Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 1, 19–29

A-Whitehead groups

Ulrich Albrecht

Abstract. This paper investigates various extensions of the notion of Whitehead modules. An Abelian group G is an A-Whitehead group if there exists an exact sequence $0 \to U \to \oplus_I A \to G \to 0$ such that $S_A(U) = U$ with respect to which A is injective. We investigate the structure of A-Whitehead groups.

Mathematics Subject Classification (2010): 20K20, 20K40.

Keywords: Whitehead modules, endomorphism rings, adjoint functors.

1. Introduction

A right R-module M is a *Whitehead module* if $\mathrm{Ext}^1_R(M,R) = 0$. It is the goal of this paper to investigate Whitehead modules in the context of A-projective and A-solvable Abelian groups. The class of *A-projective groups*, which consists of all groups P which are isomorphic to a direct summand of $\oplus_I A$ for some index-set I, was introduced by Arnold, Lady and Murley ([6] and [7]). An A-projective group P has *finite A-rank* if I can be chosen to be finite. A-projective groups are usually investigated using the adjoint pair (H_A, T_A) of functors between the category *Ab* of Abelian groups and the category M_E of right E-modules defined by $H_A(G) = \mathrm{Hom}(A, G)$ and $T_A(M) = M \otimes_E A$ for all $G \in Ab$ and all $M \in M_E$. Here, $E = E(A)$ denotes the endomorphism ring of A. These functors induce natural maps $\theta_G : T_A H_A(G) \to G$ and $\phi_M : M \to H_A T_A(M)$ defined by $\theta_G(\alpha \otimes a) = \alpha(a)$ and $[\phi_M(x)](a) = x \otimes a$. An Abelian group G is *A-solvable* if θ_G is an isomorphism. If A is self-small, then all A-projective groups are A-solvable. Here, A is self-small if the natural map $H_A(\oplus_I A) \to \Pi_I E$ actually maps into $\oplus_I E$ for all index-sets I [7].

An Abelian group G is *(finitely, κ-) A-generated* if it is an epimorphic image of $\oplus_I A$ for some index-set I (with $|I| < \infty$, $|I| < \kappa$ respectively). It is easy to see that G is A-generated iff $S_A(G) = G$ where $S_A(G) = im(\theta_G)$. The group G is *A-presented* if there exists an exact sequence $0 \to U \to F \to G \to 0$ in which F is A-projective and U is A-generated. A sequence $0 \to G \to H \to L \to 0$ is *A-cobalanced (A-balanced)* if A is injective (projective) with respect to it. For a self-small group A, the A-solvable groups can be described as those groups G for which we can find an A-balanced exact sequence $0 \to U \to F \to G \to 0$ in which F is A-projective and U is A-generated [4].

The functor Ext_R^1 can be defined either in terms of equivalence classes of exact sequences or via projective resolutions. We thus call an A-generated group W an *A-Whitehead splitter* if every exact sequence $0 \to A \to G \to W \to 0$ with $S_A(G) = G$ splits. On the other hand, a group W is an *A-Whitehead group* if it admits an A-cobalanced resolution $0 \to U \to F \to W \to 0$ in which F is A-projective and U is A-generated. Section 2 investigates how A-Whitehead groups and A-Whitehead splitters are related. While all A-presented A-Whitehead splitters are A-Whitehead groups, the converse surprisingly fails in general. Several examples demonstrate the differences between the classic concepts and our more general situation. We show that all A-Whitehead groups are A-Whitehead-splitters if E has injective dimension at most 1 as a right and left E-module. In particular, all countably A-generated A-Whitehead groups are A-projective if A has a right and left Noetherian, hereditary endomorphism ring. By [10], strongly κ-projective and Whitehead modules are closely related. The last results of this paper show that this relation extends to A-Whitehead groups.

2. A-Whitehead Groups

An Abelian group A is *(faithfully) flat* if it is flat (and faithful) as a left E-module. Since every exact sequence $0 \to U \to G \to A \to 0$ with $S_A(G) = G$ splits if A is faithfully flat [2], A is an A-Whitehead splitter in this case. However, this may not be true without the faithfulness condition as the next result shows.

Example 2.1. There exists a flat torsion-free Abelian group A of finite rank such that A is not an A-Whitehead splitter.

Proof. Let p, q, and r be distinct primes, and select subgroups A_1, A_2, and A_3 of $\mathbb{Q}$ such that A_1 is divisible by all primes except p and q, A_2 is divisible by all primes except p and r, and A_3 is divisible by all primes except q and r. By [8, Section 2], there exists a strongly indecomposable subgroup G of $\mathbb{Q} \oplus \mathbb{Q}$ which is generated by $A_1(1,0)$, $A_2(0,1)$, and $A_3(1,1)$. Moreover, $A_4 = G/A_1(1,0)$ is a subgroup of $\mathbb{Q}$ which is divisible by all primes except q. The group $A = \mathbb{Z} \oplus A_1 \oplus A_2 \oplus A_3 \oplus A_4$ is flat as a left E-module by Ulmer's Theorem [16]. Since $A_1 + A_3 = A_4$, A is not faithful. However, the exact sequence $0 \to A \to G \oplus A \oplus A_2 \oplus A_3 \oplus \mathbb{Z} \oplus \mathbb{Z} \to A \to 0$ cannot split since otherwise G would be completely decomposable. Because G is A-generated, A is not an A-Whitehead splitter. □

