Stud. Univ. Babeş-Bolyai Math. 61(2016), No. 4, 443–451

Abstract method of upper and lower solutions and application to singular boundary value problems

Radu Precup

Dedicated to Professor Ioan A. Rus on the occasion of his 80th anniversary

Abstract. The method of upper and lower solutions is presented for the fixed point problem associated to operators which are compositions of a linear operator and a nonlinear mapping. Spectral properties of the linear part together with growth and monotonicity properties of the nonlinear part are involved. An application to singular boundary value problems is included.

Mathematics Subject Classification (2010): 47J25, 47H07, 34B16, 34B18.

Keywords: Upper and lower solution, monotone iterative principle, fixed point, cone, positive solution, singular boundary value problem.

1. Introduction

One of the most useful methods for solving nonlinear equations arising from mathematical modeling of real processes is the method of upper and lower solutions (see [1], [2], [5], [6], [8], [10], [12], [13], [14]). It consists in localizing solutions of an operator equation

$$u = Tu$$

in an order interval $[u_0, v_0]$, where u_0 is a *lower solution*, i.e.

$$u_0 \leq Tu_0,$$

v_0 is an *upper solution*, i.e.

$$v_0 \geq Tv_0,$$

and u_0, v_0 are *comparable* in the sense of order, that is $u_0 \leq v_0$. Thus a basic problem is to find comparable lower and upper solutions. In this paper we present such type of results for the abstract Hammerstein equation

$$u = ANu \tag{1.1}$$

in an ordered Banach space X. Here A is a linear operator and N is a nonlinear mapping from X to X. Although the main motivation is in applications to real processes from science and engineering, a general abstract method is essential in order to understand unitarily particular results and to make clear the applicability of the method to a specific problem.

2. Main results

The first result guarantees that the solutions of an equation $u = A\Phi u$ simpler than (1.1), are upper (lower) solutions for (1.1) provided that Φ (respectively, N) dominates N (respectively, Φ).Throughout this paper we shall use the same symbol $\leq$ to denote the order relation in different ordered sets.

Theorem 2.1. *Let X and Y be two ordered sets, $N : X \to Y$ be any mapping and $A : Y \to X$ be an increasing operator. Assume that there are $D \subset X$ and $\Phi : D \to Y$ such that*

$$Nu \leq \Phi u \quad (\textit{respectively, } Nu \geq \Phi u) \tag{2.1}$$

for all $u \in D$. Then any solution $u \in D$ of the equation

$$u = A\Phi u, \tag{2.2}$$

if there is one, is an upper (respectively, lower) solution of the equation $u = ANu$.

Proof. Assume $v_0 \in D$ solves (2.2). Then, from (2.1) we have

$$Nv_0 \leq \Phi v_0$$

and since A is increasing,

$$ANv_0 \leq A\Phi v_0 = v_0.$$

Hence v_0 is an upper solution. Similarly, if $Nu \geq \Phi u$ on D, then any solution of (2.2) is a lower solution of the equation $u = ANu$. □

If in Theorem 2.1 we add linearity, then we obtain the following result.

Corollary 2.2. *Let X, Y be ordered linear spaces, $N : X \to Y$ any mapping and $A : Y \to X$ a linear increasing operator. Let K_X be the cone of all elements u of X with $u \geq 0$. Assume there are $c \in \mathbf{R}_+$ and $w_0 \in Y$ such that*

$$Nu \leq cu + w_0 \tag{2.3}$$

for all $u \in K_X \cap Y$. Then any solution $v_0 \in K_X$ of the equation

$$u - cAu = Aw_0 \tag{2.4}$$

is an upper solution of (1.1). *If in addition,*

$$-N(-u) \leq cu + w_0 \tag{2.5}$$

for all $u \in K_X \cap Y$, then $u_0 := -v_0$ is a lower solution of (1.1).

Proof. In Theorem 2.1 take $D = K_X \cap Y$ and $\Phi u = cu + w_0$. For the second part of the corollary, take $D = (-K_X) \cap Y$, $\Phi u = cu - w_0$. □

Equation (2.4) suggests that more applicable results can be established if we take into account the spectral properties of A.

