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On products of self-small abelian groups

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Abstract. An abelian group A is called self-small if direct sums of copies of A commute with the covariant $\mathrm{Hom}\,(A, -)$ functor. The paper presents an elementary example of a non-self-small countable product of self-small abelian groups without non-zero homomorphisms between different ones. A criterion of self-smallness of a finite product of modules is given.

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1. Introduction

The notion of self-small module as a generalization of the finitely generated module appears as a useful tool in the study of splitting properties [1], groups of homomorphisms of graded modules [10] or representable equivalences between subcategories of module categories [8].

The paper [4] in which the topic of self-small modules is introduced contains a mistake in the proof of [4, Corollary 1.3], which states when the product of (infinite) system $(A_i \mid i \in I)$ of self-small modules is self-small. A counterexample and correct version of the hypothesis were presented in [12] for a system of modules over a non-steady abelian regular ring. In the present paper an elementary counterexample in the category of $\mathbb{Z}$-modules, i.e. abelian groups, is constructed and as a consequence, an elementary example of two self-small abelian groups such that their product is not self-small is presented.

Throughout the paper a *module* means a right module over an associative ring with unit. If A and B are two modules over a ring R, $\mathrm{Hom}_R(A, B)$ denotes the abelian group of all R-homomorphisms $A \to B$. The set of all prime numbers is denoted by $\mathbb{P}$, for given $p \in \mathbb{P}$, $\mathbb{Z}_p$ means the cyclic group of order p and $\mathbb{Q}$ is the group of rational numbers. $E(A)$ denotes the injective envelope of the module A. Recall that injective $\mathbb{Z}$-modules, i.e. abelian groups, are precisely the divisible ones. For non-explained terminology we refer to [9].

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Definition 1.1. *An R-module A is self-small, if for arbitrary index set I and each $f \in \mathrm{Hom}_R(A, \bigoplus_{i\in I} A_i)$, where $A_i \cong A$, there exists a finite $I^{'} \subseteq I$ such that $f(A) \subseteq \bigoplus_{i\in I'} A_i$.*

Properties of self-small modules and mainly of self-small groups are thoroughly investigated in [2], [3], [4], [5] and [6] revealing several characterizations of self-small groups and discussing the properties of the category of self-small groups and modules.

For our purpose the following notation will be of use:

Definition 1.2. *For an R-module A and $B \subseteq A$ we define the annihilator of B*

$$B^* := \{f \mid f \in End_R(A), f(a) = 0 \ for \ each \ a \in B\} .$$

The first (negative) characterization of self-small modules is given in [4] and it describes non-self-small modules via annihilators and chains of submodules:

Theorem 1.3. [4, Proposition 1.1] *For an R-module A the following conditions are equivalent:*

1. *A is not self-small*
2. *there exists a chain $A_1 \subseteq A_2 \subseteq ... \subseteq A_n \subseteq ... \subsetneq A$ of proper submodules in A such that $\bigcup_{n=1}^{\infty} A_n = A$ and for each $n \in \mathbb{N}$ we have $A_n^* \neq \{0\}$.*

2. Examples

The key tool for constructions of this paper is the following well-known lemma:

Lemma 2.1. $\prod_{p\in\mathbb{P}} \mathbb{Z}_p / \bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p \cong \mathbb{Q}^{(2^\omega)}$.

Proof. Since $\bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p$ is the torsion part of $\prod_{p\in\mathbb{P}} \mathbb{Z}_p$, the group $\prod_{p\in\mathbb{P}} \mathbb{Z}_p / \bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p$ is torsion-free. Now the assertion follows from [7, Exercises S 2.5 and S 2.7]. □

Let $B = \prod_{p\in\mathbb{P}} \mathbb{Z}_p / \bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p$. By the previous lemma it is easy to see that there exists an infinite countable chain of subgroups $B_i \subseteq B_{i+1}$ of B such that $B = \bigcup_n B_n$ and $\mathrm{Hom}_{\mathbb{Z}}(B/B_n, \mathbb{Q}) \neq 0$ for each n.