Proposition 2.2. *Let A be a self-small Abelian group. If W is an A-presented A-Whitehead splitter, then W is an A-Whitehead group.*

Proof. Consider an exact sequence $0 \to U \xrightarrow{\alpha} F \xrightarrow{\beta} W \to 0$, where F is A-projective and $U = S_A(U)$. For $\psi \in \mathrm{Hom}(U, A)$, we obtain the push-out diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \xrightarrow{\alpha} & F & \xrightarrow{\beta} & W & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi_1} & & \downarrow{\scriptstyle 1_W} & & \\ 0 & \longrightarrow & A & \xrightarrow{\alpha_1} & X & \longrightarrow & W & \longrightarrow & 0. \end{array}$$

As a push-out, X is A-generated being an epimorphic image of $A \oplus F$. Since W is an A-Whitehead splitter, the bottom sequence splits, say $\delta\alpha_1 = 1_A$. Now it is easy to see that $\delta\psi_1\alpha = \psi$. □

However the converse of the last result fails in general:

Example 2.3. There exists a self-small faithfully flat Abelian group A for which we can find an A-Whitehead group G which is not an A-Whitehead splitter.

Proof. Let $\mathcal{P}$ be the set of primes, and consider the groups $A = \Pi_{\mathcal{P}}\mathbb{Z}_p$ and $U = \oplus_{\mathcal{P}}\mathbb{Z}_p$. Then, A is a self-small [18, Proposition 1.6], faithfully flat Abelian group, and U is an A-generated subgroup of A such that $A/U \cong \mathbb{Q}^{(2^{\aleph_0})}$. The sequence $0 \to U \to A \to A/U \to 0$ is A-cobalanced since each $\mathbb{Z}_p$ is fully invariant in A and U. Therefore, A/U is an A-Whitehead group and $S_A(X_p) = X_p$.

Fix a a prime p, and choose a group X_p with $E(X_p) = \mathbb{Z}_p$ and $X_p/\mathbb{Z}_p \cong \mathbb{Q}$. This is possible by Corner's Theorem [12]. Then, the induced sequence $0 \to \mathbb{Z}_p^{(2^{\aleph_0})} \to X_p^{(2^{\aleph_0})} \to \mathbb{Q}^{(2^{\aleph_0})} \to 0$ does not split although $A/U \cong \mathbb{Q}^{(2^{\aleph_0})}$ is an A-Whitehead group. □

Moreover, A-Whitehead splitters need not be A-presented. To see this, let p be a prime. If A is any torsion-free Abelian group with $pA = A$, then $\mathbb{Z}(p^\infty)$ is an epimorphic image of A. Moreover, $\mathrm{Ext}(\mathbb{Z}(p^\infty), A) = 0$ because $pA = A$ [12]. Therefore, $\mathbb{Z}(p^\infty)$ is an A-Whitehead splitter. However, no p-group can be A-presented since all A-generated groups are p-divisible.

If A is faithfully flat, then every exact sequence $0 \to U \to G \to H \to 0$ with G and H A-solvable is A-balanced and $S_A(U) = U$ [2]. If U is a submodule of $H_A(G)$, let $UA = \langle \phi(A) | \phi \in U \rangle$.

Lemma 2.4. *If A is a faithfully flat Abelian group, then the following hold for an A-solvable group G:*

a) *If U is a submodule of $H_A(G)$, then the evaluation map $\theta : T_A(U) \to UA$ defined by $\theta(u \otimes a) = u(a)$ is an isomorphism.*
b) *If U and V are submodules of $H_A(G)$ with $UA = VA$, then $U = V$.*

Proof. a) Clearly, θ is onto. To see that it is one-to-one, consider the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & T_A(U) & \longrightarrow & T_AH_A(G) \\ & & \downarrow \theta & & \wr\downarrow \theta_G \\ 0 & \longrightarrow & UA & \longrightarrow & G \end{array}$$

whose top-row is exact since A is flat.

b) Since $UA = VA = (U+V)A$, it suffices to consider the case $U \subseteq V$. By a), the evaluation maps $T_A(U) \to UA$ and $T_A(V) \to VA$ in the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_A(U) & \longrightarrow & T_A(V) & \longrightarrow & T_A(V/U) & \longrightarrow & 0 \\ & & \wr\downarrow & & \wr\downarrow & & & & \\ 0 & \longrightarrow & UA & \xrightarrow{=} & VA & \longrightarrow & 0 & & \end{array}$$

are isomorphisms. Thus, $T_A(V/U) = 0$ which yields $V/U = 0$ since A is faithfully flat. □

Theorem 2.5. *Let A be a self-small faithfully flat Abelian group. The following are equivalent for an A-generated Abelian group W:*

a) *W is an A-Whitehead group.*
b) *There exists a Whitehead-module M with $W \cong T_A(M)$.*

Proof. *a)* ⇒ *b)*: Consider an A-cobalanced exact sequence $0 \to U \overset{\alpha}{\to} F \overset{\beta}{\to} W \to 0$ in which U is A-generated and F is A-projective. It induces the sequence $0 \to H_A(U) \overset{H_A(\alpha)}{\longrightarrow} H_A(F) \overset{H_A(\beta)}{\longrightarrow} M \to 0$ where $M = Im(H_A(\beta))$ is a submodule of $H_A(W)$. We obtain the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_AH_A(U) & \xrightarrow{T_AH_A(\alpha)} & T_AH_A(F) & \xrightarrow{T_AH_A(\beta)} & T_A(M) & \longrightarrow & 0 \\ & & \wr\downarrow \theta_U & & \wr\downarrow \theta_F & & \downarrow \theta & & \\ 0 & \longrightarrow & U & \xrightarrow{\alpha} & F & \xrightarrow{\beta} & W & \longrightarrow & 0. \end{array}$$

By the 3-Lemma, the induced map θ is an isomorphism, and it remains to show that M is a Whitehead-module.

