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Nonlinear differential polynomial sharing a nonzero polynomial with certain degree

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Dedicated to Professor Ioan A. Rus on the occasion of his 80th anniversary

Abstract. In this paper we study the uniqueness problems of meromorphic functions when certain non-linear differential polynomial sharing a nonzero polynomial with certain degree. We obtain some results which will rectify, improve and generalize some recent results of C. Wu and J. Li [15] in a large extent. Our results will also improve and generalize some recent results due to Fang [5], Zhang-Zhang [24], Zhang [22], Xu et al. [16], Qi-Yang [14], Dou-Qi-Yang [4], Zhang-Xu [26] and Liu-Yang [13].

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1. Introduction, definitions and results

In this paper by meromorphic functions we shall always mean meromorphic functions in the complex plane.

Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, provided that $f - a$ and $g - a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f - a$ and $g - a$ have the same zeros ignoring multiplicities. In addition we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM.

We adopt the standard notations of value distribution theory (see [7]). For a non-constant meromorphic function f, we denote by $T(r, f)$ the Nevanlinna characteristic of f and by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ possibly outside a set of finite linear measure.

A meromorphic function $a(z)$ is called a small function with respect to f, provided that $T(r, a) = S(r, f)$.

We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r)=o(T(r))$ as $r \longrightarrow \infty$, outside of a possible exceptional set of finite linear measure.

A finite value z_0 is said to be a fixed point of $f(z)$ if $f(z_0)=z_0$.

For the sake of simplicity we also use the notations $m^* := \chi_\mu m$, where $\chi_\mu = 0$ if $\mu = 0$, $\chi_\mu = 1$ if $\mu \neq 0$.

In 1959, W.K. Hayman (see [7], Corollary of Theorem 9) proved the following theorem.

Theorem A. *Let f be a transcendental meromorphic function and $n(\geq 3)$ is an integer. Then $f^n f' = 1$ has infinitely many solutions.*

Theorem A was extended by Chen [3] in the following manner:

Theorem B. *Let f be a transcendental entire function, n, k two positive integers with $n \geq k+1$. Then $(f^n)^{(k)} - 1$ has infinitely many zeros.*

Latter Fang [5] obtained the following two uniqueness theorem corresponding to *Theorem B.*

Theorem C. *Letf and g be two non-constant entire functions, and let n, k be two positive integers with $n > 2k+4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$ or $f \equiv tg$ for a constant t such that $t^n = 1$.*

Theorem D. *Let f and g be two non-constant entire functions, and let n, k be two positive integers with $n > 2k+8$. If $(f^n(z)(f(z)-1))^{(k)}$ and $(g^n(z)(g(z)-1))^{(k)}$ share 1 CM, then $f(z) \equiv g(z)$.*

In 2008, improving the above results J. F. Zhang and X. Y. Zhang [24] obtained the following theorem:

Theorem E. *Let f and g be two non-constant entire functions and let n, k be two positive integers with $n > 5k+7$. If $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share 1 IM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1 c_2)^n [nc]^{2k} = 1$ or $f \equiv tg$ for a constant t such that $t^n = 1$.*

Theorem F. *Let f and g be two non-constant entire functions, and let n, k be two positive integers with $n > 5k+13$. If $(f^n(z)(f(z)-1))^{(k)}$ and $(g^n(z)(g(z)-1))^{(k)}$ share 1 IM, then $f(z) \equiv g(z)$.*

In 2008 Zhang [22] obtained similar type of result as mentioned in *Theorem E* in the the following way:

Theorem G. *Let f and g be two non-constant entire functions, and let n, k be two positive integers with $n > 2k+4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, then either*

(1) *$k = 1$, $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(c_1 c_2)^n (nc)^2 = -1$ or*
(2) *$f \equiv tg$ for a constant t such that $t^n = 1$.*

Recently Xiao-Bin Zhang and Jun-Feng Xu [26] proved the following result for meromorphic function.

Theorem H. [26] *Let f and g be two non-constant meromorphic functions, and $a(z)$ $(\not\equiv 0,\infty)$ be a small function with respect to f. Let n, k and m be three positive integers with $n > 3k+m+8$ and let $P(w) = a_m w^m + a_{m-1}w^{m-1} + \ldots + a_1 w + a_0$ or $P(w) \equiv c_0$ where $a_0(\neq 0), a_1, \ldots, a_{m-1}, a_m(\neq 0), c_0(\neq 0)$ are complex constants. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share a CM, then*

(I) *when $P(w) = a_m w^m + a_{m-1}w^{m-1} + \ldots + a_1 w + a_0$, one of the following three cases holds:*

(I1) *$f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where*

$$d = GCD(n+m, \ldots, n+m-i, \ldots, n),\ a_{m-i} \neq 0$$

for some $i = 1, 2, \ldots, m$,

(I2) *f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where*

$$R(\omega_1,\omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \ldots + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \ldots + a_0),$$

(I3) *$[f^n P(f)]^{(k)}[g^n P(g)]^{(k)} \equiv a^2$;*

(II) *when $P(w) \equiv c_0$, one of the following two cases holds:*

(II1) *$f \equiv tg$ for some constant t such that $t^n = 1$,*

(II2) *$c_0^2[f^n]^{(k)}[g^n]^{(k)} \equiv a^2$.*

Generalized results in the above directions for entire function were obtained by Qi-Yang [14] and Dou-Qi-Yang [4] in the following manner:

Theorem I. *Let f and g be two transcendental entire functions, and let n, k and m be three positive integers with $n > 2k + m^* + 4$, λ, μ be two constants such that $|\lambda| + |\mu| \neq 0$. If $[f^n(\lambda f^m + \mu)]^{(k)}$ and $[g^n(\lambda g^m + \mu)]^{(k)}$ share z CM, then one of the following conclusions hold:*

(1) *If $\lambda\mu \neq 0$, then $f^d \equiv g^d$, where $d = gcd(n,m)$; in particular $f \equiv g$, when $d = 1$;*

(2) *If $\lambda\mu = 0$, then $f \equiv cg$, where c is a constant satisfying $c^{n+m^*} = 1$, or $k = 1$ and $f(z) = b_1 e^{bz^2}$, $g(z) = b_2 e^{-bz^2}$, for some constants b_1, b_2 and b that satisfy $4(\lambda+\mu)^2(b_1b_2)^{n+m^*}[(n+m^*)b]^2 = -1$.*

Theorem J. *Let $P(z) = a_m z^m + a_{m-1}z^{m-1} + \ldots + a_1 z + a_0$ or $P(z) = C$, where $a_0, a_1, \ldots, a_{m-1}$, $a_m(\neq 0), C(\neq 0)$ are complex constants. Suppose that f and g be two transcendental entire functions, and let n, k and m be three positive integers with $n > 2k + m^{**} + 4$, where $m^{**} = 0$, if $P(z) \equiv C$, otherwise $m^{**} = m$. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share z CM then the following conclusions hold:*

(i) *If $P(w) = a_m w^m + a_{m-1}w^{m-1} + \ldots + a_1 w + a_0$ is not a monomial, then $f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = gcd(n+m, \ldots, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, 2, \ldots, m$, or f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where*

$$R(\omega_1,\omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \ldots + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \ldots + a_0);$$

(ii) *If* $P(z) \equiv C$ *or* $P(z) = a_m z^m$ *then* $f \equiv tg$ *for some constant* t *such that* $t^{n+m^{**}} = 1$, *or then* $f = b_1 e^{bz^2}$, $g = b_2 e^{-bz^2}$, *for three constants* b_1, b_2 *and* b *that satisfy* $4a_m^2(b_1b_2)^{n+m}[(n+m)b]^2 = -1$ *or* $4C^2(b_1b_2)^n[nb]^2 = -1$.

In 2013, Liu and Yang [13] replaced the CM value sharing concept by IM fixed point sharing one in the above two theorems. They proved the following results:

Theorem K. *Let* f *and* g *be two transcendental entire functions, and let* n, k *and* m *be three positive integers with* $n > 5k + 4m^* + 7$, λ, μ *be two constants such that* $|\lambda| + |\mu| \neq 0$. *If* $[f^n(\lambda f^m + \mu)]^{(k)}$ *and* $[g^n(\lambda g^m + \mu)]^{(k)}$ *share* z *IM, then the conclusion of* Theorem I *holds*

Theorem L. *Let* $P(\omega) = a_m\omega^m + a_{m-1}\omega^{m-1} + \ldots + a_1\omega + a_0$ *or* $P(\omega) = C$, *where* $a_0, a_1, \ldots, a_{m-1}$, $a_m(\neq 0), C(\neq 0)$ *are complex constants. Suppose that* f *and* g *be two transcendental entire functions, and let* n, k *and* m *be three positive integers with* $n > 5k + 4m^{**} + 7$. *If* $[f^nP(f)]^{(k)}$ *and* $[g^nP(g)]^{(k)}$ *share* z *IM then the conclusion of* Theorem J *holds*

In this paper, we always use $P(\omega)$ denoting an arbitrary polynomial of degree n as follows:

$$P(\omega) = a_n\omega^n + a_{n-1}\omega^{n-1} + \ldots + a_0 = a_n(\omega - c_{l_1})^{l_1}(\omega - c_{l_2})^{l_2}\ldots(\omega - c_{l_s})^{l_s}, \quad (1.1)$$

where $a_i (i = 0, 1, \ldots, n-1)$, $a_n \neq 0$ and $c_{l_j} (j = 1, 2, \ldots, s)$ are distinct finite complex numbers and $l_1, l_2, \ldots, l_s$, s, n and k are all positives integers satisfying

$$\sum_{i=1}^{s} l_i = n.$$

Also we let

$$l = \max\{l_1, l_2, \ldots, l_s\}$$

and from (1.1) we have

$$P(w) = (w - c_l)^l P_*(w),$$

where $P_*(w)$ is a polynomial in degree $n - l = r(say)$.

We also use $P_1(\omega_1)$ as an arbitrary non-zero polynomial defined by

$$P_1(\omega_1) = a_n \prod_{\substack{i=1 \\ l_i \neq l}}^{s} (\omega_1 + c_l - c_{l_i}) = b_m\omega_1^m + b_{m-1}\omega_1^{m-1} + \ldots + b_0, \quad (1.2)$$

where $\omega_1 = \omega - c_l$ and $m = n - l$.
Obviously

$$P(\omega) = \omega_1^l P_1(\omega_1).$$

If we observe the above theorems carefully we see that all the investigations were done on the basis of sharing of the expression of the particular form $h^nP(h)$ $(h = f$ or $g)$. So it will be quiet natural to investigate all the results for the most standard form $P(h)$ instead of $h^nP(h)$ $(h = f$ or $g)$.

