k = 0, ..., n-1.$$

This property is obtained from the uniqueness of the Rodrigues coefficients and from the following property of the Cayley transform

$$U\mathrm{Cay}(A)U^{-1} = \mathrm{Cay}(UAU^{-1}).$$

To justify the last relation just observe that we have successively

$$\begin{aligned} U\mathrm{Cay}(A)U^{-1} &= U(I_n + A)(I_n - A)^{-1}U^{-1} = U(I_n + A)U^{-1}U(+I_n - A)^{-1}U^{-1} \\ &= (I_n + UAU^{-1})(U^{-1})^{-1}(I_n - A)^{-1}U^{-1}(I_n + UAU^{-1})(U(I_n - A)U^{-1})^{-1} \\ &= (I_n + UAU^{-1})(I_n + UAU^{-1})^{-1} = \mathrm{Cay}(UAU^{-1}). \end{aligned}$$

Theorem 3.1. *Let $\lambda_1, ..., \lambda_n$ be the eigenvalues of the matrix $A \in \mathfrak{so}(\mathfrak{n})$.*

1) Rodrigues coefficients of A relative to the application Cay *are solutions of the system*

$$\sum_{k=0}^{n-1} S_{k+j} b_k = \sum_{s=1}^{n} \lambda_s^j \frac{1+\lambda_s}{1-\lambda_s}, j = 0, ..., n-1, \tag{3.2}$$

where $S_j = \lambda_1^j + ... + \lambda_n^j$.

2) If the eigenvalues $\lambda_1, ..., \lambda_n$ of the matrix A are pairwise distinct, then the Rodrigues coefficients $b_0, ..., b_{n-1}$ are perfectly determined by this system and are rational functions of $\lambda_1, ..., \lambda_n$.

Proof. 1) By multiplying the relation (3.1) by the power $A^j, j = 0, ..., n-1$, we obtain the matrix relations

$$A^j\mathrm{Cay}(A) = \sum_{k=0}^{n-1} b_k A^{k+j}, j = 0, ..., n-1.$$

Now, considering the trace in both sides of the above relations, it follows

$$\sum_{k=0}^{n-1} tr(A^{k+j})b_k = tr(A^j\mathrm{Cay}(A)), j = 0, ..., n-1. \tag{3.3}$$

The matrix A^{k+j} has the eigenvalues $\lambda_1^{k+j}, ..., \lambda_n^{k+j}$, and the matrix $A^j \mathrm{Cay}(A)$ has the eigenvalues $\lambda_1^j \frac{1+\lambda_1}{1-\lambda_1}, ..., \lambda_n^j \frac{1+\lambda_n}{1-\lambda_n}$ and the system (3.3) is equivalent to the system (3.2).

2) For the second statement, observe that the determinant of the system (3.2) can be written as

$$D_n = \det \begin{pmatrix} S_0 & S_1 & \dots & S_{n-1} \\ S_1 & S_2 & \dots & S_n \\ \dots & \dots & \dots & \dots \\ S_{n-1} & S_n & \dots & S_{2n-1}, \end{pmatrix}$$

where $S_l = S_l(\lambda_1, \dots, \lambda_n) = \lambda_1^l + \dots + \lambda_n^l, l = 0, \dots, 2n-1$.

It is clear that

$$D_n = \det \begin{pmatrix} 1 & \dots & 1 \\ \lambda_1 & \dots & \lambda_n \\ \dots & \dots & \dots \\ \lambda_1^{n-1} & \dots & \lambda_n^{n-1} \end{pmatrix} \cdot \det \begin{pmatrix} 1 & \lambda_1 & \dots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \dots & \lambda_2^{n-1} \\ \dots & \dots & \dots & \dots \\ 1 & \lambda_n & \dots & \lambda_n^{n-1} \end{pmatrix}$$

$$= V_n^2(\lambda_1, ..., \lambda_n) = \prod_{1 \le i < j \le n} (\lambda_i - \lambda_j)^2,$$

where $V_n = V_n(\lambda_1, \dots, \lambda_n)$ is the Vandermonde determinant of order n. According to the well-known formulas giving the solution $b_0, \dots, b_{n-1}$ to the system (3.2), the conclusion follows. $\square$

We will continue to illustrate the particular cases $n = 2$ and $n = 3$. If $A = O_n$, then $\mathrm{Cay}(A) = I_n$ and so $b_0(O_n) = 1, b_1(O_n) = ... = b_{n-1}(O_n) = 0$.