For $\psi \in \mathrm{Hom}_E(H_A(U), E)$, consider $T_A(\psi) : T_AH_A(U) \to T_A(E)$. Let $\sigma : T_A(E) \to A$ be an isomorphism. By a), there is $\lambda : F \to A$ with $\lambda\alpha = \sigma T_A(\psi)\theta_U^{-1}$. An application of H_A gives

$$\begin{aligned} H_A(\sigma^{-1}\lambda\theta_F)H_AT_AH_A(\alpha)) &= H_A(\sigma^{-1}\lambda\theta_F T_AH_A(\alpha)) \\ &= H_A(\sigma^{-1}\lambda\alpha)\theta_U = H_AT_A(\psi). \end{aligned}$$

Since $H_AT_A(\psi)\phi_{H_A(U)} = \phi_E\psi$, we have

$$\begin{aligned} \phi_E^{-1}H_A(\sigma^{-1}\lambda\theta_F)\phi_{H_A(F)}H_A(\alpha) &= \phi_E^{-1}H_A(\sigma^{-1}\lambda\theta_F)H_AT_AH_A(\alpha)\phi_{H_A(U)} \\ &= \phi_E^{-1}H_AT_A(\psi)\phi_{H_A(U)} = \psi, \end{aligned}$$

and M is a Whitehead-module.

b) ⇒ *a)*: Consider an exact sequence $0 \to U \overset{\alpha}{\to} F \overset{\beta}{\to} M \to 0$ in which F is a free right E-module. Since A is faithfully flat, ϕ_U is an isomorphism by [4]. It remains to show that the induced sequence $0 \to T_A(U) \overset{T_A(\alpha)}{\longrightarrow} F \overset{T_A(\beta)}{\longrightarrow} T_A(M) \to 0$ is A-cobalanced. For this, consider a map $\psi \in \mathrm{Hom}(T_A(U), A)$. Because $\mathrm{Ext}_E^1(M, E) = 0$, there exists $\lambda : F \to E$ with $H_A(\psi)\phi_U = \lambda\alpha$. Then,

$$\begin{aligned} \theta_AT_A(\lambda)T_A(\alpha) &= \theta_AT_AH_A(\psi)T_A(\phi_U) \\ &= \psi\theta_{T_A(U)}T_A(\phi_U) = \psi \end{aligned}$$

since $\theta_{T_A(U)}T_A(\phi_U)(u \otimes a) = \theta_{T_A(U)}(\phi_U(u) \otimes a) = u \otimes a$ for all $u \in U$ and $a \in A$. □

Example 2.6. There exists a self-small faithfully flat Abelian group A and a A-Whitehead group W such that $W \cong T_A(M)$ for some right E-module M with $\mathrm{Ext}_R^1(M, E) \neq 0$.

Proof. Let A and U be as in Example 2.3, and consider the A-Whitehead-group $W = A/U$. In view of the proof of Theorem 2.5, it suffices to construct an exact sequence $0 \to V \to P \to W \to 0$ such that P is A-projective and V is A-generated which is not A-cobalanced.

Since A/U is a $\mathbb{Z}_p$-module, there are index-sets I and J and an exact sequence $0 \to \oplus_I \mathbb{Z}_p \to \oplus_J \mathbb{Z}_p \to A/U \to 0$. Because of $\mathrm{Ext}_{\mathbb{Z}_p}(\mathbb{Q}, \mathbb{Z}_p) \neq 0$, this sequence cannot be A-cobalanced. It is easy to see that it cannot be A-balanced either. □

If G and H are A-solvable, and A is a self-small faithfully flat Abelian group, then the equivalence classes of exact sequences $0 \to H \to X \to G \to 0$ with $S_A(X) = X$ form a subgroup of $Ext(G, H)$ denoted by $A - Bext(G, H)$ [3].