Recently, C. Wu and J. Li [15] obtained the following results in this direction:

Theorem M. *Let f and g be two non-constant meromorphic functions, and let n, k and l be three positive integers satisfying $4lk + 12l > 4kn + 11n + 9k + 14$. If $[P(f)]^{(k)}$ and $[P(g)]^{(k)}$ share 1 IM, then either $f = b_1e^{bz} + c$, $g = b_2e^{-bz} + c$, or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where b_1, b_2, b are three constants satisfying $(-1)^k(b_1b_2)^n(nb)^{2k} = 1$ and $R(\omega_1, \omega_2) = P(\omega_1) - P(\omega_2)$.*

Theorem N. *Let f and g be two non-constant meromorphic functions, and let n, k and l be three positive integers satisfying $kl + 6l > nk + 5n + 3k + 8$. If $[P(f)]^{(k)}$ and $[P(g)]^{(k)}$ share 1 CM, then conclusion of* Theorem M *holds.*

Theorem O. *Let f and g be two non-constant entire functions, and let n, k and l be three positive integers satisfying $5l > 4n + 5k + 7$. If $[P(f)]^{(k)}$ and $[P(g)]^{(k)}$ share 1 IM, then conclusion of* Theorem M *holds.*

Theorem P. *Let f and g be two non-constant entire functions, and let n, k and l be three positive integers satisfying $2l > n + 2k + 4$. If $[P(f)]^{(k)}$ and $[P(g)]^{(k)}$ share 1 CM, then conclusion of* Theorem M *holds.*

Remark 1.1. *The results* [15] *are new and seem fine. However in the proof of Theorem 11* [15]*, one can easily point out a number of gaps.*

We first consider p. 299 *under the* **case 1.1.2** *fifth line from top.*
The calculations are true only when $p_{j_0} > k$, : A question arises: When $p_{j_0} \leq k$? Actually the author did not consider this case.

In the same page the author used the inequality

$$\overline{N}(r, \infty; f) \leq \sum_{j=1}^{s} \overline{N}(r, c_j; g) + \overline{N}(r, 0; g^{'}),$$

which is not true for any arbitrary k and the situation when

$$[L(f)]^{(k)}[L(g)]^{(k)} \equiv 1.$$

Remark 1.2. The authors declare that *Lemma 11* in [15] can be obtained from [17]. But in [17] there is no such lemma. One can easily verify that the *lemma 11* in [15] is actually *Lemma 2.12* of [25]. Also one can easily observe that to prove *Lemma 2.12* in [25], *Lemma 2.8* plays a vital role [see p.8 last line in [25]]. But the following example shows that *Lemma 2.8* of [25] is invalid.

Example 1.1. Let $F = ze^z$, $G = \frac{1}{ze^z}$, then F and G share 1 and -1 and satisfy

$$N(r, 0; F) + N(r, \infty; F) = S(r, F)$$

and

$$N(r, 0; G) + N(r, \infty; G) = S(r, G).$$

It is clear that F and G share neither 0 nor ∞.

So the very existence of *Lemma 11* in [15] and proof of *Theorem 11*, where *Lemma 11* plays a vital role is questionable. In this paper we tackle the problem by obtaining the correct proof of *Lemma 11* as well as *Theorem 11.* We also observe that in *Theorems M* and *N* as $n = l+r$, the relation $4lk+12l > 4kn+11n+9k+14$ ($kl+6l >$

$nk+5n+3k+8$) ultimately produce $l>(4k+11)r+9k+14$ ($l>(k+5)r+3k+8$) which are very much stronger result in-comparison to the lower bound of l obtained by the previous authors. In that sense in this paper we shall decrease the lower bound of l to a large extent. Not only that our results will largely improve and generalize all the previous results mentioned earlier. To proceed further we require the following definition. In 2001 an idea of gradation of sharing of values was introduced in {[8], [9]} which measures how close a shared value is to being share CM or to being shared IM. This notion is known as weighted sharing and is defined as follows.

Definition 1.1. [8, 9] Let k be a nonnegative integer or infinity. For $a\in\mathbb{C}\cup\{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m\le k$ and $k+1$ times if $m>k$. If $E_k(a;f)=E_k(a;g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity m ($\le k$) if and only if it is an a-point of g with multiplicity m ($\le k$) and z_0 is an a-point of f with multiplicity m ($>k$) if and only if it is an a-point of g with multiplicity n ($>k$), where m is not necessarily equal to n.

We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k), then f, g share (a,p) for any integer p, $0\le p<k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

The main results of the paper are as follows.

Theorem 1.1. *Let f and g be two transcendental meromorphic functions and $p(z)$ be a nonzero polynomial with $deg(p)\le l-1$, where n, k, r and l be four positive integers with $n=l+r$ such that $l>3k+r+8$. Suppose $[P(f)]^{(k)}$ and $[P(g)]^{(k)}$ share $(p,2)$, where $P(\omega)$ be defined as in (1.1). Now*

- (I) *when $s\ge 2$ then one of the following three cases holds:*
  - (I1) *$f\equiv tg$ for a constant t such that $t^d=1$, where $d=GCD(n,\ldots,n-i,\ldots,1)$, $a_{n-i}\neq 0$ for some $i\in\{1,2,\ldots,n-1\}$;*
  - (I2) *f and g satisfy the algebraic equation $R(f,g)\equiv 0$, where*

$$R(\omega_1,\omega_2)=(a_n\omega_1^n+a_{n-1}\omega_1^{n-1}+\ldots+a_1\omega_1)-(a_n\omega_2^n+a_{n-1}\omega_2^{n-1}+\ldots+a_1\omega_2);$$

  - (I3) *$[P(f)]^{(k)}[P(g)]^{(k)}\equiv p^2$;*
- (II) *when $s=1$ then one of the following two cases holds:*
  - (II1) *$f\equiv tg$ for some constant t such that $t^n=1$,*
  - (II2) *if $p(z)$ is not a constant, then $f=c_1e^{cQ(z)}+c_l$, $g=c_2e^{-cQ(z)}+c_l$, where*

$$Q(z)=\int_0^z p(z)dz,$$

c_1, c_2 and c are constants such that $b_i^2(c_1c_2)^{l+i}[(l+i)c]^2=-1$, if $p(z)$ is a nonzero constant b, then $f=c_3e^{cz}+c_l$, $g=c_4e^{-cz}+c_l$, where c_3, c_4 and c are constants such that $(-1)^kb_i^2(c_3c_4)^{l+i}[(l+i)c]^{2k}=b^2$.

In particular when $l_i>k(i=1,2,\ldots,s)$ and

$$\Theta(0;f)+\Theta(\infty;f)>\frac{n(3-s)-2ks+4k}{n+2k},$$

then $(I3)$ *does not hold.*

Theorem 1.2. *Let f and g be two transcendental meromorphic functions and $p(z)$ be a nonzero polynomial with $deg(p) \leq l-1$, where n, k, r and l be four positive integers with $n = l+r$ such that $l > 9k+4r+14$. Suppose $[P(f)]^{(k)}$ and $[P(g)]^{(k)}$ share $p(z)$ IM, where $P(\omega)$ be defined as in (1.1). Then the conclusion of Theorem 1.1 holds.*

Remark 1.3. Theorems 1.1 and 1.2 both hold for two non-constant meromorphic functions f and g when $p(z)$ is a non-zero constant.

Remark 1.4. When $l = n$, $c_l = 0$ from Theorem 1.1 we can easily get an improved version of *Theorem H*.

Corollary 1.1. *Let f and g be two transcendental entire functions and $p(z)$ be a nonzero polynomial with $deg(p) \leq l-1$, where n, k, r and l be four positive integers with $n = l+r$ such that $l > 2k+r+4$. Suppose $[P(f)]^{(k)}$ and $[P(g)]^{(k)}$ share $(p, 2)$, where $P(\omega)$ be defined as in (1.1). Then one of the following three cases holds:*

(1) *$f \equiv tg$ for a constant t such that $t^d = 1$,where $d = GCD(n, \ldots, n-i, \ldots, 1)$, $a_{n-i} \neq 0$ for some $i \in \{1, 2, \ldots, n-1\}$;*
(2) *f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where*

$$R(\omega_1, \omega_2) = (a_n\omega_1^n + a_{n-1}\omega_1^{n-1} + \ldots + a_1\omega_1) - (a_n\omega_2^n + a_{n-1}\omega_2^{n-1} + \ldots + a_1\omega_2);$$

(3) *if $p(z)$ is not a constant, then $f = c_1e^{cQ(z)} + c_l$, $g = c_2e^{-cQ(z)} + c_l$, where $Q(z) = \int_0^z p(z)dz$, c_1, c_2 and c are constants such that $b_i^2(c_1c_2)^{l+i}[(l+i)c]^2 = -1$, if $p(z)$ is a nonzero constant b, then $f = c_3e^{cz} + c_l$, $g = c_4e^{-cz} + c_l$, where c_3, c_4 and c are constants such that $(-1)^k b_i^2(c_3c_4)^{l+i}[(l+i)c]^{2k} = b^2$.*

Corollary 1.2. *Let f and g be two transcendental entire functions and $p(z)$ be a nonzero polynomial with $deg(p) \leq l-1$, where n, k, r and l be four positive integers with $n = l+r$ such that $l > 5k+4r+7$. Let $P(\omega)$ be defined as in (1.1). If $[P(f)]^{(k)}$ and $[P(g)]^{(k)}$ share $p(z)$ IM then the conclusion of Corollary 1.1 holds.*

Remark 1.5. Corollaries 1.1 and 1.2 both hold for two non-constant entire functions f and g when $p(z)$ is a non-zero constant.

Remark 1.6. When $l = n$, $c_l = 0$, from Corollary 1.1 and Corollary 1.2 we can easily get the improved version of *Theorems C, G* and *Theorem E* respectively.

Remark 1.7. When $l = n_1$, $n = n_1+1$ and $c_l = 0$, from Corollary 1.1 and Corollary 1.2 we can easily obtain the improved version of *Theorem D* and *Theorem F* respectively.