Theorem 2.3. *Let X be a Banach space ordered by a normal cone K. Assume that $A : X \to X$ is a completely continuous linear operator whose non-zero eigenvalues are positive and that A satisfies the weak maximum principle*

$$u - \alpha Au = Aw, \ w \in K \quad \textit{implies} \ \ u \in K \tag{2.6}$$

for every $\alpha \in (-\infty, |A|^{-1})$. In addition assume that $N : X \to X$ is a continuous mapping such that

$$Nu \le cu + w_0, \qquad N(-u) \ge -cu - w_0 \tag{2.7}$$

for all $u \in K$ and some $0 < c < |A|^{-1}$, $w_0 \in K$, and there exists $a \in \mathbf{R}_+$ such that the operator

$$Nu + au \quad \textit{is increasing on} \quad [-v_0, v_0],$$

where v_0 is the (unique) solution of the equation $u - cAu = Aw_0$.

Then equation (1.1) has at least one solution. Moreover, if the set $\mathcal{S}_+$ ($\mathcal{S}_-$) of all solutions $u \ge 0$ (respectively, $u \le 0$) is nonempty, then it has a maximal (respectively, minimal) element.

Proof. First note that for any constant $\alpha < |A|^{-1}$, the operator $I - \alpha A$ is injective (equivalently, bijective, according to the Fredholm's alternative [4, p. 92]). Indeed, otherwise for some $u \in X \setminus \{0\}$ one has $u - \alpha Au = 0$. For $\alpha < 0$ this is impossible since all non-zero eigenvalues are assumed to be positive (here $1/\alpha$ is a non-zero eigenvalue). If $\alpha = 0$, this equality is obviously impossible. It remains to discuss the case $\alpha > 0$. Then $|u| = \alpha |Au| \le \alpha |A| |u|$, whence $\alpha \ge |A|^{-1}$, a contradiction. Thus our claim is proved.

Let v_0 be the unique solution of the equation $u - cAu = Aw_0$. From (2.6) one has $v_0 \ge 0$. Now, (2.7) guarantees both (2.3), (2.5). Thus, by Corollary 2.2, v_0 is an upper solution and $u_0 := -v_0$ is a lower solution. Let

$$N_a u = Nu + au.$$

The equation $u = ANu$ is equivalent to

$$u = (I + aA)^{-1} AN_a u.$$

Let

$$T_a = (I + aA)^{-1} AN_a.$$

Clearly T_a is completely continuous on $[u_0, v_0]$. Also T_a is increasing on $[u_0, v_0]$ since N_a is increasing by our hypothesis and $(I + aA)^{-1} A$ is increasing as well. Indeed, if $w \in K$ and $u := (I + aA)^{-1} Aw$, then $u + aAu = Aw$ and by the weak maximum principle $u \in K$. Hence the linear operator $(I + aA)^{-1} A$ is increasing. In addition

$$T_a v_0 \le v_0. \tag{2.8}$$

To prove this denote $u := T_a v_0$. Then

$$\begin{aligned} u + aAu &= ANv_0 + aAv_0 = A(cv_0 + w_0 - h) + aAv_0 \\ &= cAv_0 + Aw_0 - Ah + aAv_0 = v_0 - Ah + aAv_0 \end{aligned}$$

where $h := cv_0 + w_0 - Nv_0 \in K$. Consequently

$$v_0 - u + aA(v_0 - u) = Ah$$

and by the weak maximum principle $v_0 - u \geq 0$ which proves (2.8). Similarly,

$$u_0 \leq T_a u_0.$$

Let u^*, v^* be the minimal, respectively maximal solution in $[u_0, v_0]$ as guaranteed by the Monotone Iterative Principle (see [9] and [13]). One has

$$-v_0 \leq u^* \leq v^* \leq v_0.$$

We now show that if $w \in K$ solves $w = ANw$, then $w \leq v_0$. Indeed, from

$$w = ANw = A(cw + w_0 - h) = cAw + Aw_0 - Ah,$$

where $h := cw + w_0 - Nw \in K$, and

$$v_0 = cAv_0 + Aw_0, \tag{2.9}$$

by subtraction, we obtain

$$v_0 - w - cA(v_0 - w) = Ah.$$

Then by the weak maximum principle, $v_0 - w \geq 0$ and so $w \in [0, v_0]$. Consequently $w \leq v^*$. Hence v^* is maximal in $\mathcal{S}_+$. Similarly, if $w \in -K$ and $w = ANw$, then $-v_0 \leq w$. Hence u^* is minimal in $\mathcal{S}_-$. □