Theorem 2.7. *Let A be a self-small faithfully flat Abelian group. The following are equivalent for an A-generated group W:*

- a) *W is an A-solvable A-Whitehead splitter.*
- b) *W is an A-solvable A-Whitehead group.*
- c) *W is A-solvable and $H_A(W)$ is a Whitehead module.*
- d) *There exists an exact sequence $0 \to U \to \oplus_I F \to W \to 0$ with $S_A(U) = U$ which is A-balanced and A-cobalanced.*
- e) *W is an A-solvable group with $A - Bext(W, A) = 0$.*

Proof. Since *a)* $\Rightarrow$ *b)* holds by Proposition 2.2, we consider an A-solvable A-Whitehead group W. As in the proof of Theorem 2.5, there exists a submodule M of $H_A(W)$ with $\mathrm{Ext}^1_E(M, E) = 0$ such that the evaluation map $\theta : T_A(M) \to W$ is an isomorphism. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_A(M) & \longrightarrow & T_A H_A(A) & \longrightarrow & T_A(H_A(W)/M) & \longrightarrow & 0 \\ & & \wr\big\downarrow \theta & & \wr\big\downarrow \theta_W & & & & \\ & & W & \xrightarrow{1_W} & W & & & & \end{array}$$

which yields $T_A(H_A(W)/M) = 0$. Since A is faithfully flat, $H_A(W) = M$ is a Whitehead-module.

c) $\Rightarrow$ *d)*: Since W is A-solvable where exists an A-balanced sequence $0 \to U \to F \to W \to 0$ with $S_A(U) = U$ and F A-projective. By the Adjoint-Functor-Theorem, there exists an isomorphism $\lambda_G : \mathrm{Hom}(G, A) \to \mathrm{Hom}_E(H_A(G), E)$ for all A-solvable groups G. We therefore obtain the commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}_E(H_A(F), E) & \longrightarrow & \mathrm{Hom}_E(H_A(U), E) & \longrightarrow & \mathrm{Ext}^1_E(H_A(W), E) = 0 \\ \wr\big\uparrow \lambda_F & & \wr\big\uparrow \lambda_U & & \\ \mathrm{Hom}(F, A) & \longrightarrow & \mathrm{Hom}(U, A) & & \end{array}$$

whose top-row is exact since the original sequence is A-balanced.

d) $\Rightarrow$ *a)*: Since there exists an A-balanced sequence $0 \to U \to F \to W \to 0$ with $S_A(U) = U$ and F A-projective, we know that W is A-solvable. Using the maps λ_G

as before, we obtain the commutative diagram

$$\begin{array}{ccccccc} \mathrm{Hom}_E(H_A(F),E) & \longrightarrow & \mathrm{Hom}_E(H_A(U),E) & \longrightarrow & \mathrm{Ext}^1_E(H_A(W),E) & \longrightarrow & 0 \\ \wr\uparrow \lambda_F & & \wr\uparrow \lambda_U & & & & \\ \mathrm{Hom}(F,A) & \longrightarrow & \mathrm{Hom}(U,A) & \longrightarrow & 0 & & \end{array}$$

from which it follows that $H_A(W)$ is a Whitehead module. Since A is faithfully flat, an exact sequence $0 \to A \to G \to W \to 0$ with $S_A(G) = G$ is A-balanced. Therefore, it induces the exact sequence $0 \to H_A(A) \to H_A(G) \to H_A(W) \to 0$ which splits because $H_A(W)$ is a Whitehead module. We therefore obtain the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_AH_A(A) & \longrightarrow & T_AH_A(G) & \longrightarrow & T_AH_A(W) & \longrightarrow & 0 \\ & & \wr\downarrow \theta_A & & \downarrow \theta_G & & \wr\downarrow \theta_W & & \\ 0 & \longrightarrow & A & \longrightarrow & G & \longrightarrow & W & \longrightarrow & 0 \end{array}$$

whose top-row splits. Since θ_G is an isomorphism by the 3-Lemma, the bottom row splits too.

Since $A{-}Bext(G,H) \cong Ext^1_E(H_A(G), H_A(H)$ whenever G and H are A-solvable [3], c) and e) are equivalent. □

3. Groups with Endomorphism Rings of Injective Dimension 1

We now discuss the Abelian groups A for which all A-Whitehead groups are A-Whitehead splitters. The nilradical of a ring R is denoted by $N = N(R)$. If A is a torsion-free Abelian group whose endomorphism ring has finite rank, then $N(E) = 0$ if and only if its quasi-endomorphism ring $\mathbb{Q}E$ is semi-simple Artinian. Moreover, $E(A)$ is right and left Noetherian in this case [8, Section 9]. An Abelian group G is *locally A-projective* if every finite subset of G is contained in an A-projective direct summand of G which has finite A-rank [7]. If $E(A)$ has finite rank, then H_A and T_A give a category equivalence between the categories of locally A-projective groups and locally projective right E-modules [7]. We want to remind the reader that *the A-radical of a group G* is $R_A(G) = \cap\{\mathrm{Ker}\ \phi \mid \phi \in \mathrm{Hom}(G,A)\}$. Clearly, $R_A(G) = 0$ if and only if G can be embedded into A^I for some index-set I.

Theorem 3.1. *The following are equivalent for a faithfully flat Abelian group A such that $\mathbb{Q}E$ is a finite-dimensional semi-simple $\mathbb{Q}$-algebra:*

a) $id(E_E) = 1$.
b) *A-generated subgroups of torsion-free A-Whitehead groups are A-Whitehead groups.*

For such an A, every A-Whitehead groups W satisfies $R_A(W) = 0$ and is A-solvable. In particular, W is an A-Whitehead splitter.

Proof. a) ⇒ b): If V is a submodule of a Whitehead module X, then we obtain an exact sequence $0 = \mathrm{Ext}^1_E(X,E) \to \mathrm{Ext}^1_E(V,E) \to \mathrm{Ext}^2_E(X/V,E) = 0$ because $id(E_E) \le 1$. Thus, V is a Whitehead module.