In the case $n = 2$, consider the antisymmetric matrix $A \neq O_2$, where

$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, a \in \mathbb{R}^*,$$

with eigenvalues $\lambda_1 = ai, \lambda_2 = -ai$. System (3.2) becomes in this case

$$\begin{cases} 2b_0 = \frac{1+ai}{1-ai} + \frac{1-ai}{1+ai} \\ -2a^2 b_1 = ai\frac{1+ai}{1-ai} - ai\frac{1-ai}{1+ai} \end{cases}$$

and we obtain

$$b_0 = \frac{1-a^2}{1+a^2}, b_1 = \frac{1}{1+a^2}.$$

Thus, the Rodrigues type formula for the Cayley transform is

$$\mathrm{Cay}(A) = \frac{1-a^2}{1+a^2} I_2 + \frac{2}{1+a^2} A. \tag{3.4}$$

For $n = 3$ any real antisymmetric matrix is of the form

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

with the characteristic polynomial $p_A(t) = t^3 + \theta^2 t$, where $\theta = \sqrt{a^2+b^2+c^2}$. The eigenvalues of the matrix A are $\lambda_1 = \theta i, \lambda_2 = -\theta i, \lambda_3 = 0$. We have $A = O_3$ if and only if $\theta = 0$, so it is enough to consider only the situation in which $\theta \neq 0$. The system 3.2 becomes

$$\begin{cases} 3b_0 - 2\theta^2 b_2 = \frac{1+\theta i}{1-\theta i} + \frac{1-\theta i}{1+\theta i} + 1 \\ -2\theta^2 b_1 = \theta i \frac{1+\theta i}{1-\theta i} - \theta i \frac{1-\theta i}{1+\theta i} \\ -2\theta^2 b_0 + \theta^4 b_2 = -\theta^2 \left(\frac{1+\theta i}{1-\theta i} + \frac{1-\theta i}{1+\theta i} \right) \end{cases}$$

with the solution

$$b_0 = 1, b_1 = \frac{2}{1+\theta^2}, b_2 = \frac{2}{1+\theta^2}.$$

It follows the Rodrigues type formula for the Cayley transform of group $\mathbf{SO}(3)$

$$\mathrm{Cay}(A) = I_3 + \frac{2}{1+\theta^2} A + \frac{2}{1+\theta^2} A^2. \tag{3.5}$$

Formula (3.5) offers the possibility to obtain another formula for the inverse of Cayley transform. Let be $R \in \mathbf{SO}(3)$ such that

$$R = I_3 + \frac{2}{1+\theta^2} A + \frac{2}{1+\theta^2} A^2,$$

where A is an antisymmetric matrix. Considering the matrix transpose in both sides of the above relation and taking into account that ${}^tA = -A$, we obtain

$$R -^t R = \frac{4}{1+\theta^2} A. \tag{3.6}$$

On the other hand, we have

$$tr(R) = 3 - \frac{4\theta^2}{1+\theta^2} = -1 + \frac{4}{1+\theta^2},$$

and by replacing in the relation (3.6), we get the formula

$$\mathrm{Cay}^{-1}(R) = \frac{1}{1+tr(R)} (R -^t R). \tag{3.7}$$

Formula (3.7) makes sense for rotations $R \in \mathbf{SO}(3)$ for which $1+tr(R) \neq 0$. If R is a rotation of angle α, then we have $tr(R) = 1 + 2\cos\alpha$, so application Cay^{-1} is not defined for the rotations of angle $\alpha = \pm\pi$. Because in the domain where is defined, the application Cay is bijective, it follows that the antisymmetric matrices from $\mathfrak{so}(3)$ can be used as coordinates for rotations. Considering the Lie algebra isomorphism "$\widehat{\ }$" between $(\mathbb{R}^3, \times)$ and $(\mathfrak{so}(3), [\cdot,\cdot])$, where " $\times$ " denote the vector product, defined by $v \in \mathbb{R}^3 \to \widehat{v} \in \mathfrak{so}(3)$, where

$$v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and

$$\widehat{v} = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix},$$

by composing the applications

$$\mathbb{R}^3 \widehat{\to} \mathfrak{so}(3) \xrightarrow{\text{Cay}} \mathbf{SO}(3)$$

we get a vectorial parameterization of rotations from $\mathbf{SO}(3)$.

4. The Cayley transform for Euclidean group $\mathbf{SE}(n)$

In this subparagraph we will define a Cayley type transformation for the special Euclidean group $\mathbf{SE}(n)$. By analogy with the special orthogonal group $\mathbf{SO}(n)$, we define the application $\text{Cay}_{n+1} : \mathfrak{se}(n) \to \mathbf{SE}(n)$, where

$$\text{Cay}_{n+1}(S) = (I_{n+1} + S)(I_{n+1} - S)^{-1}. \tag{4.1}$$

We will call this application *Cayley transform* of the group $\mathbf{SE}(n)$. First we show that it is well defined. Let be $S \in \mathfrak{se}(n)$, a matrix defined in blocks

$$S = \begin{pmatrix} A & u \\ 0 & 0 \end{pmatrix},$$

where $A \in \mathfrak{so}(n)$ and $u \in \mathbb{R}^n$. A simple computation shows that we have the formula

$$(I_{n+1} + S)(I_{n+1} - S)^{-1} = \begin{pmatrix} R & (R + I_n)u \\ 0 & 1 \end{pmatrix},$$

where $R = (I_n + A)(I_n - A)^{-1} = \text{Cay}(A) \in \mathbf{SO}(n)$, that is the desired formula.