For our next theorem, an existence and localization result of a nonnegative nonzero solution, we assume that X is a Hilbert space with inner product and norm $(.,.)$, $|.|$ ordered by a normal cone K, which is also a vector lattice with respect to the order relation introduced by K. Then any element $x \in X$ can be written as a difference of two elements $x^+, x^-$of K, that is $x = x^+ - x^-$, where $x^+ = x \vee 0$ and $x^- = (-x) \vee 0$. Thus for an element x one has $x \geq 0$, if and only if $x^- = 0$. We also assume that

$$(x,\ y) \geq 0 \ \text{ for all } \ x, y \in K \ \text{ and } \ \left(x^+,\ x^-\right) = 0 \ \text{ for every } \ x \in X. \tag{2.10}$$

We also note that if $A : X \to X$ is a completely continuous positive (with $(Au,\ u) \geq 0$ for all $u \in X$) self-adjoint linear operator, then there exists $u_1 \in X$, $|u_1| = 1$, such that

$$|A| = (Au_1,\ u_1). \tag{2.11}$$

This follows from the characterization of the norm of self-adjoint linear operators:

$$|A| = \sup_{u \neq 0} \frac{|(Au,\ u)|}{|u|^2}.$$

Since $(x^+,\ x^-) = 0$ for every $x \in X$, we have $\left|u_1^+ + u_1^-\right| = \left|u_1^+ - u_1^-\right| = 1$. Also if $A(K) \subset K$, then

$$\begin{aligned} |A| &= \left(A\left(u_1^+ - u_1^-\right),\ u_1^+ - u_1^-\right) \\ &= \left(Au_1^+,\ u_1^+\right) + \left(Au_1^-,\ u_1^-\right) - 2\left(Au_1^+,\ u_1^-\right) \\ &\le \left(Au_1^+,\ u_1^+\right) + \left(Au_1^-,\ u_1^-\right) + 2\left(Au_1^+,\ u_1^-\right) \\ &= \left(A\left(u_1^+ + u_1^-\right),\ u_1^+ + u_1^-\right) \le |A| \left|u_1^+ + u_1^-\right|^2 \\ &= |A|. \end{aligned}$$

Hence in (2.11) we may assume that $u_1 \ge 0$ (otherwise, replace u_1 by $u_1^+ + u_1^-$).

Theorem 2.4. *Let* $A : X \to X$ *be a completely continuous positive self-adjoint linear operator such that weak maximum principle* (2.6) *holds, and let* $N : X \to X$ *be any continuous mapping such that* $N(0) = 0$,

$$Nu \le cu + w_0 \tag{2.12}$$

for all $u \in K$ *and some* $0 \le c < |A|^{-1}$, $w_0 \ge u_1$, *and*

$$N(\varepsilon u_1) \ge \varepsilon |A|^{-1} u_1 \tag{2.13}$$

for all $\varepsilon \in [0, \varepsilon_0]$ *and some* $\varepsilon_0 > 0$. *Here* $u_1 \in K$, $|u_1| = 1$ *and* $(Au_1,\ u_1) = |A|$. *In addition assume that there exists* $a \in \mathbf{R}_+$ *with*

$$Nu + au \quad \textit{increasing on} \quad [0, v_0],$$

where v_0 *is the* (*unique*) *solution of the equation* $u - cAu = Aw_0$. *Then equation* (1.1) *has a maximal solution in* $K \setminus \{0\}$.

Proof. First note that the non-zero eigenvalues of A are positive since A is positive. As above, the unique solution v_0 of the equation $u - cAu = Aw_0$ is an upper solution of the equation $u = ANu$. Since $N(0) = 0$, the null element is a solution, and so a lower solution. Now we apply the Monotone Iterative Principle to deduce the existence of a maximal fixed point v^* in $[0, v_0]$ of the operator

$$T_a = (I + aA)^{-1} AN_a.$$

As in the proof of Theorem 2.3 we can show that v^* is maximal in the set of all nonnegative solutions. To show that $v^* \ne 0$, we prove that v^* is the maximal fixed point of T_a in an order subinterval $[u_0, v_0] \subset [0, v_0]$ with $u_0 \ne 0$.