Let W be a torsion-free A-Whitehead group. To see $R_A(W) = 0$, observe that there is a Whitehead module M with $W \cong T_A(M)$ by Theorem 2.5. Since A is flat, the sequence $0 \to T_A(tM) \to T_A(M) \cong W$ is exact. Hence, $T_A(tM) = 0$, which yields $tM = 0$ because A is a faithful E-module. The submodule $U = \cap\{\text{Ker } \phi \mid \phi \in \text{Hom}_E(M, E)\}$ of M is a Whitehead module by the first paragraph.

We consider the exact sequence

$$\begin{aligned} 0 \to \text{Hom}_E(M/U, E) &\xrightarrow{\pi^*} \text{Hom}_E(M, E) \to \text{Hom}_E(U, E) \\ &\to \text{Ext}^1_E(M/U, E) \to \text{Ext}^1_E(M, E) = 0. \end{aligned}$$

Since π^* is onto, $\text{Hom}_E(U, E) \cong \text{Ext}^1_E(M/U, E)$. Because U is pure in M as an Abelian group, multiplication by a non-zero integer n induces an exact sequence $\text{Ext}^1_E(M/U, E) \xrightarrow{n\times} \text{Ext}^1_E(M/U, E) \to \text{Ext}^2_E(., E) = 0$, from which we obtain that $\text{Ext}^1_E(M/U, E) \cong \text{Hom}_E(U, E)$ is divisible. However, this is only possible if $\text{Hom}_E(U, E) = 0$ since $\text{Hom}_E(U, E)$ is reduced.

Let D be the injective hull of U. Since $\mathbb{Q}E$ is semi-simple Artinian, $D \cong \mathbb{Q} \otimes_{\mathbb{Z}} U$ by [15]. Hence, D/U is torsion as an Abelian group, and we can find an index-set I, non-zero integers $\{n_i \mid i \in I\}$, and an exact sequence $0 \to X \to \oplus_I E/n_i E \to D/U \to 0$. It induces

$$\begin{aligned} 0 = \text{Hom}_E(X, E) &\to \text{Ext}^1_E(D/U, E) \\ &\to \text{Ext}^1_E(\oplus_I E/n_i E, E) \cong \Pi_I \text{Ext}^1_E(E/n_i E, E). \end{aligned}$$

Therefore, $\text{Ext}^1_E(D/U, E)$ is reduced since the exact sequence $\text{Hom}_E(E, E) \xrightarrow{n_i\times} \text{Hom}_E(E, E) \to \text{Ext}^1_E(E/n_i E, E) \to 0$ yields $\text{Ext}^1_E(E/n_i E, E) \cong E/n_i E$. On the other hand, we have the induced sequence $0 = \text{Hom}_E(U, E) \to \text{Ext}^1_E(D/U, E) \to \text{Ext}^1_E(D, E) \to \text{Ext}^1_E(U, E) = 0$ where the last Ext-group vanishes since U is a Whitehead module. Since D is torsion-free and divisible, the same holds for $\text{Ext}^1_E(D/U, E)$. However, this is only possible if $\text{Ext}^1_E(D/U, E) \cong \text{Ext}^1_E(D, E) = 0$.

If $D \neq 0$, then it has a direct summand S which is simple as a $\mathbb{Q}E$-module since $\mathbb{Q}E$ is semi-simple Artinian. In particular, $\text{Ext}^1_E(S, E) = 0$. Using Corner's Theorem [12], we can find a reduced Abelian group B with $End(B) \cong E^{op}$ which fits into an exact sequence $0 \to E^{op} \to B \to \mathbb{Q}E^{op} \to 0$ as a left E^{op}-module. Then, B can be viewed as a right E-module fitting into an exact sequence $0 \to E \to B \to \mathbb{Q}E \to 0$. We can find an E-submodule $E \subseteq V$ of B with $V/E \cong S$. Since $\text{Ext}^1_E(S, E) = 0$, we have $V \cong E \oplus S$. However, S is divisible as an Abelian group, while V is reduced, a contradiction. Therefore, $D = 0$; and $M \subseteq E^J$ for some index-set J. Since E is Noetherian as mentioned before, E^J is locally projective [1]. In particular, ϕ_{E^J} is an isomorphism by [7]. Because A is faithfully flat, ϕ_M has to be an isomorphism too by [4]. Therefore, $W \cong T_A(M)$ is A-solvable as a subgroup of the locally A-projective group $T_A(E^J)$ and $R_A(W) = 0$. By Theorem 2.7, W is an A-Whitehead splitter, and $H_A(W)$ is a Whitehead module.