Remark 1.8. When $l = n_1$, $n = n_1 + m^*$ and $c_l = 0$, from Corollary 1.1, Lemmas 2.16 and 2.17 we can easily obtained the improvement of *Theorem I* where as from Corollary 1.2 we get the improved version of *Theorem K*.

Remark 1.9. When $l = n_1$, $n = n_1+m^{**}$ and $c_l = 0$, from Corollary 1.1 and Corollary 1.2 we can easily get an improved version of *Theorem J* and *Theorem L* respectively.

We now explain some definitions and notations which are used in the paper.

Definition 1.2. [11] Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$.

(i) $N(r,a;f \mid\geq p)$ ($\overline{N}(r,a;f \mid\geq p)$)denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than p.
(ii) $N(r,a;f \mid\leq p)$ ($\overline{N}(r,a;f \mid\leq p)$)denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than p.

Definition 1.3. {11, cf.[19]} For $a \in \mathbb{C}\cup\{\infty\}$ and a positive integer p we denote by $N_p(r,a;f)$ the sum $\overline{N}(r,a;f) + \overline{N}(r,a;f \mid\geq 2) + \ldots + \overline{N}(r,a;f \mid\geq p)$. Clearly $N_1(r,a;f) = \overline{N}(r,a;f)$.

Definition 1.4. Let $a, b \in \mathbb{C}\cup\{\infty\}$. Let p be a positive integer. We denote by $\overline{N}(r,a;f \mid \geq p \mid g = b)$ ($\overline{N}(r,a;f \mid \geq p \mid g \neq b)$) the reduced counting function of those a-points of f with multiplicities $\geq p$, which are the b-points (not the b-points) of g.

Definition 1.5. {cf.[1], 2} Let f and g be two non-constant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_L(r,1;f)$ the counting function of those 1-points of f and g where $p > q$, by $N_E^{1)}(r,1;f)$ the counting function of those 1-points of f and g where $p = q = 1$ and by $\overline{N}_E^{(2}(r,1;f)$ the counting function of those 1-points of f and g where $p = q \geq 2$, each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r,1;g)$, $N_E^{1)}(r,1;g)$, $\overline{N}_E^{(2}(r,1;g)$.

Definition 1.6. {cf.[1], 2} Let k be a positive integer. Let f and g be two non-constant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_{f>k}(r,1;g)$ the reduced counting function of those 1-points of f and g such that $p > q = k$. $\overline{N}_{g>k}(r,1;f)$ is defined analogously.

Definition 1.7. [8, 9] Let f, g share a value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.
Clearly $\overline{N}_*(r,a;f,g) \equiv \overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g) = \overline{N}_L(r,a;f) + \overline{N}_L(r,a;g)$.

2. Lemmas

Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We denote by H the function as follows:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{2.1}$$

Lemma 2.1. [17] *Let f be a non-constant meromorphic function and let $a_n(z)(\not\equiv 0)$, $a_{n-1}(z)$,..., $a_0(z)$ be meromorphic functions such that $T(r,a_i(z)) = S(r,f)$ for $i = 0,1,2,\ldots,n$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \ldots + a_1 f + a_0) = nT(r,f) + S(r,f).$$

Lemma 2.2. [23] *Let f be a non-constant meromorphic function, and p, k be positive integers. Then*

$$N_p\left(r,0;f^{(k)}\right) \leq T\left(r,f^{(k)}\right) - T(r,f) + N_{p+k}(r,0;f) + S(r,f),$$

$$N_p\left(r,0;f^{(k)}\right) \leq k\overline{N}(r,\infty;f) + N_{p+k}(r,0;f) + S(r,f).$$

Lemma 2.3. [10] *If $N(r,0;f^{(k)}|f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity then*

$$N(r,0;f^{(k)}|f \neq 0) \leq k\overline{N}(r,\infty;f) + N(r,0;f|<k) + k\overline{N}(r,0;f|\geq k) + S(r,f).$$

Lemma 2.4. *Let f be a non-constant meromorphic function. Let n, k and l be three positive integers such that $l > k+2$ and $P(\omega)$ be defined as in (1.1), $a(z)(\not\equiv 0,\infty)$ be a small function with respect to f. Then $[P(f)]^{(k)} - a(z)$ has infinitely many zeros.*

Proof. Let us take $F = P(f)$.
In view of *Lemmas* 2.1, 2.2 and by the second theorem for small functions (see [18]) we get

$$\begin{aligned}
& nT(r,f) \\
= \; & T(r,F) + O(1) \\
\leq \; & T(r,F^{(k)}) - \overline{N}(r,0;F^{(k)}) + N_{k+1}(r,0;F) + S(r,f) \\
\leq \; & \overline{N}(r,0;F^{(k)}) + \overline{N}(r,\infty;F^{(k)}) + \overline{N}(r,a(z);F^{(k)}) - \overline{N}(r,0;F^{(k)}) + N_{k+1}(r,0;F) \\
& +(\varepsilon + o(1))\,T(r,f) \\
\leq \; & \overline{N}(r,\infty;f) + (k+1)\,\overline{N}(r,c_l;f) + N(r,0;P(f)|f \neq c_l) + \overline{N}(r,a(z);F^{(k)}) \\
& +(\varepsilon + o(1))\,T(r,f) \\
\leq \; & (n-l+k+2)\,T(r,f) + \overline{N}(r,a(z);F^{(k)}) + (\varepsilon + o(1))\,T(r,f),
\end{aligned}$$

for all $\varepsilon > 0$. Take $\varepsilon < 1$. Since $l > k+2$ from above one can easily say that $F^{(k)} - a(z)$ has infinitely many zeros. □

Lemma 2.5. ([20], Lemma 6) *If $H \equiv 0$, then F, G share 1 CM. If further F, G share ∞ IM then F, G share ∞ CM.*

Lemma 2.6. [12] *Let f_1 and f_2 be two non-constant meromorphic functions satisfying $\overline{N}(r,0;f_i) + \overline{N}(r,\infty;f_i) = S(r;f_1,f_2)$ for $i = 1,2$. If $f_1^s f_2^t - 1$ is not identically zero for arbitrary integers s and $t(|s|+|t|>0)$, then for any positive ε, we have*

$$N_0(r,1;f_1,f_2) \leq \varepsilon T(r) + S(r;f_1,f_2),$$

where $N_0(r,1;f_1,f_2)$ denotes the reduced counting function related to the common 1-points of f_1 and f_2 and $T(r) = T(r,f_1) + T(r,f_2)$, $S(r;f_1,f_2) = o(T(r))$ as $r \longrightarrow \infty$ possibly outside a set of finite linear measure.

Lemma 2.7. *Let f and g be two non-constant meromorphic functions. Let n, k and l be three positive integers such that $2l > n+3k$. If $[P(f)]^{(k)} \equiv [P(g)]^{(k)}$, then $P(f) \equiv P(g)$, where $P(\omega)$ be defined as in (1.1).*

Proof. We have $[P(f)]^{(k)} \equiv [P(g)]^{(k)}$.
Integrating we get

$$[P(f)]^{(k-1)} \equiv [P(g)]^{(k-1)} + c_{k-1}.$$

If possible suppose $c_{k-1} \neq 0$.
Now in view of *Lemma* 2.2 for $p = 1$ and using the second fundamental theorem we get

$$\begin{aligned}
& n\,T(r,f) \\
&= T(r,P(f)) + O(1) \\
&\leq T(r,[P(f)]^{(k-1)}) - \overline{N}(r,0;[P(f)]^{(k-1)}) + N_k(r,0;P(f)) + S(r,f) \\
&\leq \overline{N}(r,0;[P(f)]^{(k-1)}) + \overline{N}(r,\infty;f) + \overline{N}(r,c_{k-1};[P(f)]^{(k-1)}) - \overline{N}(r,0;[P(f)]^{(k-1)}) \\
&+ N_k(r,0;P(f)) + S(r,f) \\
&\leq \overline{N}(r,\infty;f) + \overline{N}(r,0;[P(g)]^{(k-1)}) + N_k(r,0;P(f)) + S(r,f) \\
&\leq \overline{N}(r,\infty;f) + (k-1)\overline{N}(r,\infty;g) + N_k(r,0;P(g)) + N_k(r,0;P(f)) + S(r,f) \\
&\leq \overline{N}(r,\infty;f) + (k-1)\overline{N}(r,\infty;g) + k\overline{N}(r,c_l;g) + N(r,0;P(g)|g \neq c_l) + k\overline{N}(r,c_l;f) \\
&+ N(r,0;P(f)|f \neq c_l) + S(r,f) \\
&\leq (n-l+k+1)\,T(r,f) + (n-l+2k-1)\,T(r,g) + S(r,f) + S(r,g) \\
&\leq (2n-2l+3k)\,T(r) + S(r).
\end{aligned}$$

Similarly we get

$$n\,T(r,g) \leq (2n-2l+3k)\,T(r) + S(r).$$

Combining these we get

$$(2l-n-3k)\,T(r) \leq S(r),$$

which is a contradiction since $2l > n+3k$.
Therefore $c_{k-1} = 0$ and so $[P(f)]^{(k-1)} \equiv [P(g)]^{(k-1)}$.
Proceeding in this way we obtain

$$[P(f)]^{'} \equiv [P(g)]^{'}.$$

Integrating we get

$$P(f) \equiv P(g) + c_0.$$

If possible suppose $c_0 \neq 0$. Now using the second fundamental theorem we get

$$\begin{aligned}
& nT(r,f) \\
= \;& T(r,P(f)) + O(1) \\
\leq \;& \overline{N}(r,0;P(f)) + \overline{N}(r,\infty;P(f)) + \overline{N}(r,c_0;P(f)) \\
\leq \;& \overline{N}(r,0;P(f)) + \overline{N}(r,\infty;f) + \overline{N}(r,0;P(g)) \\
\leq \;& (n-l+2)\,T(r,f) + (n-l+1)\,T(r,g) + S(r,f) \\
\leq \;& (2n-2l+3)\,T(r) + S(r).
\end{aligned}$$