Recall that $\mathbb{Q}$ is torsion-free of rank 1 and each nontrivial factor of $\mathbb{Q}$ is a torsion group, hence there is no nonzero non-injective endomorphism $\mathbb{Q}$, which by Theorem 1.3 implies well-known fact that $\mathbb{Q}$ is self-small.

Using the previous observations, the counterexample to [4, Corollary 1.3] can be constructed:

Example 2.2. Since $\mathbb{Z}_p$ is finite for every $p \in \mathbb{P}$, it is a self-small group. Now, all homomorphisms between $\mathbb{Z}_p$'s for different $p \in \mathbb{P}$, or $\mathbb{Q}$ are trivial:

$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_p, \mathbb{Q}) = \{0\}$, since $\mathbb{Z}_p$ is a torsion group, whereas $\mathbb{Q}$ is torsion-free. $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}_p) = \{0\}$, since every factor of $\mathbb{Q}$ is divisible and 0 is the only divisible subgroup of $\mathbb{Z}_p$. Obviously, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_q, \mathbb{Z}_p) = \{0\}$.

Let $A = \mathbb{Q} \times \prod_{p\in\mathbb{P}} \mathbb{Z}_p$ and $B = A/\left(\mathbb{Q} \times \bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p\right) \cong \prod_{p\in\mathbb{P}} \mathbb{Z}_p / \bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p$. Then by Lemma 2.1 there exists a countable chain of subgroups $B_i \subseteq B_{i+1}$ of B, $i < \omega$, such that $B = \bigcup_{i<\omega} B_i$ and $\mathrm{Hom}_{\mathbb{Z}}(B/B_n, \mathbb{Q}) \neq 0$ for each n, where $\mathbb{Q}$ may be viewed as a subgroup of A. Now put A_n to be the preimage of B_n in A under factorization by $\mathbb{Q} \times \bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p$. Then the subgroups $A_n, n \in \mathbb{N}$ form a chain of subgroups and $A = \bigcup_{n\in\mathbb{N}} A_n$. At the same time the composition of the factorization by $\mathbb{Q}\times\bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p$ and $\bar{\nu_n}$ is an endomorphism φ_n of the group A such, that $A_n \subseteq Ker\,\varphi_n$. Therefore the condition of Theorem 1.3 is satisfied, hence the group A is not self-small.

The previous example shows that for two different primes p, q

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_q, \mathbb{Z}_p) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}_p) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_p, \mathbb{Q}) = \{0\},$$

all the groups $\mathbb{Z}_p$, $p \in \mathbb{P}$ and $\mathbb{Q}$ are self-small, but the group $\mathbb{Q} \times \prod_{p\in\mathbb{P}} \mathbb{Z}_p$ is not self-small.

Finally, as a consequence of Example 2.2 an elementary example of two self-small abelian groups such that their product is not self-small may be constructed. It illustrates that the assumption $\mathrm{Hom}_{\mathbb{Z}}(M_j, M_i) = 0$ for each $i \neq j$ cannot be omitted even in the category of $\mathbb{Z}$-modules.

Example 2.3. By [12, Example 2.7] the group $\prod_{p\in\mathbb{P}} \mathbb{Z}_p$ is self-small as well as the group $\mathbb{Q}$. Moreover, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \prod_{p\in\mathbb{P}} \mathbb{Z}_p) = \prod_{p\in\mathbb{P}} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}_p) = 0$. Nevertheless, the product $\mathbb{Q}\times\prod_{p\in\mathbb{P}} \mathbb{Z}_p$ is not self-small by Example 2.2. Note that it is not surprising in view of Corollary 2.6 that the structure of $\mathrm{Hom}_{\mathbb{Z}}(\prod_{p\in\mathbb{P}} \mathbb{Z}_p, \mathbb{Q})$ is quite rich as shown in Lemma 2.1.

Recall that classes of small modules, i.e. modules over which the covariant Hom-functor commutes with all direct sums, are closed under homomorphic images and extensions [11, Proposition 1.3]. Obviously, self-small modules do not satisfy this closure property and, moreover, although any class of self-small modules is closed under direct summands, the last example illustrates that it is not closed under finite direct sums.