An A-generated subgroup C of W is A-solvable since A is flat. By Theorem 2.7, C is an A-Whitehead group if the can show that $H_A(C)$ is a Whitehead module. However, this holds because the class of Whitehead modules is closed with respect to submodule if $id(E_E) = 1$ by the first paragraph.

b) $\Rightarrow$ *a)*: Clearly, $id(E_E) = 1$ if and only if $\mathrm{Ext}^1_E(E/I, \mathbb{Q}E/E) = 0$ for all right ideals I of E. Standard homological arguments show $\mathrm{Ext}^1_E(E/I, \mathbb{Q}E/E) \cong \mathrm{Ext}^1_E(I, E)$. To see that I is a Whitehead module, observe that $IA \cong T_A(I)$ is an A-solvable A-Whitehead module by b) because A is an A-Whitehead splitter since A is faithfully flat. By Theorem 2.7, IA is an A-Whitehead splitter, and $H_A(IA)$ is a Whitehead module. But, $I \cong H_A T_A(I) \cong H_A(IA)$ since A is faithfully flat. □

Corollary 3.2. *Let A be an Abelian group such that $\mathbb{Q}E$ is a finite dimensional semi-simple $\mathbb{Q}$-algebra and $id(_EE) = id(E_E) = 1$. Every A-Whitehead group is torsion-free, A-solvable and an A-Whitehead splitter.*

Proof. If p is a prime with $pA = A$, then $(E/J)_p = 0$ for every essential right ideal J of E since E/J is bounded and p-divisible. By Theorem 3.1, it remains to show that every A-Whitehead group W is torsion-free. Suppose that W is not torsion-free, and select a Whitehead module M with $W \cong T_A(M)$. Since A is faithfully flat, $tW \cong T_A(tM)$. Select a cyclic submodule U of M with U^+ torsion. Because $id(E_E) = 1$, U is a Whitehead module. There is a right ideal I of E with $E/I \cong U$ which is a reflexive E-module by [14]. The exact sequence $0 = \mathrm{Hom}_E(U, E) \to \mathrm{Hom}_E(E, E) \to \mathrm{Hom}_E(I, E) \to \mathrm{Ext}^1_E(U, E) = 0$ yields $\mathrm{Hom}_E(I, E) \cong E$. Hence, $I \cong \mathrm{Hom}_E(\mathrm{Hom}_E(I, E), E) \cong E$. Thus, U fits into an exact sequence $0 \to E \to E \to U \to 0$, from which we get $E \cong E \oplus U$, which is a contradiction unless $U = 0$. □

Moreover, if E is right and left Noetherian and hereditary, then A is self-small and faithfully flat, and E is semi-prime [4].

Corollary 3.3. *Let A be a self-small faithfully flat Abelian group such that E is a right and left Noetherian, hereditary ring with $r_0(E) < \infty$. If W is an A-Whitehead group, then W is locally A-projective. In particular, every countably A-generated A-Whitehead group is A-projective.*

Proof. Select a finite subset X of $H_A(W)$ and a finitely generated submodule U of $H_A(W)$ containing X. The $\mathbb{Z}$-purification V of U in $H_A(W)$ is countable. Since E is hereditary, $\mathrm{Ext}_E(H_A(W)/V, E)$ is divisible as an Abelian group. On the other hand, we have an exact sequence $\mathrm{Hom}_E(H_A(W), E) \to \mathrm{Hom}_E(V, E) \to \mathrm{Ext}_E(H_A(W)/V, E) \to Ext_E(H_A(W), E) = 0$, because $H_A(W)$ is a Whitehead module. Since $\mathrm{Hom}_E(V, E)$ is a finitely generated right E-module, the same holds for $\mathrm{Ext}_E(H_A(W)/V, E)$. Thus, $\mathrm{Ext}_E(H_A(W)/V, E) \cong P' \oplus T$ where P' is projective and T^+ is bounded. Because A is reduced, $\mathrm{Ext}_E(H_A(W)/V, E)$ is reduced, which is not possible unless $\mathrm{Ext}_E(H_A(W)/V, E) = 0$.

Since $R_A(W) = 0$ by Theorem 3.1, $H_A(W) \subseteq E^I$ for some index-set I. Because E is left Noetherian, E^I is a locally projective module. Thus, its countable submodule V has to be projective. Since V contains a finitely generated essential submodule, it is finitely generated by Sandomierski's Lemma [9]. But then, there is $n < \omega$ such that $\mathrm{Ext}_E(H_A(W)/V, V) = 0$ since $\mathrm{Ext}_E(H_A(W)/V, E) = 0$. Consequently, V is a finitely generated projective direct summand of $H_A(W)$, and $H_A(W)$ is locally projective. By [7], $W \cong T_A H_A(W)$ is locally A-projective.

If G is an epimorphic image of $\oplus_\omega A$, then $H_A(G)$ is an image of $\oplus_\omega E$ since G is A-solvable. However, a countably generated locally projective module is projective. □

4. κ-A-Projective Groups

Let κ be an uncountable cardinal, and assume that A is a torsion-free Abelian with $|A| < \kappa$ whose endomorphism ring is right and left Noetherian and hereditary. An A-generated group G is *κ-A-projective* if every κ-A-generated subgroup of G is A-projective. Since every finitely A-generated subgroup of G is A-projective in this case, κ-A-projective groups are A-solvable. An A-projective subgroup U of an $\aleph_0$-A-projective group G is κ-A-closed if $(U + V)/U$ is A-projective for all κ-A-generated subgroups V of G. If $|U| < \kappa$, then this is equivalent to the condition that W/U is A-projective for all κ-A-generated subgroups W of G with $U \subseteq W$. Finally, G is *strongly κ-A-projective* if it is κ-A-projective and every κ-A-generated subgroup of G is contained in a κ-A-generated, κ-A-closed subgroup of G. Our first result reduces the investigation of strongly κ-A-projective groups to that of strongly κ-projective modules.