Similarly we get

$$n\,T(r,g) \leq (2n-2l+3)\,T(r) + S(r).$$

Combining these we get

$$(2l-n-3)\,T(r)\le S(r),$$

which is a contradiction since $2l>n+3$.
Therefore $c_0=0$ and so

$$P(f)\equiv P(g).$$

This proves the Lemma. □

Lemma 2.8. *Let f, g be two non-constant meromorphic functions. Let n, k and l be three positive integers such that $l>k+2$ and $P(\omega)$ be defined as in (1.1). If $[P(f)]^{(k)}$ and $[P(g)]^{(k)}$ share α IM, where $\alpha(\not\equiv 0,\infty)$ is a small function of f and g, then $T(r,f)=O(T(r,g))$ and $T(r,g)=O(T(r,f))$.*

Proof. Let $F=P(f)$. By the second fundamental theorem for small functions {see [18]}, we have

$$T(r,F^{(k)})\le \overline{N}(r,\infty;F^{(k)})+\overline{N}(r,0;F^{(k)})+\overline{N}(r,\alpha;F^{(k)})+(\varepsilon+o(1))T(r,F),$$

for all $\varepsilon>0$.
Now in the view of *Lemmas* 2.1 and 2.2 for $p=1$ and using above we get

$$\begin{aligned}
&n\,T(r,f)\\
&\le T(r,F^{(k)})-\overline{N}(r,0;F^{(k)})+N_{k+1}(r,0;P(f))+(\varepsilon+o(1))T(r,f)\\
&\le \overline{N}(r,0;F^{(k)})+\overline{N}(r,\infty;f)+\overline{N}(r,\alpha;F^{(k)})-\overline{N}(r,0;F^{(k)})+N_{k+1}(r,0;P(f))\\
&+(\varepsilon+o(1))T(r,f)\\
&\le \overline{N}(r,\infty;f)+\overline{N}(r,\alpha;[P(f)]^{(k)})+(k+1)\overline{N}(r,c_l;f)+N(r,0;P(f)|f\ne c_l)\\
&+(\varepsilon+o(1))T(r,f)\\
&\le (n-l+k+2)\,T(r,f)+\overline{N}(r,\alpha;[P(g)]^{(k)})+(\varepsilon+o(1))T(r,f)\\
&\le (n-l+k+2)\,T(r,f)+(k+1)n\,T(r,g)+(\varepsilon+o(1))T(r,f),
\end{aligned}$$

i.e.,

$$(l-k-2)\,T(r,f)\le (k+1)n\,T(r,g)+(\varepsilon+o(1))T(r,f).$$

Since $l>k+2$, take $\varepsilon<1$ and we have $T(r,f)=O(T(r,g))$. Similarly we have $T(r,g)=O(T(r,f))$. This completes the proof of Lemma. □

Lemma 2.9. *Let f, g be two non-constant meromorphic functions and let*

$$F=[P(f)]^{(k)}/\alpha(z),\ G=[P(g)]^{(k)}/\alpha(z),$$

where $P(\omega)$ be defined as in (1.1), $\alpha(z)$ be a small function with respect to f, g and n, k and l be positive integers such that $2l>n+3k+3$. Suppose $H\equiv 0$. Then one of the following holds:

(i) $[P(f)]^{(k)}[P(g)]^{(k)}\equiv\alpha^2$;
(ii) *$f\equiv tg$ for a constant t such that $t^d=1$, where $d=GCD(n,\ldots,n-i,\ldots,1)$, $a_{n-i}\ne 0$ for some $i\in\{1,2,\ldots,n-1\}$;*

(iii) *f and g satisfy the algebraic equation $R(f,g)\equiv 0$, where*

$$R(\omega_1,\omega_2)=(a_n\omega_1^n+a_{n-1}\omega_1^{n-1}+\ldots+a_1\omega_1)-(a_n\omega_2^n+a_{n-1}\omega_2^{n-1}+\ldots+a_1\omega_2).$$

Proof. Since $H\equiv 0$, by *Lemma* 2.5 we get F and G share 1 CM.
On integration we get

$$\frac{1}{F-1}\equiv\frac{bG+a-b}{G-1}, \tag{2.2}$$

where a,b are constants and $a\neq 0$. We now consider the following cases.
Case 1. Let $b\neq 0$ and $a\neq b$.
If $b=-1$, then from (2.2) we have

$$F\equiv\frac{-a}{G-a-1}.$$

Therefore

$$\overline{N}(r,a+1;G)=\overline{N}(r,\infty;F)=\overline{N}(r,\infty;f).$$

So in view of *Lemma* 2.2 and the second fundamental theorem we get

$$\begin{aligned}
&n\,T(r,g)\\
&=T(r,P(f))+O(1)\\
&\leq T(r,G)+N_{k+1}(r,0;P(g))-\overline{N}(r,0;G)\\
&\leq \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}(r,a+1;G)+N_{k+1}(r,0;P(g))-\overline{N}(r,0;G)+S(r,g)\\
&\leq \overline{N}(r,\infty;g)+(k+1)\overline{N}(r,c_l;g)+N(r,0;P(g)|g\neq c_l)+\overline{N}(r,\infty;f)+S(r,g)\\
&\leq\ T(r,f)+(n-l+k+2)\,T(r,g)+S(r,f)+S(r,g).
\end{aligned}$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,f)\leq T(r,g)$ for $r\in I$.
So for $r\in I$ we have

$$(l-k-3)\,T(r,g)\leq S(r,g),$$

which is a contradiction since $l>k+3$.
If $b\neq -1$, from (2.2) we obtain that

$$F-(1+\frac{1}{b})\equiv\frac{-a}{b^2[G+\frac{a-b}{b}]}.$$

So

$$\overline{N}(r,\frac{(b-a)}{b};G)=\overline{N}(r,\infty;F)=\overline{N}(r,\infty;f).$$

Using *Lemma* 2.2 and the same argument as used in the case when $b=-1$ we can get a contradiction.
Case 2. Let $b\neq 0$ and $a=b$.
If $b=-1$, then from (2.2) we have

$$FG\equiv\alpha^2,$$

i.e.,

$$[P(f)]^{(k)}[P(g)]^{(k)}\equiv\alpha^2,$$

where $[P(f)]^k$ and $[P(g)]^k$ share α CM.
If $b \neq -1$, from (2.2) we have

$$\frac{1}{F} \equiv \frac{bG}{(1+b)G-1}.$$

Therefore

$$\overline{N}(r, \frac{1}{1+b}; G) = \overline{N}(r, 0; F).$$

So in view of *Lemma* 2.2 and the second fundamental theorem we get

$$\begin{aligned}
&n\,T(r,g) \\
&\leq T(r,G) + N_{k+1}(r,0;P(g)) - \overline{N}(r,0;G) + S(r,g) \\
&\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,\frac{1}{1+b};G) + N_{k+1}(r,0;P(g)) - \overline{N}(r,0;G) + S(r,g) \\
&\leq \overline{N}(r,\infty;g) + N_{k+1}(r,0;P(g)) + \overline{N}(r,0;F) + S(r,g) \\
&\leq \overline{N}(r,\infty;g) + N_{k+1}(r,0;P(g)) + N_{k+1}(r,0;P(f)) + k\overline{N}(r,\infty;f) + S(r,f) + S(r,g) \\
&\leq (n-l+k+2)\,T(r,g) + (n-l+2k+1)\,T(r,f) + S(r,f) + S(r,g).
\end{aligned}$$

So for $r \in I$ we have

$$(2l-n-3k-3)\,T(r,g) \leq S(r,g),$$

which is a contradiction since $2l > n+3k+3$.
Case 3. Let $b=0$. From (2.2) we obtain

$$F \equiv \frac{G+a-1}{a}. \tag{2.3}$$

If $a \neq 1$ then from (2.3) we obtain

$$\overline{N}(r, 1-a; G) = \overline{N}(r, 0; F).$$

We can similarly deduce a contradiction as in *Case 2*. Therefore $a=1$ and from (2.3) we obtain

$$F \equiv G,$$

i.e.,

$$[P(f)]^{(k)} \equiv [P(g)]^{(k)}.$$

Then by Lemma 2.7 we have

$$P(f) \equiv P(g). \tag{2.4}$$

Let $h = \frac{f}{g}$. If h is a constant, by putting $f = hg$ in (2.4) we get

$$a_n g^{n-1}(h^n-1) + a_{n-1}g^{n-2}(h^{n-1}-1) + \ldots + a_1(h-1) \equiv 0,$$

which implies that $h^d = 1$, where $d = GCD(n, \ldots, n-i, \ldots, 1)$, $a_{n-i} \neq 0$ for some $i \in \{1, 2, \ldots, n-1\}$. Thus $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = GCD(n, \ldots, n-i, \ldots, 1)$, $a_{n-i} \neq 0$ for some $i \in \{1, 2, \ldots, n-1\}$.
If h is not constant then f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(\omega_1, \omega_2) = (a_n\omega_1^n + a_{n-1}\omega_1^{n-1} + \ldots + a_1\omega_1) - (a_n\omega_2^n + a_{n-1}\omega_2^{n-1} + \ldots + a_1\omega_2)$. $\square$

Lemma 2.10. [6] *Let $f(z)$ be a non-constant entire function and let $k \geq 2$ be a positive integer. If $f(z)f^{(k)}(z) \neq 0$, then $f(z) = e^{az+b}$, where $a \neq 0, b$ are constant.*

Lemma 2.11. [[7], Theorem 3.10] *Suppose that f is a non-constant meromorphic function, $k \geq 2$ is an integer. If*

$$N(r, \infty, f) + N(r, 0; f) + N(r, 0; f^{(k)}) = S(r, \frac{f^{'}}{f}),$$

then $f = e^{az+b}$, where $a \neq 0$, b are constants.

Lemma 2.12. [[21], Theorem 1.24] *Let f be a non-constant meromorphic function and let k be a positive integer. Suppose that $f^{(k)} \not\equiv 0$, then*

$$N(r, 0; f^{(k)}) \leq N(r, 0; f) + k\overline{N}(r, \infty; f) + S(r, f).$$

Lemma 2.13. *Let f, g be two transcendental meromorphic functions and $p(z)$ be a non-zero polynomial with $deg(p) \leq n - 1$, where n and k be two positive integers such that $n > \max\{2k, k + 2\}$. Suppose $[f^n]^{(k)}[g^n]^{(k)} \equiv p^2$, where $[f^n]^{(k)} - p(z)$ and $[g^n]^{(k)} - p(z)$ share 0 CM. Now*

(i) if $p(z)$ is not a constant, then $f = c_1e^{cQ(z)}$, $g = c_2e^{-cQ(z)}$, where

$$Q(z) = \int_0^z p(z)dz,$$

c_1, c_2 and c are constants such that $(nc)^2(c_1c_2)^n = -1$,

(ii) if $p(z)$ is a nonzero constant b, then $f = c_3e^{dz}$, $g = c_4e^{-dz}$, where c_3, c_4 and d are constants such that $(-1)^k(c_3c_4)^n(nd)^{2k} = b^2$.