For any fixed $v \in X$ we consider the function

$$g(t) = \frac{(A(u_1 + tv),\ u_1 + tv)}{|u_1 + tv|^2},$$

which can be defined on a neighborhood of $t = 0$. This function attains its maximum $|A|$ at $t = 0$, so $g'(0) = 0$. Notice

$$g'(0) = 2\left[(Au_1,\ v) - |A|(u_1,\ v)\right].$$

Hence

$$u_1 = |A|^{-1} Au_1$$

(i.e., $|A|$ is the largest eigenvalue of A and u_1 is an eigenvector). Let $u_0 = \varepsilon u_1$, where $0 < \varepsilon \leq \varepsilon_0$. Clearly

$$u_0 \geq 0, \quad u_0 \neq 0, \quad u_0 = |A|^{-1} Au_0.$$

Using (2.13), we deduce

$$\begin{aligned} u_0 &= |A|^{-1} Au_0 = A\left(|A|^{-1} u_0\right) \\ &\leq ANu_0. \end{aligned}$$

Thus u_0 is a lower solution of $u = ANu$. Also, from

$$v_0 = cAv_0 + Aw_0, \quad u_0 = |A|^{-1} Au_0,$$

we have

$$\begin{aligned} v_0 - u_0 &= cA(v_0 - u_0) + \left(c - |A|^{-1}\right) Au_0 + Aw_0. \\ &= cA(v_0 - u_0) + A\left[\left(c - |A|^{-1}\right) u_0 + w_0\right]. \end{aligned}$$

Since $w_0 \geq u_1$, we may write $w_0 = u_1 + h$, where $h = w_0 - u_1 \in K$. Then

$$v_0 - u_0 = cA(v_0 - u_0) + A\left[\left(\left(c - |A|^{-1}\right)\varepsilon + 1\right) u_1 + h\right].$$

Now we choose $\varepsilon > 0$ small enough so that $\left(c - |A|^{-1}\right)\varepsilon + 1 \geq 0$. Then

$$\left(\left(c - |A|^{-1}\right)\varepsilon + 1\right) u_1 \in K$$

and

$$\left(\left(c - |A|^{-1}\right)\varepsilon + 1\right) u_1 + h \in K$$

too, and by the maximum principle, $v_0 - u_0 \geq 0$. Next we apply the Monotone Iterative Principle to deduce the existence of a maximal fixed point in $[u_0, v_0]$ of T_a. Clearly it is equal to v^*. □

Remark 2.5. *Under the assumptions on* X *from Theorem 2.4, the weak maximum principle holds for* A *on* $\left(-\infty, |A|^{-1}\right)$ *if it holds on* $(-\infty, 0]$.

Indeed, if (2.6) *holds on* $(-\infty, 0]$, *then, in particular (take* $\alpha = 0$ *in* (2.6)) $A(K) \subset K$. *Furthermore, assume* $\alpha \in \left(0, |A|^{-1}\right)$ *and* $u := v - \alpha Av \in K$. *We have to show that* $v \geq 0$, *equivalently* $v^- = 0$. *Assume the contrary, i.e.* $v^- \neq 0$. *Then if we multiply by* v^- *and we use* (2.10), *we obtain*

$$\begin{aligned} 0 &\leq (v^-,\ u) = (v^-,\ v) - \alpha(v^-,\ Av) \\ &= (v^-,\ v^+) - |v^-|^2 - \alpha(v^-,\ Av^+) + \alpha(v^-,\ Av^-) \\ &\leq -|v^-|^2 + \alpha(v^-,\ Av^-). \end{aligned}$$

It follows that

$$\alpha \geq \frac{|v^-|^2}{(v^-,\ Av^-)}.$$

But $(v^-, \ A v^-) \leq |A| \, |v^-|^2$. *Then* $\alpha \geq \dfrac{1}{|A|}$, *a contradiction. Thus* $v^- = 0$.

3. Application to singular boundary value problems

We shall apply the above results to the boundary value problem for a singular second order differential equation

$$\begin{cases} -\frac{1}{p}\left(pu'\right)' = q\left(t\right) f\left(u\right), & 0 < t < 1 \\ \left(pu'\right)(0) = 0 \\ u\left(1\right) = 0 \end{cases} \tag{3.1}$$

where $p \in C\left[0,1\right] \cap C^1\left(0,1\right)$ with $p > 0$ on $(0,1)$, $\displaystyle\int_0^1 \frac{1}{p\left(t\right)} dt < \infty$ and $q \in L^{\infty}\left(\left[0,1\right], \mathbf{R}_+\right)$. The equation is singular if p is zero at $t = 0$ or/and $t = 1$. Such kind of problems are in connection with radial solutions to stationary diffusion and waves equations and arise from mathematical modelling of many processes in physics and biology [3], [7], [11].