Proposition 4.1. *Let κ be a regular uncountable cardinal. If A is a torsion-free Abelian group with $|A| < \kappa$ whose endomorphism ring is right and left Noetherian and hereditary, then the following are equivalent for a κ-A-projective group G with $|G| \geq \kappa$:*

a) *G is strongly κ-A-projective.*
b) *$H_A(G)$ is a strongly κ-projective right E-module.*

Proof. Consider an exact sequence $0 \to U \to \oplus_I A \overset{\beta}{\to} G \to 0$ with $|I| \geq \kappa$. Since A is faithfully flat, the sequence is A-balanced and $S_A(U) = U$. Thus, $H_A(G)$ is an epimorphic image of $\oplus_I E$. Moreover, $G \cong T_A H_A(G)$ yields $|H_A(G)| = |G| \geq \kappa$.

a) $\Rightarrow$ *b)*: Suppose that U is a submodule of $H_A(G)$ with $|U| < \kappa$. By Lemma 2.4, the evaluation map $\theta : T_A(U) \to UA$ is an isomorphism since G is A-solvable and A is faithfully flat. Then, $|UA| < \kappa$, and there is a κ-A-generated κ-closed subgroup V of G with $UA \subseteq V$. Observe that V is A-projective. Therefore, $H_A(UA) \subseteq H_A(V)$ is projective since E is right hereditary. However, $U \cong H_A T_A(U) \cong H_A(UA)$ since $U \subseteq H_A(G)$ and $\phi_{H_A(G)}$ is an isomorphism by [4]. Thus, $H_A(G)$ is κ-projective.

We now show that $H_A(V)$ is κ-closed in $H_A(G)$. Let W be a submodule of $H_A(G)$ with $|W| < \kappa$ which contains $H_A(V)$. Since $|WA| < \kappa$ and $V \subseteq WA$, we obtain that WA/V is A-projective. Hence, V is a direct summand of WA by [2] since E is right and left Noetherian and hereditary. Applying the functor H_A yields that $H_A(V)$ is a direct summand of $H_A(WA)$. By Lemma 2.4, $H_A(WA) = W$, and we are done.

b) $\Rightarrow$ *a)*: For a κ-A-generated subgroup U of G, choose an exact sequence $\oplus_I A \overset{\pi}{\to} U \to 0$. Since G is A-solvable, the same holds for U, and the last sequence is A-balanced. Therefore, $H_A(U)$ is a κ-generated submodule of $H_A(G)$. We can find a κ-closed submodule W of $H_A(G)$ containing $H_A(U)$ with $|W| < \kappa$. Then, $U = H_A(U)A \subseteq WA$ has cardinality less than κ, and it remains to show that WA is κ-A-closed. For this, let V be a κ-A-generated subgroup of G containing WA. Since $W = H_A(WA)$ by Lemma 2.4, $H_A(V)/H_A(WA)$ is projective. Consider the commutative diagram

$$\begin{array}{ccccccc} T_AH_A(WA) & \longrightarrow & T_AH_A(V) & \longrightarrow & T_A(H_A(V)/H_A(WA)) & \longrightarrow & 0 \\ \wr\uparrow\theta_{WA} & & \wr\uparrow\theta_V & & \uparrow & & \\ WA & \longrightarrow & V & \longrightarrow & V/WA & \longrightarrow & 0. \end{array}$$

Since WA and V are A-solvable, V/WA is A-projective . □

We now can prove the main result of this section.

Theorem 4.2. *Let κ be a regular, uncountable cardinal which is not weakly compact, and suppose that A is a torsion-free Abelian group with $|A| < \kappa$ such that E is right and left Noetherian and hereditary.*

a) *If we assume $V = L$, then there exists a strongly κ-A-projective group G with* $\mathrm{Hom}(G, A) = 0$.

b) *Let $\kappa = \aleph_1$, and assume $MA + \aleph_1 < 2^{\aleph_0}$. Every strongly $\aleph_1$-A-projective group G with $|G| < 2^{\aleph_0}$ is an A-Whitehead splitter.*

Proof. a) By [13], there exists strongly κ-free left E^{op}-module M of cardinality κ with $End_{\mathbb{Z}}(M) = E^{op}$. Therefore, $End_{E^{op}}(M) = C(E)$, the center of E. Viewing M as an E-module yields a strongly κ-free right E-module M with $End_E(M) = C(E)$. We consider $G = T_A(M)$. If $\phi_1, \ldots, \phi_n \in H_AT_A(M)$, then there is a κ-generated submodule U of M such that $\phi_1(A) + \ldots + \phi_n(A) \subseteq T_A(U)$ since $|A| < \kappa$. However, since U is contained in a free submodule P of M, we obtain that $\phi_1(A) + \ldots + \phi_n(A)$ is A-projective. Thus, G is A-solvable, and $\phi_{H_AT_A(M)}$ is an isomorphism. By [4], ϕ_M is an isomorphism since A is faithfully flat. Consequently, $M \cong H_A(G)$ is strongly κ-projective. By Proposition 4.1, G is strongly κ-A-projective. Moreover, every subset of G of cardinality less than κ is contained in an A-free subgroup of G.