Proof. Suppose

$$[f^n]^{(k)}[g^n]^{(k)} \equiv p^2. \tag{2.5}$$

We consider the following cases:

Case 1. Let $deg(p(z)) = l(\geq 1)$.

Let $z_0(p(z_0) \neq 0)$ be a zero of f with multiplicity q. Note that z_0 is a zero of $[f^n]^{(k)}$ with multiplicity $nq - k$. Obviously z_0 will be a pole of g with multiplicity q_1, say. Note that z_0 is a pole of $[g^n]^{(k)}$ with multiplicity $nq_1 + k$ and so $nq - k = nq_1 + k$. Now

$$nq - k = nq_1 + k$$

implies that

$$n(q - q_1) = 2k. \tag{2.6}$$

Since $n > 2k$, we get a contradiction from (2.6).

This shows that z_0 is a zero of $p(z)$ and so we have $N(r, 0; f) = O(logr)$. Similarly we can prove that $N(r, 0; g) = O(logr)$.

Thus in general we can take $N(r, 0; f) + N(r, 0; g) = O(logr)$.

We know that

$$N(r, \infty; [f^n]^{(k)}) = nN(r, \infty; f) + k\overline{N}(r, \infty; f).$$

Also by *Lemma* 2.12 we have

$$\begin{aligned} N(r,0;[g^n]^{(k)}) &\le nN(r,0;g)+k\overline{N}(r,\infty;g)+S(r,g) \\ &\le k\overline{N}(r,\infty;g)+O(logr)+S(r,g). \end{aligned}$$

From (2.5) we get

$$N(r,\infty;[f^n]^{(k)})=N(r,0;[g^n]^{(k)}),$$

i.e.,

$$nN(r,\infty;f)+k\overline{N}(r,\infty;f)\le k\overline{N}(r,\infty;g)+O(logr)+S(r,g). \tag{2.7}$$

Similarly we get

$$nN(r,\infty;g)+k\overline{N}(r,\infty;g)\le k\overline{N}(r,\infty;f)+O(logr)+S(r,f). \tag{2.8}$$

Since f and g are transcendental, it follows that

$$S(r,f)+O(logr)=S(r,f),\quad S(r,g)+O(logr)=S(r,g).$$

Now combining (2.7) and (2.8) we get

$$N(r,\infty;f)+N(r,\infty;g)=S(r,f)+S(r,g).$$

By *Lemma* 2.8 we have $S(r,f)=S(r,g)$ and so we obtain

$$N(r,\infty;f)=S(r,f),\quad N(r,\infty;g)=S(r,g). \tag{2.9}$$

Let

$$F_1=\frac{[f^n]^{(k)}}{p}\quad and\quad G_1=\frac{[g^n]^{(k)}}{p}. \tag{2.10}$$

Note that $T(r,F_1)\le n(k+1)T(r,f)+S(r,f)$ and so $T(r,F_1)=O(T(r,f))$. Also by *Lemma* 2.2, one can obtain $T(r,f)=O(T(r,F_1))$. Hence $S(r,F_1)=S(r,f)$. Similarly we get $S(r,G_1)=S(r,g)$. Hence we get $S(r,F_1)=S(r,G_1)$. From (2.5) we get

$$F_1G_1\equiv 1. \tag{2.11}$$

If $F_1\equiv cG_1$, where c is a nonzero constant, then F_1 is a constant and so f is a polynomial, which contradicts our assumption. Hence $F_1\not\equiv cG_1$ and so in the view of (2.11) we see that F_1 and G_1 share -1 IM.
Now by *Lemma* 2.12 we have

$$N(r,0;F_1)\le nN(r,0;f)+k\overline{N}(r,\infty;f)+S(r,f)\le S(r,F_1).$$

Similarly we have

$$N(r,0;G_1)\le nN(r,0;g)+k\overline{N}(r,\infty;g)+S(r,g)\le S(r,G_1).$$

Also we see that

$$N(r,\infty;F_1)=S(r,F_1),\quad N(r,\infty;G_1)=S(r,G_1).$$

It is clearly that $T(r,F_1)=T(r,G_1)+O(1)$. Let

$$f_1 = \frac{F_1}{G_1}.$$

and

$$f_2=\frac{F_1-1}{G_1-1}.$$

Clearly f_1 is non-constant. If f_2 is a nonzero constant then F_1 and G_1 share ∞ CM and so from (2.11) we conclude that F_1 and G_1 have no poles.

Next we suppose that f_2 is non-constant. We see that

$$F_1 = \frac{f_1(1-f_2)}{f_1 - f_2}, \quad G_1 = \frac{1-f_2}{f_1-f_2}.$$

Clearly

$$T(r, F_1) \leq 2[T(r, f_1) + T(r, f_2)] + O(1)$$

and

$$T(r, f_1) + T(r, f_2) \leq 4T(r, F_1) + O(1).$$

These give $S(r, F_1) = S(r; f_1, f_2)$. Also we note that

$$\overline{N}(r, 0; f_i) + \overline{N}(r, \infty; f_i) = S(r; f_1, f_2)$$

for $i = 1, 2$.

Next we suppose $\overline{N}(r, -1; F_1) \neq S(r, F_1)$, otherwise by the second fundamental theorem F_1 will be a constant.

Also we see that

$$\overline{N}(r, -1; F_1) \leq N_0(r, 1; f_1, f_2).$$

Thus we have

$$T(r, f_1) + T(r, f_2) \leq 4\ N_0(r, 1; f_1, f_2) + S(r, F_1).$$

Then by *Lemma* 2.6 there exist two mutually prime integers s and $t(|s|+|t|>0)$ such that

$$f_1^s f_2^t \equiv 1,$$

i.e.,

$$\left[\frac{F_1}{G_1}\right]^s \left[\frac{F_1 - 1}{G_1 - 1}\right]^t \equiv 1. \tag{2.12}$$

If either s or t is zero then we arrive at a contradiction and so $st \neq 0$.
We now consider following cases:
Case (i). Suppose $s > 0$ and $t = -t_1$, where $t_1 > 0$. Then we have

$$\left[\frac{F_1}{G_1}\right]^s \equiv \left[\frac{F_1 - 1}{G_1 - 1}\right]^{t_1}. \tag{2.13}$$

Let z_1 be a pole of F_1 of multiplicity p. Then from (2.11) we see that z_1 must be a zero of G_1 of multiplicity p. Now from (2.13) we get $2s = t_1$, which is impossible. Hence F_1 has no pole. Similarly we can prove that G_1 also has no poles.
Case (ii). Suppose either $s > 0$ and $t > 0$ or $s < 0$ and $t < 0$. Then from (2.13) one can easily prove that F_1 and G_1 have no poles.

Consequently from (2.11) we see that F_1 and G_1 have no zeros. So we deduce from (2.10) that both f and g have no pole.

Since F_1 and G_1 have no zeros and poles, we have

$$F_1 \equiv e^{\gamma_1} G_1,$$

i.e.,

$$[f^n]^{(k)} \equiv e^{\gamma_1}[g^n]^{(k)},$$

where γ_1 is a non-constant entire function. Then from (2.5) we get

$$[f^n]^{(k)} \equiv ce^{\frac{1}{2}\gamma_1}p(z), \quad [g^n]^{(k)} \equiv ce^{-\frac{1}{2}\gamma_1}p(z),$$

where c is a nonzero constant. This shows that $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share 0 CM.
Also we deduce from (2.10) that both f and g are transcendental entire functions.
Since $N(r,0;f) = O(logr)$ and $N(r,0;g) = O(logr)$, so we can take

$$f(z) = h_1(z)e^{\alpha(z)}, \quad g(z) = h_2(z)e^{\beta(z)}, \tag{2.14}$$

where h_1 and h_2 are nonzero polynomials and α, β are two non-constant entire functions.
We deduce from (2.5) and (2.14) that either both α and β are transcendental entire functions or both α and β are polynomials.
We consider the following cases:
Subcase 1.1: Let $k \geq 2$.
First we suppose both α and β are transcendental entire functions.
Let $\alpha_1 = \alpha^{'} + \frac{h_1^{'}}{h_1}$ and $\beta_1 = \beta^{'} + \frac{h_2^{'}}{h_2}$. Clearly both α_1 and β_1 are transcendental entire functions.
Note that

$$S(r, n\alpha_1) = S(r, \frac{[f^n]^{'}}{f^n}), \quad S(r, n\beta_1) = S(r, \frac{[g^n]^{'}}{g^n}).$$

Moreover we see that

$$N(r, 0; [f^n]^{(k)}) \leq N(r, 0; p^2) = O(logr).$$

$$N(r, 0; [g^n]^{(k)}) \leq N(r, 0; p^2) = O(logr).$$

From these and using (2.14) we have

$$N(r, \infty; f^n) + N(r, 0; f^n) + N(r, 0; [f^n]^{(k)}) = S(r, n\alpha_1) = S(r, \frac{[f^n]^{'}}{f^n}) \tag{2.15}$$

and

$$N(r, \infty; g^n) + N(r, 0; g^n) + N(r, 0; [g^n]^{(k)}) = S(r, n\beta_1) = S(r, \frac{[g^n]^{'}}{g^n}). \tag{2.16}$$