By a solution of (3.1) we mean a function $u \in C\left[0,1\right] \cap C^1\left(0,1\right)$, with $pu' \in AC\left[0,1\right]$ which satisfies the differential equation for almost every $t \in (0,1)$.

Let $X = L^2_p\left[0,1\right]$ with inner product and norm

$$(u,v) = \int_0^1 p \; uv \; dt, \quad |u| = \left(\int_0^1 p \; u^2 dt \right)^{1/2}.$$

Clearly, X is vector lattice ordered by the regular (hence, normal) cone K of all nonnegative functions, with the additional property (2.10).

Denote $Lu = -\frac{1}{p}\left(pu'\right)'$, where

$$\begin{aligned} D\left(L\right) &= \{u \in C\left[0,1\right] \cap C^1\left(0,1\right) : \; pu' \in AC\left[0,1\right], \\ Lu &\in L^2_p\left[0,1\right], \; \left(pu'\right)(0) = u\left(1\right) = 0\}. \end{aligned}$$

It is easy to see that for every $h \in L^2_p\left[0,1\right]$ there is a unique $u \in D\left(L\right)$ with $Lu = h$, and

$$u\left(t\right) = \int_t^1 \frac{1}{p\left(s\right)} \int_0^s p\left(\tau\right) h\left(\tau\right) d\tau ds.$$

Let A be the inverse of L, more exactly

$$A : L^2_p\left[0,1\right] \to L^2_p\left[0,1\right], \quad \left(Ah\right)(t) = \int_t^1 \frac{1}{p\left(s\right)} \int_0^s p\left(\tau\right) h\left(\tau\right) d\tau ds.$$

We note that A has all the required properties, i.e., it is completely continuous, positive, self-adjoint (see e.g. [11]) and satisfies the weak maximum principle. To prove the last property, according to Remark 2.5, it is sufficient to show that (2.6) holds for $\alpha \leq 0$. For $\alpha = 0$ this trivially holds as follows looking at the expression of A. Let $\alpha < 0$ and let $u - \alpha Au = Aw$ for some $w \in K$. Then

$$\begin{cases} -\frac{1}{p}\left(pu'\right)' - \alpha u = w, & 0 < t < 1 \\ u\left(1\right) = \left(pu'\right)(0) = 0. \end{cases} \tag{3.2}$$

Suppose that $u \notin K$. Then there would be an interval $[a, b]$, $0 \leq a < b \leq 1$ such that

$$\begin{aligned} u &< 0 \text{ in } (a,b), \quad u(b) = 0 \text{ and} \\ \text{either } a &= 0, \quad \text{or} \quad 0 < a \text{ and } u(a) = 0. \end{aligned}$$

Then on $[a, b]$, one has $-\frac{1}{p}(pu')' \geq 0$, i.e. $(pu')' \leq 0$. Hence pu' is decreasing on $[a, b]$ and since $(pu')(b) \geq 0$, we must have $pu' \geq 0$ on $[a, b]$. Then u is increasing and since $u(b) = 0$, we have $u(a) < 0$. Hence $a = 0$ and $u < 0$ in $(0, b)$. Now integration from 0 to b in (3.2) gives

$$-(pu')(b) - \alpha \int_0^b pudt = \int_0^b pwdt.$$

Since $\int_0^b pwdt \geq 0$ and $\alpha \int_0^b pudt > 0$, we deduce $(pu')(b) < 0$, a contradiction. Therefore $u \geq 0$ in $[0, 1]$.

Theorem 3.1. *Assume* $f \in C^1(\mathbf{R})$ *and*

$$\limsup_{|x| \to \infty} \left| \frac{f(x)}{x} \right| < |A|^{-1} |q|^{-1}_{L^\infty[0,1]}. \tag{3.3}$$

Then problem (3.1) *has at least one solution. Moreover, if the set* $\mathcal{S}_+$ $(\mathcal{S}_-)$ *of all solutions* $u \geq 0$ (*respectively,* $u \leq 0$) *is nonempty, then it has a maximal* (*respectively, minimal*) *element.*