Since E is Noetherian, it does not have any infinite family of orthogonal idempotent, and the same holds for $C(E)$. By the Adjoint-Functor-Theorem, we have $End_{\mathbb{Z}}(T_A(M)) \cong End_E(M) = C(E)$ since $T_A(M)$ is A-solvable. Therefore, $End_{\mathbb{Z}}(G)$ is commutative, and $G = G_1 \oplus \ldots \oplus G_m$ where each G_j is indecomposable and $\mathrm{Hom}(G_i, G_j) = 0$ for $i \neq j$. Since G_i is A-generated and indecomposable, G_i is either A-projective of finite A-rank, or $\mathrm{Hom}(G_i, A) = 0$ since $E(A)$ is right and left Noetherian and hereditary. Consequently, $G = B \oplus C$ where C is A-projective of finite A-rank, and $\mathrm{Hom}(B, A) = \mathrm{Hom}(B, C) = \mathrm{Hom}(C, B) = 0$.

Since $|A| < \kappa$, G contains a subgroup U isomorphic to $\oplus_\omega A$. We can find a subgroup V of G which is A-free and contains C and U, say $V \cong \oplus_I A$ for some infinite index-set I. Since A is discrete in the finite topology, it is self-small. Therefore, we can find a finite subset J of I such that $\alpha(A) \subseteq \oplus_J A$. Since C is a direct summand of G, we have $V = C \oplus (B \cap V)$ and $B \cap V \cong (\oplus_J A)/C \oplus (\oplus_{I \setminus J} A)$. But then, $\mathrm{Hom}(C, B) \neq 0$, which results in a contradiction unless $C = 0$. This shows, $\mathrm{Hom}(G, A) = 0$.

b) If G is a strongly $\aleph_1$-projective group with $\aleph_1 \leq |G| < 2^{\aleph_0}$, then G is A-solvable. By Proposition 4.1, $H_A(G)$ is a strongly $\aleph_1$-projective right E-module. Arguing as in the case $A = \mathbb{Z}$ (e.g. see [10, Chapter 12] or [11]), we obtain that $H_A(G)$ is a Whitehead-module. By Theorem 2.7, G is an A-Whitehead splitter. □

References

[1] Albrecht, U., *Endomorphism Rings and A-Projective Torsion-Free Abelian Groups*, Abelian Group Theory, Proceedings Honolulu 1982/83, Springer Lecture Notes in Mathematics 1006, Springer Verlag, Berlin, New York, Heidelberg, 1983, 209-227.

[2] Albrecht, U., *Baer's Lemma and Fuchs' Problem 84a*, Transactions of the American Mathematical Society, **293**(1986), 565-582.

[3] Albrecht, U., *Extension Functors on the Category of A-Solvable Abelian Groups*, Czechoslovak Journal of Mathematics, **41**(116)(1991), 685-694.

[4] Albrecht, U., *Endomorphism Rings, Tensor products, and Fuchs' Problem 47*, Warfield Memorial Volume; Contemporary Mathematics, **130**(1992), 17 - 31.

[5] Albrecht, U., *Abelian Groups with Semi-Simple Artinian Quasi-Endomorphism Rings*, Rocky Mountain Journal of Mathematics, **24**(1994), 853-866.

[6] Arnold, D.M., Lady, L., *Endomorphism rings and direct sums of torsion-free Abelian groups*, Trans. Amer. Math. Soc., **211**(1975), 225-237.

[7] Arnold, D.M., and Murley, C.E., *Abelian groups A, such that* $\mathrm{Hom}(A,-)$ *preserves direct sums of copies of A*, Pac. J. Math., **56**(1975), 7-20.

[8] Arnold, D. M., *Finite Rank Torsion Free Abelian Groups and Rings*, Lecture Notes in Mathematics, Vol. 931, Springer-Verlag Berlin-Heidelberg-New York, 1982.

[9] Chatters, A.W., Hajarnavis, C.R., *Rings with Chain Conditions*, Pitman Advanced Publishing 44, Boston, London, Melbourne, 1980.

[10] Eklof, P.C., Mekler, A.H., *Almost free Modules*, Vol. **46**, North Holland Mathematical Library, 1990.

[11] Eklof, P.C, Mekler, A.H., Shelah, S., *Uniformization and the diversity of Whitehead groups*, Israel J. Math., **80**(1992), 301-321.

[12] Fuchs, L., *Infinite Abelian Groups*, vol. 1 and 2, Academic Press, New York and London, 1970/73.

[13] Göbel, R., Trlifaj, J., *Approximations and Endomorphism Algebras of Modules*, vol. I and II, Expositions in Mathematics 41, De Gruyter, 2012.

[14] Jans, J., *Rings and Homology*, Holt, Rinehart, and Winston, 1963.

[15] Stenström, B., *Rings of Quotients*, Lecture Notes in Math. 217, Springer Verlag, Berlin, Heidelberg, New York, 1975.

[16] Ulmer, F., *A flatness criterion in Grothendieck categories*, Inv. Math., **19**(1973), 331-336.

[17] Warfield, R.B., Jr., *Homomorphisms and duality for torsion-free groups*, Math. Z., **107**(1968), 189-200.

[18] Žemlička, J., *When products of self-small modules are self-small*, Comm. Alg., **36**(2008), no. 7, 2570-2576.

Ulrich Albrecht
Department of Mathematics, Auburn University
Auburn, AL 36849, U.S.A.
e-mail: `albreuf@@mail.auburn.edu`