Then from (2.15), (2.16) and Lemma 2.9 we must have

$$f = e^{az+b}, \quad g = e^{cz+d}, \tag{2.17}$$

where $a \neq 0$, b, $c \neq 0$ and d are constants. But these types of f and g do not agree with the relation (2.5).
Next we suppose α and β are both polynomials.
From (2.5) we get $\alpha + \beta \equiv C$ i.e., $\alpha^{'} \equiv -\beta^{'}$. Therefore $deg(\alpha) = deg(\beta)$.
We deduce from (2.14) that

$$[f^n]^{(k)} \equiv Ah_1^{n-k}[h_1^k(\alpha^{'})^k + P_{k-1}(\alpha^{'}, h_1^{'})]e^{n\alpha} \equiv p(z)e^{n\alpha}, \tag{2.18}$$

and

$$[g^n]^{(k)} = Bh_2^{n-k}[h_2^k(\beta^{'})^k + Q_{k-1}(\beta^{'}, h_2^{'})]e^{n\beta} \equiv p(z)e^{n\beta}, \tag{2.19}$$

where A, B are nonzero constants, $P_{k-1}(\alpha^{'}, h_1^{'})$ and $Q_{k-1}(\beta^{'}, h_2^{'})$ are differential polynomials in $\alpha^{'}, h_1^{'}$ and $\beta^{'}, h_2^{'}$ respectively.
Since $deg(p) \leq n-1$, from (2.17) and (2.19) we conclude that both h_1 and h_2 are nonzero constant.
So we can rewrite f and g as follows:

$$f = e^{\gamma_2}, \quad g = e^{\delta_2}, \tag{2.20}$$

where $\gamma_2 + \delta_2 \equiv C$ and $deg(\gamma_2) = deg(\delta_2)$.
If $deg(\gamma_2) = deg(\delta_2) = 1$, then we again get a contradiction from (2.5).
Next we suppose $deg(\gamma_2) = deg(\delta_2) \geq 2$.
We deduce from (2.20) that

$$[f^n]^{(k)} = A_1[(\gamma_2^{'})^k + P_{k-1}(\gamma_2^{'})]e^{n\gamma_2}, \quad [g^n]^{(k)} = B_1[(\delta_2^{'})^k + Q_{k-1}(\delta_2^{'})]e^{n\delta_2},$$

where A_1, B_1 are nonzero constants, $P_{k-1}(\gamma_2^{'})$ and $Q_{k-1}(\delta_2^{'})$ are differential polynomials in $\gamma_2^{'}$ and $\delta_2^{'}$ of degree atmost $k-1$ respectively.
Since $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share 0 CM, it follows that

$$[(\gamma_2^{'})^k + P_{k-1}(\gamma_2^{'})] \equiv D[(\delta_2^{'})^k + Q_{k-1}(\delta_2^{'})],$$

where D is a nonzero constant, which is impossible as $k \geq 2$.

Actually $[(\gamma_2^{'})^k + P_{k-1}(\gamma_2^{'})]$ and $[(\delta_2^{'})^k + Q_{k-1}(\delta_2^{'})]$ contain the terms $(\gamma_2^{'})^k + K(\gamma_2^{'})^{k-2}\gamma_2^{''}$ and $(\delta_2^{'})^k + K(\delta_2^{'})^{k-2}\delta_2^{''}$ respectively, where K is a suitably positive integer. But these two terms are not identical.
Subcase 1.2: Let $k = 1$.
Now from (2.5) we get

$$f^{n-1}f^{'}g^{n-1}g^{'} \equiv p_1^2, \tag{2.21}$$

where $p_1^2 = \frac{1}{n^2}p^2$.
First we suppose both α and β are transcendental entire functions.
Let $h = fg$. Clearly h is a transcendental entire function. Then from (2.21) we get

$$\left(\frac{g^{'}}{g} - \frac{1}{2}\frac{h^{'}}{h}\right)^2 \equiv \frac{1}{4}\left(\frac{h^{'}}{h}\right)^2 - h^{-n}p_1^2. \tag{2.22}$$

Let

$$\alpha_2 = \frac{g^{'}}{g} - \frac{1}{2}\frac{h^{'}}{h}.$$

From (2.22) we get

$$\alpha_2^2 \equiv \frac{1}{4}\left(\frac{h^{'}}{h}\right)^2 - h^{-n}p_1^2. \tag{2.23}$$

First we suppose $\alpha_2 \equiv 0$. Then we get $h^{-n}p_1^2 \equiv \frac{1}{4}\left(\frac{h'}{h}\right)^2$ and so $T(r,h) = S(r,h)$, which is impossible. Next we suppose that $\alpha_2 \not\equiv 0$. Differentiating (2.23) we get

$$2\alpha_2\alpha_2' \equiv \frac{1}{2}\frac{h'}{h}\left(\frac{h'}{h}\right)' + n\, h' h^{-n-1}p_1^2 - 2h^{-n}p_1p_1'.$$

Applying (2.23) we obtain

$$h^{-n}\left(-n\frac{h'}{h}p_1^2 + 2p_1p_1' - 2\frac{\alpha_2'}{\alpha_2}p_1^2\right) \equiv \frac{1}{2}\frac{h'}{h}\left(\left(\frac{h'}{h}\right)' - \frac{h'}{h}\frac{\alpha_2'}{\alpha_2}\right). \tag{2.24}$$

First we suppose

$$-n\frac{h'}{h}p_1^2 + 2p_1p_1' - 2\frac{\alpha_2'}{\alpha_2}p_1^2 \equiv 0.$$

Then there exist a non-zero constant c such that $\alpha_2^2 \equiv ch^{-n}p_1^2$ and so from (2.23) we get

$$(c+1)h^{-n}p_1^2 \equiv \frac{1}{4}\left(\frac{h'}{h}\right)^2.$$

If $c = -1$, then h will be a constant. If $c \neq -1$, then we have $T(r,h) = S(r,h)$, which is impossible. Next we suppose that

$$-n\frac{h'}{h}p_1^2 + 2p_1p_1' - 2\frac{\alpha_2'}{\alpha_2}p_1^2 \not\equiv 0.$$

Then by (2.24) we have

$$\begin{aligned}
& n\,T(r,h) && (2.25)\\
= \;& n\,m(r,h) \\
\leq \;& m\left(r, h^n\frac{1}{2}\frac{h'}{h}\left(\left(\frac{h'}{h}\right)' - \frac{h'}{h}\frac{\alpha_2'}{\alpha_2}\right)\right) + m\left(r, \frac{1}{\frac{1}{2}\frac{h'}{h}\left(\left(\frac{h'}{h}\right)' - \frac{h'}{h}\frac{\alpha_2'}{\alpha_2}\right)}\right) + O(1) \\
\leq \;& T\left(r, \frac{1}{2}\frac{h'}{h}\left(\left(\frac{h'}{h}\right)' - \frac{h'}{h}\frac{\alpha_2'}{\alpha_2}\right)\right) + m\left(r, n\frac{h'}{h}p_1^2 - 2p_1p_1' + 2\frac{\alpha_2'}{\alpha_2}p_1^2\right) \\
\leq \;& \overline{N}(r,0;\alpha_2) + \overline{N}(r,\infty;\alpha_2) + S(r,h) + S(r,\alpha_2) \\
\leq \;& T(r,\alpha_2) + S(r,h). && (2.26)
\end{aligned}$$

From (2.23) we get

$$T(r,\alpha_2) \leq \frac{1}{2}n\,T(r,h) + S(r,h).$$

Now from (2.25) we get

$$\frac{1}{2}n\,T(r,h) \leq S(r,h),$$

which is impossible .
Thus α and β are both polynomials. Also from (2.5) we can conclude that $\alpha(z)+\beta(z) \equiv C$ for a constant C and so $\alpha^{'}(z) + \beta^{'}(z) \equiv 0$. We deduce from (2.5) that

$$[f^n]^{'} \equiv n[h_1^n\alpha^{'} + h_1^{n-1}h_1^{'}]e^{n\alpha} \equiv p(z)e^{n\alpha}, \tag{2.27}$$

and

$$[g^n]^{'} = n[h_2^n\beta^{'} + h_2^{n-1}h_2^{'}]e^{n\beta} \equiv p(z)e^{n\beta}. \tag{2.28}$$

Since $deg(p) \leq n-1$, from (2.27) and (2.28) we conclude that both h_1 and h_2 are nonzero constant.
So we can rewrite f and g as follows:

$$f = e^{\gamma_3}, \quad g = e^{\delta_3}. \tag{2.29}$$

Now from (2.5) we get

$$n^2\gamma_3^{'}\delta_3^{'}e^{n(\gamma_3+\delta_3)} \equiv p^2. \tag{2.30}$$

Also from (2.30) we can conclude that $\gamma_3(z) + \delta_3(z) \equiv C$ for a constant C and so $\gamma_3^{'}(z) + \delta_3^{'}(z) \equiv 0$. Thus from (2.30) we get $n^2e^{nC}\gamma_3^{'}\delta_3^{'} \equiv p^2(z)$. By computation we get

$$\gamma_3^{'} = cp(z), \quad \delta_3^{'} = -cp(z). \tag{2.31}$$

Hence

$$\gamma_3 = cQ(z) + b_1, \quad \delta_3 = -cQ(z) + b_2, \tag{2.32}$$

where $Q(z) = \int_0^z p(z)dz$ and b_1, b_2 are constants. Finally we take f and g as

$$f(z) = c_1e^{cQ(z)}, \quad g(z) = c_2e^{-cQ(z)},$$

where c_1, c_2 and c are constants such that $(nc)^2(c_1c_2)^n = -1$.
Case 2. Let $p(z)$ be a nonzero constant b. Since $n > 2k$, one can easily prove that f and g have no zeros. Now proceeding in the same way as done in the proof of the **Case 1** we get $f = e^{\alpha}$ and $g = e^{\beta}$, where α and β are two non-constant entire functions.
We now consider the following two subcases:
Subcase 2.1: Let $k \geq 2$.
We see that

$$N(r, 0; [f^n]^{(k)}) = 0$$

and

$$f^n(z)[f^n(z)]^{(k)} \neq 0. \tag{2.33}$$

Similarly we have

$$g^n(z)[g^n(z)]^{(k)} \neq 0. \tag{2.34}$$

Then from (2.33), (2.34) and Lemma 2.10 we must have

$$f = e^{az+b}, \quad g = e^{cz+d}, \tag{2.35}$$

where $a \neq 0$, b, $c \neq 0$ and d are constants.
Subcase 2.1: Let $k = 1$.
Considering **Subcase 1.2** one can easily get

$$f = e^{az+b}, \quad g = e^{cz+d}, \tag{2.36}$$

where $a \neq 0$, b, $c \neq 0$ and d are constants.
Finally we can take f and g as

$$f = c_3 e^{dz}, \quad g = c_4 e^{-dz},$$

where c_3, c_4 and d are nonzero constants such that $(-1)^k(c_3c_4)^n(nd)^{2k} = b^2$.
This completes the proof of Lemma. □

Lemma 2.14. *Let f, g be two transcendental meromorphic functions, $p(z)$ be a nonzero polynomial with $deg(p) \leq n-1$, where n and k be two positive integers such that $n > \max\{2k, k+2\}$.*
Let $[(f-a)^n]^{(k)}$, $[(g-a)^n]^{(k)}$ share p CM and $[(f-a)^n]^{(k)}[(g-a)^n]^{(k)} \equiv p^2$. Now

(i) if $p(z)$ is not a constant, then $f = c_1e^{cQ(z)} + a$, $g = c_2e^{-cQ(z)} + a$, where

$$Q(z) = \int_0^z p(z)dz,$$

c_1, c_2 and c are constants such that $(nc)^2(c_1c_2)^n = -1$,

(ii) if $p(z)$ is a nonzero constant b, then $f = c_3e^{dz} + a$, $g = c_4e^{-dz} + a$, where c_3, c_4 and d are constants such that $(-1)^k(c_3c_4)^n(nd)^{2k} = b^2$.