The connection between the transform $\text{Cay} : \mathfrak{so}(n) \to \mathbf{SO}(n)$ and $\text{Cay}_{n+1} : \mathfrak{se}(n) \to \mathbf{SE}(n)$ is given by the formula

$$\text{Cay}_{n+1}(S) = \begin{pmatrix} \text{Cay}(A) & (R + I_n)u \\ 0 & 1 \end{pmatrix}.$$

As for the classical transform $\text{Cay} : \mathfrak{so}(n) \to \mathbf{SO}(n)$ we can get effective Rodrigues type formulas for transform $\text{Cay}_{n+1} : \mathfrak{se}(n) \to \mathbf{SE}(n)$, for small values of n. Using the observation from section 5.1 in the paper of R.-A. Rohan [5], we obtain that for a matrix $S \in \mathfrak{se}(n)$ defined in blocks as above, its characteristic polynomial p_S satisfy the relation $p_S(t) = tp_A(t)$. The Rodrigues formula for the transform $\text{Cay}_{n+1} : \mathfrak{se}(n) \to \mathbf{SE}(n)$ is of the form

$$\text{Cay}_{n+1}(S) = c_0 I_{n+1} + c_1 S + ... + c_n S^n,$$

where the coefficients $c_0, c_1, ..., c_n$ depend on the matrix S.

For $n = 2$, consider the antisymmetric matrix $A \neq O_2$, where

$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, a \in \mathbb{R}^*.$$

Using the above observation, it follows that the matrix $S \in \mathfrak{se}(2)$ has eigenvalues $\lambda_1 = ai, \lambda_2 = -ai, \lambda_3 = 0$, and the corresponding Rodrigues formula has the form

$$\text{Cay}_3(S) = c_0 I_3 + c_1 S + c_2 S^2.$$

We have a result analogous to that of Theorem 3.1, which is reduced to the system

$$\begin{cases} S_0c_0 + S_1c_1 + S_2c_2 = 1 + \frac{1+\lambda_1}{1-\lambda_1} + \frac{1+\lambda_2}{1-\lambda_2} \\ S_1c_0 + S_2c_1 + S_3c_2 = \lambda_1\frac{1+\lambda_1}{1-\lambda_1} + \lambda_2\frac{1+\lambda_2}{1-\lambda_2} \\ S_2c_0 + S_3c_1 + S_4c_2 = \lambda_1^2\frac{1+\lambda_1}{1-\lambda_1} + \lambda_2^2\frac{1+\lambda_2}{1-\lambda_2} \end{cases}$$

where in our case we have $S_0 = 3, S_1 = 0, S_2 = -2a^2, S_3 = 0, S_4 = 2a^2$. This system is equivalent to

$$\begin{cases} 3c_0 - 2a^2c_2 = 1 + \frac{2(1-a^2)}{1+a^2} \\ -2a^2c_1 = -\frac{4a^2}{1+a^2} \\ -2a^2c_0 + 2a^4c_2 = -2a^2\frac{1-a^2}{1+a^2} \end{cases}$$

with solution

$$c_0 = 1, c_1 = \frac{1}{1+a^2}, c_2 = \frac{1}{1+a^2}.$$

So Rodrigues formula for transformation Cay_3 is

$$\mathrm{Cay}_3(S) = I_3 + \frac{1}{1+a^2}S + \frac{1}{1+a^2}S^2. \tag{4.2}$$

For $n = 3$ we consider an antisymmetric matrix of the form

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

with the characteristic polynomial $p_A(t) = t^3 + \theta^2 t$, where $\theta = \sqrt{a^2+b^2+c^2}$. The matrix $S \in \mathfrak{se}(3)$ has the characteristic polynomial $p_S(t) = tp_A(t) = t^4 + \theta^2t^2$, and the eigenvalues of its are $\lambda_1 = \theta i, \lambda_2 = -\theta i, \lambda_3 = 0, \lambda_4 = 0$. Rodrigues formula has the form

$$\mathrm{Cay}_4(S) = c_0I_4 + c_1S + c_2S^2 + c_3S^3.$$

After a similar computation, we obtain the formula

$$\mathrm{Cay}_3(S) = I_3 + 2S + \frac{2}{1+\theta^2}S^2 + \frac{2}{1+\theta^2}S^3. \tag{4.3}$$

As for Cayley transform of the group $\mathbf{SO}(n)$, denote by $\sum_{n+1}$ the set of matrices from $\mathbf{SE}(n)$ that has -1 as eigenvalue. Clearly we have $M \in \mathbf{SE}(n)$ if and only if the matrix $I_{n+1} + M$ is singular. With a similar proof as in Theorem 3.1, we get

Theorem 4.1. *The map* $\mathrm{Cay}_{n+1} : \mathfrak{se}(n) \to SE(n) \setminus \sum_{n+1}$ *is bijective and its inverse is given by*

$$\mathrm{Cay}_{n+1}^{-1}(M) = \begin{pmatrix} \mathrm{Cay}^{-1}(M) & (R+I_n)^{-1}\mathbf{t} \\ 0 & 0 \end{pmatrix},$$

where the matrix M *is defined in blocks by*

$$S = \begin{pmatrix} R & \mathbf{t} \\ 0 & 1 \end{pmatrix}.$$

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