Proof. From (3.3) we can find a $c_0 \in \left(0, |A|^{-1} |q|^{-1}_{L^\infty[0,1]}\right)$ and a $\mu > 0$ such that

$$|f(x)| \leq c_0 |x| \quad \text{for} \quad |x| > \mu.$$

Next the continuity of f on $[-\mu, \mu]$ guarantees the existence of a $c_1 > 0$ with

$$|f(x)| \leq c_1 \quad \text{on} \quad [-\mu, \mu].$$

Thus

$$|f(x)| \leq c_0 |x| + c_1 \quad \text{for all } x \in \mathbf{R}. \tag{3.4}$$

This implies that the mapping

$$N(u)(t) = q(t) f(u(t))$$

is well-defined and continuous from $L^2_p[0, 1]$ to itself and

$$|Nu|_{L^2_p[0,1]} \leq |q|_{L^\infty[0,1]} \left(c_0 |u|_{L^2_p[0,1]} + c_1 |1|_{L^2_p[0,1]} \right).$$

On the other hand, if $u \in K = L^2_p([0,1], \mathbf{R}_+)$, then (3.4) guarantees

$$N(u) \leq cu + w_0 \quad \text{and} \quad -N(-u) \leq cu + w_0,$$

where $c = c_0 |q|_{L^\infty[0,1]} < |A|^{-1}$ and $w_0 = c_1 |q|_{L^\infty[0,1]}$.

If v_0 is the (unique) solution of the equation $u - cAu = Aw_0$, then $v_0 \in C([0,1], \mathbf{R}_+)$ and so $0 \leq v_0(t) \leq M$ for all $t \in [0, 1]$ and some $M > 0$. Function f being C^1, there is a number $a \in \mathbf{R}_+$ such that

$$|q|_{L^\infty[0,1]} f'(x) + a \geq 0 \quad \text{for all} \quad x \in [-M, M].$$

Consequently, $N(u)+au$ is an increasing operator on $[-v_0, v_0]$ and we may apply Theorem 2.3. □

Finally Theorem 2.4 yields the following result.

Theorem 3.2. *Assume* $q \equiv 1,\ f \in C^1(\mathbf{R}_+, \mathbf{R}),\ f(0)=0$ *and*

$$\limsup_{x\to\infty} \frac{f(x)}{x} < |A|^{-1} < f'(0).$$

Then problem (3.1) *has a maximal positive solution.*

References

[1] Agarwal, R.P., O'Regan, D., Precup, R., *Construction of upper and lower solutions with applications to singular boundary value problems*, J. Comput. Anal. Appl., **7**(2005), 205-221.

[2] Amman, H., *Fixed point equations and nonlinear eigenvalues problems in ordered Banach spaces,* SIAM Review, **18**(1976), 620-709.

[3] Anderson, N., Arthurs, A.M., *Analytic bounding functions for diffusion problems with Michaelis-Menten kinetics,* Bull. Math. Biol., **47**(1985), 145-153.

[4] Brezis, H., *Analyse fonctionnelle,* Masson, Paris, 1983.

[5] Cabada, A., *An overview of the lower and upper solutions method with nonlinear boundary value conditions*, Boundary Value Problems, **2011**(2011), Article ID 893753, 18 pages.

[6] Carl, S., Heikkilä, S., *Nonlinear Differential Equations in Ordered Spaces*, Chapman & Hall/CRC, Boca Raton, 2000.

[7] Chan, C.Y., Hon, Y.C., *Computational methods for generalized Emden Fowler models of neutral atoms,* Quart. Appl. Math., **46**(1988), 711-726.

[8] De Coster, C., Habets, P., *Two-Point Boundary Value Problems: Lower and Upper Solutions,* Elsevier, Amsterdam, 2006.

[9] Deimling, K., *Nonlinear Functional Analysis,* Springer, Berlin, 1985.

[10] Guo, D., Lakshmikantham, V., *Nonlinear Problems in Abstract Cones,* Academic Press, Boston, 1988.

[11] O'Regan, D., *Theory of Singular Boundary Value Problems,* World Scientific, Singapore, 1994.

[12] Precup, R., *Monotone technique to the initial values problem for a delay integral equation from biomathematics,* Studia Univ. Babeş-Bolyai Math., **40**(1995), no. 2, 63-73.

[13] Precup, R., *Methods in Nonlinear Integral Equations,* Kluwer, Dordrecht, 2002.

[14] Rus, I.A., *Fixed points, upper and lower fixed points: abstract Gronwall lemmas,* Carpathian J. Math., **20**(2004), 125-134.

Radu Precup
Department of Mathematics
Babeş-Bolyai University
400084 Cluj, Romania
e-mail: r.precup@math.ubbcluj.ro