Proof. The Lemma follows from *Lemma* 2.13. □

Lemma 2.15. *Let f, g be two transcendental entire functions and $P(\omega)$ be defined as in (1.1), $p(z)$ be a nonzero polynomial such that $deg(p) \leq l-1$, where n, k and l be three positive integers such that $2l > n+3k+3$. Suppose $[P(f)]^{(k)}[P(g)]^{(k)} \equiv p^2$. Then*

(i) if $p(z)$ is not a constant, then $f = c_1e^{cQ(z)} + c_l$, $g = c_2e^{-cQ(z)} + c_l$, where

$$Q(z) = \int_0^z p(z)dz,$$

c_1, c_2 and c are constants such that $(nc)^2(c_1c_2)^n = -1$,

(ii) if $p(z)$ is a nonzero constant b, then $f = c_3e^{dz} + c_l$, $g = c_4e^{-dz} + c_l$, where c_3, c_4 and d are constants such that $(-1)^k(c_3c_4)^n(nd)^{2k} = b^2$.

Proof. Suppose

$$[P(f)]^{(k)}[P(g)]^{(k)} \equiv p^2. \tag{2.37}$$

Since $l > k$, we can take

$$f(z) - c_l = h(z)e^{\alpha(z)}, \tag{2.38}$$

where h is a nonzero polynomial and α is a non-constant entire function.
Let $f_1 = f - c_l$, $g_1 = g - c_l$.
Clearly $P(f) = f_1^l P_1(f_1)$ and $P(g) = g_1^l P_1(g_1)$,
i.e.,

$$P(f) = f_1^l[b_m f_1^m + b_{m-1}f_1^{m-1} + \ldots + b_0]$$

and

$$P(g) = g_1^l[b_m g_1^m + b_{m-1}g_1^{m-1} + \ldots + b_0].$$

We now consider the following two cases:

Case 1. Let $s \geq 2$, where s denotes the number of distinct zeros of $P(\omega) = 0$. In this case $m \geq 1$ and so atleast two of b_i, where $i \in \{0, 1, \ldots, m\}$ are nonzero. Since $f_1 = he^{\alpha}$, then by induction we get

$$(b_i f_1^{l+i})^{(k)} = t_i(\alpha^{'}, \alpha^{''}, \ldots, \alpha^{(k)}, h, h^{'}, \ldots, h^{(k)})e^{(l+i)\alpha}, \tag{2.39}$$

where $t_i(\alpha^{'}, \alpha^{''}, \ldots, \alpha^{(k)}, h, h^{'}, \ldots, h^{(k)})$ $(i = 0, 1, 2, \ldots, m)$ are differential polynomials in
$\alpha^{'}, \alpha^{''}, \ldots, \alpha^{(k)}, h, h^{'}, \ldots, h^{(k)}$.
Obviously

$$t_i(\alpha^{'}, \alpha^{''}, \ldots, \alpha^{(k)}, h, h^{'}, \ldots, h^{(k)}) \not\equiv 0$$

and $[P(f)]^{(k)} \not\equiv 0$.
From (2.37) and (2.39) we obtain

$$\overline{N}(r, 0; t_m e^{m\alpha(z)} + t_{m-1}e^{(m-1)\alpha(z)} + \ldots + t_0) \leq N(r, 0; p^2) = S(r, f). \tag{2.40}$$

Since α is an entire function, we obtain $T(r, \alpha^{(j)}) = S(r, f)$ for $j = 1, 2, \ldots, k$. Hence $T(r, t_i) = S(r, f)$ for $i = 0, 1, 2, \ldots, m$. So from (2.40) and using second fundamental theorem for small functions{see [18]}, we obtain

$$\begin{aligned}
& m\,T(r, f) \\
= \;& T(r, t_m e^{m\alpha} + \ldots + t_1 e^{\alpha}) + S(r, f) \\
\leq \;& \overline{N}(r, 0; t_m e^{m\alpha} + \ldots + t_1 e^{\alpha}) + \overline{N}(r, 0; t_m e^{m\alpha} + \ldots + t_1 e^{\alpha} + t_0) \\
& + \overline{N}(r, \infty; t_m e^{m\alpha} + \ldots + t_1 e^{\alpha}) + (\varepsilon + o(1))\,T(r, f) \\
\leq \;& \overline{N}(r, 0; t_m e^{(m-1)\alpha} + \ldots + t_1) + (\varepsilon + o(1))\,T(, f) \\
\leq \;& (m-1)T(r, f) + (\varepsilon + o(1))\,T(r, f),
\end{aligned}$$

for all $\varepsilon > 0$. Take $\varepsilon < 1$ and we obtain a contradiction.
Subcase 2.2: Let $s = 1$.
In this case $l = n$. From (2.37) we get

$$[(f_1)^n]^{(k)}[(g_1)^n]^{(k)} \equiv p^2. \tag{2.41}$$

Finally Lemma follows from *Lemma* 2.14.
This completes the proof of the Lemma. □

Lemma 2.16. [14] *Let f and g be two non-constant entire functions and λ, μ be two constants such that $\lambda\mu \neq 0$. Let n, m and k be three positive integers such that $n > 2k+m$. If $[f^n(\lambda f^m + \mu)]^{(k)} \equiv [g^n(\lambda g^m + \mu)]^{(k)}$, then $f^d(z) \equiv g^d(z)$, $d = GCD(n, m)$.*

Lemma 2.17. [16] *Let f and g be two non-constant meromorphic functions, k, $n > 2k+1$ be two positive integers. If $[f^n]^{(k)} \equiv [g^n]^{(k)}$, then $f \equiv tg$ for a constant t such that $t^n = 1$.*

Lemma 2.18. *Let f and g be two non-constant meromorphic functions and $a(z)(\not\equiv 0, \infty)$ be a small functions of f and g. Let n, k and $s \geq 2$ be three positive integers such that $n > 2ks + k$ and $P(\omega)$ be defined as in (1.1). If $l_i > k (i = 1, 2, \ldots, s)$ and $\Theta(0; f) + \Theta(\infty; f) > \frac{n(3-s)-2ks+4k}{n+2k}$ then*

$$[P(f)]^{(k)}[P(g)]^{(k)} \not\equiv a^2,$$

Proof. First suppose that

$$[P(f)]^{(k)}[P(g)]^{(k)} \equiv a^2,$$

i.e.,

$$[(f-c_1)^{l_1}(f-c_2)^{l_2}\dots(f-c_s)^{l_s}]^{(k)}[(g-c_1)^{l_1}(g-c_2)^{l_2}\dots(g-c_s)^{l_s}]^{(k)} \equiv a^2. \quad (2.42)$$

Now by *Lemma* 2.8, we have

$$S(r,f) = S(r,g).$$

Now by the second fundamental theorem for f and g we get respectively

$$s\,T(r,f) \le \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + \sum_{i=1}^{s}\overline{N}(r,c_i;f) - \overline{N}_0(r,0;f^{'}) + S(r,f) \quad (2.43)$$

and

$$s\,T(r,g) \le \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \sum_{i=1}^{s}\overline{N}(r,c_i;g) - \overline{N}_0(r,0;g^{'}) + S(r,g), \quad (2.44)$$

where $\overline{N}_0(r,0;f^{'})$ denotes the reduced counting function of those zeros of $f^{'}$ which are not the zeros of f and $f-c_i$, $i=1,2,\dots,s$ and $\overline{N}_0(r,0;g^{'})$ can be similarly defined.

Let $z_1(a(z_1)\neq 0,\infty)$ be a zero of $f-c_i$ with multiplicity q_i, $i=1,2,\dots,s$. Obviously z_1 must be a pole of g with multiplicity r. Then from (2.42) we get $l_iq_i-k = nr+k$. This gives $q_i \ge \frac{n+2k}{l_i}$ for $i=1,2,\dots,s$ and so we get

$$\overline{N}(r,c_i;f) \le \frac{l_i}{n+2k}\,N(r,c_i;f) \le \frac{l_i}{n+2k}\,T(r,f).$$

Clearly

$$\sum_{i=1}^{s}\overline{N}(r,c_i;f) \le \frac{n}{n+2k}\,T(r,f). \quad (2.45)$$

Similarly we have

$$\sum_{i=1}^{s}\overline{N}(r,c_i;g) \le \frac{n}{n+2k}\,T(r,g). \quad (2.46)$$

Then by (2.43) and (2.45) we get

$$\begin{aligned} &s\,T(r,f) \\ \le\;& \left(2+\frac{n}{n+2k}-\Theta(0;f)-\Theta(\infty;f)+\varepsilon\right)T(r,f)+S(r,f). \end{aligned} \quad (2.47)$$

Then from (2.47) we get

$$\left(s-2-\frac{n}{n+2k}+\Theta(0;f)+\Theta(\infty;f)-\;\varepsilon\right)\;T(r,f) \le S(r,f).$$

Since $\Theta(0;f)+\Theta(\infty;f) > \frac{n(3-s)-2ks+4k}{n+2k}$, we arrive at a contradiction.
This completes the proof. $\square$

Lemma 2.19. [2] *Let f and g be two non-constant meromorphic functions sharing 1 IM. Then*

$$\begin{aligned} & \overline{N}_L(r,1;f)+2\overline{N}_L(r,1;g)+\overline{N}_E^{(2}(r,1;f)-\overline{N}_{f>1}(r,1;g)-\overline{N}_{g>1}(r,1;f) \\ \leq\ & N(r,1;g)-\overline{N}(r,1;g). \end{aligned}$$

Lemma 2.20. [2] *Let f, g share 1 IM. Then*

$$\overline{N}_L(r,1;f)\leq \overline{N}(r,0;f)+\overline{N}(r,\infty;f)+S(r,f)$$

Lemma 2.21. [2] *Let f, g share 1 IM. Then*

$$\begin{aligned} (i)\quad & \overline{N}_{f>1}(r,1;g)\leq \overline{N}(r,0;f)+\overline{N}(r,\infty;f)-N_0(r,0;f^{'})+S(r,f) \\ (ii)\quad & \overline{N}_{g>1}(r,1;f)\leq \overline{N}(r,0;g)+\overline{N}(r,\infty;g)-N_0(r,0;g^{'})+S(r,g). \end{aligned}$$

3. Proofs of the Theorems

Proof of Theorem 1.1. Let $F=\frac{[P(f)]^{(k)}}{p(z)}$ and $G=\frac{[P(g)]^{(k)}}{p(z)}$. Note that since f and g are transcendental meromorphic functions, $p(z)$ is a small function with respect to both $[P(f)]^{(k)}$ and $[P(g)]^{(k)}$. Also F and G share $(1,2)$ except the zeros of $p(z)$.
Case 1. Let $H\not\equiv 0$.