Proposition 2.4. *The following conditions are equivalent for a finite system of self-small R-modules $(M_i|\ i \le k)$:*

1. $\prod_{i\le k} M_i$ *is not self-small*
2. *there exist $i, j \le k$ and a chain $N_1 \subseteq N_2 \subseteq ... \subseteq N_n \subseteq ...$ of proper submodules of M_i such that $\bigcup_{n=1}^{\infty} N_n = M_i$ and $\mathrm{Hom}_R(M_i/N_n, M_j) \neq 0$ for each $n \in \mathbb{N}$.*

Proof. Put $M = \prod_{i\le k} M_i$.

(1)→(2) If M is not self-small, there exists a chain $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq \ldots$ of proper submodules of M for which $\bigcup_{n=1}^{\infty} A_n = M$ and $\mathrm{Hom}_R(M/A_n, M) \neq 0$ for each $n \in \mathbb{N}$. Put $A_n^i = M_i \cap A_n$ for each $i \le k$ and $n \in \mathbb{N}$. Then $M_i = \bigcup_n A_n^i$ for each $i \le k$ and there exists at least one index i such that the chain $A_1^i \subseteq A_2^i \subseteq \ldots \subseteq A_n^i \subseteq \ldots$ consists of proper submodules of M_i (or else the condition on the original chain is broken) and further on we consider only such i's.

Since for each $n \in \mathbb{N}$ there exist $0 \neq b_n \in M \setminus A_n$ and $f_n : M/A_n \to M$ such that $f_n(b_n + A_n) \neq 0$, for each n we can find an index $i(n) \le k$ with $f_n \pi_{A_n} \nu_{i(n)} \pi_{i(n)}(b_n) \neq 0$ (where $\pi_{i(n)}$, resp. $\nu_{i(n)}$ are the natural projection, resp. injection and π_{A_n} is is the natural projection $M \to M/A_n$). Now, by pigeonhole principle, there must exist at least one index i_0 such that $S := \{n \in \mathbb{N} \,|\, i(n) = i_0\}$ is infinite. By the same principle, there must exist at least one index j_0 such that $T := \{n \in S \mid \pi_{j_0} f_n \pi_{A_n} \nu_{i_0} \pi_{i_0} (b_n) \neq 0\}$ is infinite. The couple i_0, j_0 proves the implication.

(2)→(1) Put $A_n = \pi_i^{-1}(N_n)$ where $\pi_i : M \to M_i$ is the natural projection, so $\bigcup_n A_n = M$. If $0 \neq f_n \in \mathrm{Hom}_R(M_i, M_j)$ such that $N_n \subseteq \ker f_n$ and $f_n(m_n) \neq 0$ for some suitable $m_n \in M_i$, then $\nu_j f_n \pi_i \in \mathrm{Hom}_R(M, M)$, where $\nu_j : M_j \to M$ is the natural injection, $A_n \subseteq \ker \nu_j f_n \pi_i$ and the nonzero element having m_n on the i-th position show that the condition of the Theorem 1.3 holds. □

Corollary 2.5. *Let $(M_i \mid i \leq k)$ be a finite system of R-modules. Then $\prod_{i \leq n} M_i$ is self-small if and only if for every i, j and every chain $N_1 \subseteq N_2 \subseteq \ldots \subseteq N_n \subseteq \ldots$ of proper submodules of M_i such that $\bigcup_{n=1}^{\infty} N_n = M_i$ there exist n for which $\mathrm{Hom}_R(M_i/N_n, M_j) = 0$.*

In consequence we see that the "finite version" of [4, Corollary 1.3] remains true:

Corollary 2.6. *Let $(M_i \mid i \leq n)$ be a finite system of self-small modules satisfying the condition $\mathrm{Hom}_R(M_j, M_i) = 0$ for each $i \neq j$. Then $\prod_{i \leq n} M_i$ is a self-small module.*

The previous results motivates the formulation of the following open problem.

Question. Let $0 \to S_0 \to S_1 \to S_2 \to 0$ be a short exact sequence in the category of abelian groups (or more generally right modules over a ring). If $i \neq j \neq k \neq i$ and S_i, S_j are self-small, can the condition that S_k is self-small be characterized by properties of the groups S_i, S_j and the corresponding homomorphisms?

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