From (2.1) it can be easily calculated that the possible poles of H occur at (i) multiple zeros of F and G, (ii) those 1 points of F and G whose multiplicities are different,(iii) those poles of F and G, (iv) zeros of $F^{'}(G^{'})$ which are not the zeros of $F(F-1)(G(G-1))$.

Since H has only simple poles we get

$$\begin{aligned} & N(r,\infty;H) \qquad (3.1)\\ \leq\ & \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_*(r,1;F,G)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2) \\ & +\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g), \end{aligned}$$

where $\overline{N}_0(r,0;F^{'})$ is the reduced counting function of those zeros of $F^{'}$ which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G^{'})$ is similarly defined.

Let z_0 be a simple zero of $F-1$ but $p(z_0)\neq 0$. Then z_0 is a simple zero of $G-1$ and a zero of H. So

$$N(r,1;F|=1)\leq N(r,0;H)\leq N(r,\infty;H)+S(r,f)+S(r,g). \qquad (3.2)$$

Now using (3.1) and (3.2) we get

$$\begin{aligned} & \overline{N}(r,1;F) \qquad (3.3)\\ \leq\ & N(r,1;F|=1)+\overline{N}(r,1;F|\geq 2) \\ \leq\ & \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}_*(r,1;F,G) \\ & +\overline{N}(r,1;F|\geq 2)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g). \end{aligned}$$

Now in view of *Lemma* 2.3 we get

$$\begin{aligned} & \overline{N}_0(r,0;G^{'})+\overline{N}(r,1;F\,|\geq 2)+\overline{N}_*(r,1;F,G) \qquad (3.4)\\ \leq\ & N(r,0;G^{'}\mid G\neq 0)\leq \overline{N}(r,0;G)+\overline{N}(r,\infty;G)+S(r,g), \end{aligned}$$

Hence using (3.3), (3.4), *Lemmas* 2.1 and 2.2 we get from the second fundamental theorem that

$$\begin{aligned}
& nT(r,f) \\
& \leq T(r,F) + N_{k+2}(r,0;P(f)) - N_2(r,0;F) + S(r,f) \\
& \leq \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,1;F) + N_{k+2}(r,0;P(f)) - N_2(r,0;F) - N_0(r,0;F^{'}) \\
& \leq 2\overline{N}(r,\infty,F) + \overline{N}(r,\infty;G) + \overline{N}(r,0;F) + N_{k+2}(r,0;P(f)) + \overline{N}(r,0;F| \geq 2) \\
& + \overline{N}(r,0;G| \geq 2) + \overline{N}_*(r,1;F,G) + \overline{N}(r,1;F| \geq 2) + \overline{N}_0(r,0;G^{'}) - N_2(r,0;F) \\
& + S(r,f) + S(r,g) \\
& \leq 2\,\overline{N}(r,\infty;P(f)) + 2\,\overline{N}(r,\infty;P(g)) + N_{k+2}(r,0;P(f)) + N_2(r,0;G) \\
& + S(r,f) + S(r,g) \\
& \leq 2\,\overline{N}(r,\infty;P(f)) + (2+k)\,\overline{N}(r,\infty;P(g)) + N_{k+2}(r,0;P(f)) + N_{k+2}(r,0;P(g)) \\
& + S(r,f) + S(r,g) \\
& \leq 2\,\overline{N}(r,\infty;f) + (2+k)\,\overline{N}(r,\infty;g) + N_{k+2}(r,0;(f-c_l)^l P_*(f)) \qquad (3.5) \\
& + N_{k+2}(r,0;(g-c_l)^l P_*(g)) + S(r,f) + S(r,g) \\
& \leq 2\overline{N}(r,\infty;f) + (k+2)\overline{N}(r,\infty;g) + (k+2)\{T(r,f)+T(r,g)\} + r\{T(r,f)+T(r,g)\} \\
& \leq (3k+2r+8)T(r) + S(r) \qquad (3.6)
\end{aligned}$$

In a similar way we can obtain

$$nT(r,g) \quad \leq \quad (3k+2r+8)T(r) + S(r). \qquad (3.7)$$

From (3.5) and (3.7) we get

$$(l-3k-r-8)\;T(r) \leq S(r),$$

which is a contradiction since $l > 3k+r+8$.

Case 2. Let $H \equiv 0$. Then the Theorem follows from *Lemmas* 2.9, 2.14 and 2.18. □

Proof of Theorem 1.2. In this case F and G share 1 IM.

Case 1. Let $H \not\equiv 0$. Here we see that

$$N_E^{1)}(r,1;F \mid= 1) \leq N(r,0;H) \leq N(r,\infty;H) + S(r,F) + S(r,G). \qquad (3.8)$$

Now using *Lemmas* 2.3, 2.19, 2.20, 2.21, (3.1) and (3.8) we get

$$\overline{N}(r,1;F) \leq N_E^{1)}(r,1;F) + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}_E^{(2}(r,1;F) \qquad (3.9)$$

$$\leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}_{*}(r,1;F,G)$$
$$+\overline{N}_{L}(r,1;F)+\overline{N}_{L}(r,1;G)+\overline{N}_{E}^{(2}(r,1;F)$$
$$+\overline{N}_{0}(r,0;F^{'})+\overline{N}_{0}(r,0;G^{'})+S(r,f)+S(r,g)$$
$$\leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+2\overline{N}_{L}(r,1;F)$$
$$+2\overline{N}_{L}(r,1;G)+\overline{N}_{E}^{(2}(r,1;F)+\overline{N}_{0}(r,0;F^{'})+\overline{N}_{0}(r,0;G^{'})+S(r,f)+S(r,g)$$
$$\leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}_{F>1}(r,1;G)$$
$$+\overline{N}_{G>1}(r,1;F)+\overline{N}_{L}(r,1;F)+N(r,1;G)-\overline{N}(r,1;G)+\overline{N}_{0}(r,0;F^{'})+\overline{N}_{0}(r,0;G^{'})$$
$$+S(r,f)+S(r,g)$$
$$\leq 3\,\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_{2}(r,0;F)+\overline{N}(r,0;F)+N_{2}(r,0;G)+N(r,1;G)$$
$$-\overline{N}(r,1;G)+\overline{N}_{0}(r,0;G^{'})+\overline{N}_{0}(r,0;F^{'})+S(r,f)+S(r,g)$$
$$\leq 3\,\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_{2}(r,0;F)+\overline{N}(r,0;F)+N_{2}(r,0;G)$$
$$+N(r,0;G^{'}|G\neq 0)+\overline{N}_{0}(r,0;F^{'})+S(r)$$
$$\leq 3\overline{N}(r,\infty;f)+3\overline{N}(r,\infty;g)+N_{2}(r,0;F)+\overline{N}(r,0;F)+N_{2}(r,0;G)+\overline{N}(r,0;G)$$
$$+\overline{N}_{0}(r,0;F^{'})+S(r).$$

Hence using (3.9), *Lemmas* 2.1 and 2.2 we get from second fundamental theorem that

$$nT(r,f)$$
$$\leq \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)+N_{k+2}(r,0;P(f))-N_{2}(r,0;F)-N_{0}(r,0;F^{'})$$
$$\leq 4\overline{N}(r,\infty,P(f))+3\overline{N}(r,\infty;P(g))+N_{2}(r,0;F)+2\,\overline{N}(r,0;F)+N_{k+2}(r,0;P(f))$$
$$+N_{2}(r,0;G)+\overline{N}(r,0;G)-N_{2}(r,0;F)+S(r,f)+S(r,g)$$
$$\leq 4\overline{N}(r,\infty;P(f))+3\overline{N}(r,\infty;P(g))+N_{k+2}(r,0;P(f))+2\,\overline{N}(r,0;F)+N_{2}(r,0;G)$$
$$+\overline{N}(r,0;G)+S(r,f)+S(r,g)$$
$$\leq 4\overline{N}(r,\infty;P(f))+3\overline{N}(r,\infty;P(g))+N_{k+2}(r,0;P(f))+2\,k\overline{N}(r,\infty;P(f))$$
$$+2\,N_{k+1}(r,0;P(f))+k\,\overline{N}(r,\infty;g)+N_{k+2}(r,0;P(g))+k\overline{N}(r,\infty;g)$$
$$+N_{k+1}(r,0;P(g))+S(r,f)+S(r,g)$$
$$\leq (2k+4)\overline{N}(r,\infty;f)+(2k+3)\overline{N}(r,\infty;g)+(3k+3r+4)T(r,f)$$
$$+(2k+2r+3)T(r,g)+S(r,f)+S(r,g)$$
$$\leq (9k+5r+14T(r)+S(r). \tag{3.10}$$

In a similar way we can obtain

$$n\,T(r,g)\leq (9k+5r+14)T(r)+S(r). \tag{3.11}$$

Combining (3.10) and (3.11) we see that

$$(l-9k-4r-14)\,T(r)\leq S(r). \tag{3.12}$$

When $l>9k+4r+14$, (3.12) leads to a contradiction.

Case 2. Let $H \equiv 0$. Then the Theorem follows from *Lemmas* 2.9, 2.14 and 2.18. This completes the proof of the Theorem. □

Proof of Corollary 1.1 *and* 1.2. From Theorem 1.1 and 1.2 one can easily prove the corollaries. So we omit the details. □

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